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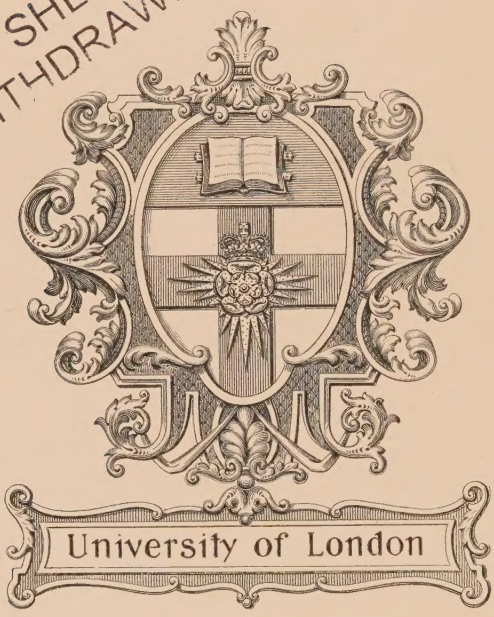

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PAPERS

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PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

A CONTRIBUTION TO THE THEORY OF INVARIANT INTEGRALS

By F. H. Murray.

[Received 22 November, 1924.—Read 11 December, 1924.]

In this paper we are concerned with the point transformations and the corresponding transformations of certain definite integrals, defined by the general solution of a differential system

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n), \tag{1}$$

in which the functions X_i are continuous, together with their partial derivatives of the first order, within a certain closed region (R); and such that a function $M(x_1, x_2, \ldots, x_n)$ exists, which is positive, and continuous, together with its partial derivatives of the first order in (R), satisfying

$$\frac{\partial}{\partial x_1}(MX_1)+\frac{\partial}{\partial x_2}(MX_2)+\ldots+\frac{\partial}{\partial x_n}(MX_n) = 0. \tag{2}$$

If the functions X_i, M were analytic, the equations (1) and (2) would imply* that the integral

$$I = \int_V \ldots \int M\, dx_1 dx_2 \ldots dx_n \tag{3}$$

is invariant for the continuous transformation defined by (1), V being an n-dimensional region in (R) bounded by a finite number of analytic

* Poincaré, *Les méthodes nouvelles de la mécanique celeste*, 3 (1899), 41–43.

surfaces. In the usual proofs of this invariance, however, it is customary to assume that the functions X_i, if not analytic, at least possess derivatives of order higher than the first. For some applications it is convenient not to make this assumption, and the present paper is chiefly concerned with a proof of the invariance of the integral (3) under the assumptions made above.*

In the first two paragraphs a type of approximation to the functions X_i and their derivatives of the first order by means of analytic functions is considered; in those remaining these results are applied to a study of the integral (3) over certain specified regions.

1. *Approximation to a continuous function by means of a definite integral.*

Suppose $f(x_1, x_2, \ldots, x_n)$ continuous, together with its partial derivatives of the first order in a region (D): $0 \leqslant x_i \leqslant 1$, $(i = 1, 2, \ldots, n)$, and suppose z a positive number. If $\omega(\delta)$ is the maximum of the quantity

$$|f(x_1', x_2', \ldots, x_n') - f(x_1'', x_2'', \ldots, x_n'')| \tag{1}$$

when $(x_1', x_2', \ldots, x_n')$, $(x_1'', x_2'', \ldots, x_n'')$ are points of (D), while $|x_i' - x_i''| \leqslant \delta$, it follows from the Heine-Borel theorem that $\lim_{\delta \to 0} \omega(\delta) = 0$.

Suppose

$$J(z) = \int_0^1 \int_0^1 \cdots \int_0^1 f(\xi_1, \xi_2, \ldots, \xi_n) \frac{\exp\left\{-\sum_{i=1}^{n} (x_i - \xi_i)^2/4z\right\}}{\{2\sqrt{(\pi z)}\}^n} d\xi_1 d\xi_2 \ldots d\xi_n. \tag{2}$$

It will be shown that the integral $J(z)$ furnishes an approximation to the function f, and that the derivatives of $J(z)$ approximate to the derivatives of f, if the constant z is chosen sufficiently small. Considered as a function of $x_1, x_2, \ldots, x_n, z$ we have evidently

$$\frac{\partial^2 J}{\partial x_1^2} + \ldots + \frac{\partial^2 J}{\partial x_n^2} - \frac{\partial J}{\partial z} = 0. \tag{3}$$

If $\phi(x)$ is any continuous function defined for $0 \leqslant x \leqslant 1$, suppose

$$J(x, z) = \int_0^1 \frac{\phi(\xi)}{2\sqrt{(\pi z)}} e^{-(x-\xi)^2/4z} d\xi. \tag{4}$$

---

* From the results of §§ 3, 4, 5 the truth of certain statements made in a preceding paper [*Bull. Amer. Math. Soc.*, 30 (1924), 18] becomes obvious.

Then, if $0<x<1,\ 0<h<x,\ 0<h<1-x,$

$$J=\int_0^{x-h}(\ldots)\,d\xi+\int_{x-h}^{x+h}(\ldots)\,d\xi+\int_{x+h}^{1}(\ldots)\,d\xi=J'+J''+J'''$$

and $$J''=\int_{x-h}^{x}(\ldots)\,d\xi+\int_{x}^{x+h}(\ldots)\,d\xi=J_1''+J_2''.$$

If M is the maximum of $|\phi(x)|$ on the given interval and $\omega'(\delta)$ the function defined for ϕ in the same manner as $\omega(\delta)$ is defined for f, then, as before, $\lim_{\delta\to 0}\omega'(\delta)=0$. Finally, if $\int_t^\infty e^{-s^2}\,ds=\sqrt{(\pi)}\,l(t)$, we obtain

$$|J'|<Ml\left(\frac{h}{2\sqrt{z}}\right),\quad |J'''|<Ml\left(\frac{h}{2\sqrt{z}}\right), \tag{5}$$

by means of the substitutions

$$u=\frac{x-\xi}{2\sqrt{z}},\quad u'=\frac{\xi-x}{2\sqrt{z}},$$

respectively. If these substitutions are made in J_1 and J_2'', we find

$$J_1''=\frac{\phi(x)}{\sqrt{\pi}}\int_0^{h/2\sqrt{z}}e^{-s^2}\,ds+\frac{1}{\sqrt{\pi}}\int_0^{h/2\sqrt{z}}[\phi(x-2\sqrt{z}\,s)-\phi(x)]\,e^{-s^2}\,ds$$

$$=\phi(x)\left[\tfrac{1}{2}-l\left(\frac{h}{2\sqrt{z}}\right)\right]+K',$$

in which $$|K'|\leqslant\frac{\omega'(h)}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^{h/2\sqrt{z}}e^{-s^2}\,ds<\frac{\omega'(h)}{2}.$$

In the same manner it is seen that

$$J_2''=\phi(x)\left[\tfrac{1}{2}-l\left(\frac{h}{2\sqrt{z}}\right)\right]+K'',\quad |K''|<\frac{\omega'(h)}{2}.$$

Combining the inequalities obtained, we find

$$|J(x,z)-\phi(x)|<4Ml\left(\frac{h}{2\sqrt{z}}\right)+\omega'(h). \tag{6}$$

From this inequality we can easily determine an upper bound for the difference between $J(z)$ and the function f. From (2), if

$$d_n = \int_0^1 \frac{e^{-(x_n-\xi_n)^2/4z}}{2\sqrt{(\pi z)}} f(\xi_1, \ldots, \xi_n)\, d\xi_n - f(\xi_1, \ldots, \xi_{n-1}, x_n),$$

$$J(z) = \int_0^1 \frac{e^{-(x_1-\xi_1)^2/4z}}{2\sqrt{(\pi z)}} \int_0^1 \frac{e^{-(x_2-\xi_2)^2/4z}}{2\sqrt{(\pi z)}} \cdots \int_0^1 \frac{e^{-(x_{n-1}-\xi_{n-1})^2/4z}}{2\sqrt{(\pi z)}} f(\xi_1, \ldots, \xi_{n-1}, x_n) d\xi_1 \ldots d\xi_{n-1}$$

$$+ \int_0^1 \frac{e^{-(x_1-\xi_1)^2/4z}}{2\sqrt{(\pi z)}} \int_0^1 \frac{e^{-(x_2-\xi_2)^2/4z}}{2\sqrt{(\pi z)}} \cdots \int_0^1 \frac{e^{-(x_{n-1}-\xi_{n-1})^2/4z}}{2\sqrt{(\pi z)}} d_n\, d\xi_1 \ldots d\xi_{n-1}$$

$$= J_{n-1} + K_n, \quad |K_n| < \text{Max}\,|d_n|,$$

since

$$\int_0^1 \frac{e^{-(x_i-\xi_i)^2/4z}}{2\sqrt{(\pi z)}} d\xi_i = \frac{1}{\sqrt{\pi}} \int_{-x_i/2\sqrt{z}}^{(1-x_i)/2\sqrt{z}} e^{-s^2} ds < \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2} ds = 1.$$

The integral J_{n-1} can be broken up into two parts in the same manner, so that finally, if $0 < a < x_i < 1-a < 1$, $h < a$,

$$J(z) = f(x_1, x_2, \ldots, x_n) + K_1 + K_2 + \ldots + K_n,$$

$$|K_i| < \text{Max}\,|d_i|,$$

d_i being defined as above, as the difference between an integral of the form (4) and the function $f(\xi_1, \xi_2, \ldots, \xi_{i-1}, x_i, x_{i+1}; x_n)$, $i = 1, \ldots, n$.

From the definition of the function $\omega(\delta)$, we obtain, if $|f| \leqslant M$ in (D),

$$|d_i| < 4Ml\left(\frac{h}{2\sqrt{z}}\right) + \omega(h),$$

and consequently

$$|J(z) - f(x_1, x_2, \ldots, x_n)| < n\left[4Ml\left(\frac{h}{2\sqrt{z}}\right) + \omega(h)\right], \tag{7}$$

the same constant h being chosen at each step.

2. *Approximation to the derivatives of f.*

Since the integrand in (2) is analytic in $x_1, x_2, \ldots, x_n$ the same is true of the integral, and differentiation may be carried out under the integral

sign. From the relation

$$\frac{\partial}{\partial x_i}\left[\exp\left\{-\sum_{i=1}^{n}(x_i-\xi_i)^2/4z\right\} f(\xi_1, \ldots, \xi_n)\right]$$

$$= -\frac{\partial}{\partial \xi_i}\left[\exp\left\{-\sum_{i=1}^{n}(x_i-\xi_i)^2/4z\right\}\right] f(\xi_1, \ldots, \xi_n)$$

$$= -\frac{\partial}{\partial \xi_i}\left[\exp\left\{-\sum_{i=1}^{n}(x_i-\xi_i)^2/4z\right\} f(\xi_1, \ldots, \xi_n)\right]$$

$$+\exp\left\{-\sum_{i=1}^{n}(x_i-\xi_i)^2/4z\right\}\frac{\partial f}{\partial \xi_i},$$

we obtain

$$(8)\quad \frac{\partial J}{\partial x_1} = \int_0^1 \cdots \int_0^1 \frac{\exp\left\{-\sum_{i=1}^{n}(x_i-\xi_i)^2/4z\right\}}{\{2\sqrt{(\pi z)}\}^n} f_{\xi_1}(\xi_1, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n$$

$$+\int_0^1 \cdots \int_0^1 \left[\frac{e^{-x_1^2/4z}}{2\sqrt{(\pi z)}} f(0, \xi_2, \ldots, \xi_n) - \frac{e^{-(1-x_1)^2/4z}}{2\sqrt{(\pi z)}} f(1, \xi_2, \ldots, \xi_n)\right]$$

$$\times \frac{\exp\left\{-\sum_{i=2}^{n}(x_i-\xi_i)^2/4z\right\}}{\{2\sqrt{(\pi z)}\}^{n-1}} d\xi_2 \ldots d\xi_n$$

$$= J' + J''.$$

Each of these integrals can be evaluated approximately by means of the results of the preceding paragraph; if $|\partial f/\partial x_i| \leqslant M_i$ in (D), and $\omega_i(\delta)$ represents the variation of $\partial f/\partial x_i$ for $|x_1' - x_1''| \leqslant \delta$; then if x_i satisfies the conditions under which (7) was derived,

$$J' = f_{x_1}(x_1, x_2, \ldots, x_n) + K', \quad |K'| < n\left[4M_1 l\left(\frac{h}{2\sqrt{z}}\right) + \omega_1(h)\right],$$

$$|J''| < \frac{M}{\sqrt{(\pi z)}} e^{-a^2/4z} \quad (e^{-x_1^2/4z},\ e^{-(1-x_1)^2/4z} < e^{-a^2/4z}),$$

and hence

$$\left|\frac{\partial J}{\partial x_1} - f_{x_1}(x_1, \ldots, x_n)\right| < \frac{Me^{-a^2/4z}}{\sqrt{(\pi z)}} + n\left[4M_1 l\left(\frac{h}{2\sqrt{z}}\right) + \omega_1(h)\right],$$

and in general

$$(9)\quad \left|\frac{\partial J}{\partial x_i} - f_{x_i}(x_1, \ldots, x_n)\right| < \frac{Me^{-a^2/4z}}{\sqrt{(\pi z)}} + n\left[4M_i l\left(\frac{h}{2\sqrt{z}}\right) + \omega_i(h)\right] \quad (i = 1, \ldots, n).$$

Now if ϵ is an arbitrarily small positive number, the constant h $(< a)$ can be chosen so small that $n\omega(h) < \epsilon/2$, $n\omega_i(h) < \epsilon/4$; with this choice the constant z can be chosen so small that

$$4nMl\left(\frac{h}{2\sqrt{z}}\right) < \frac{\epsilon}{2}, \quad 4nM_i l\left(\frac{h}{2\sqrt{z}}\right) < \frac{\epsilon}{4}, \quad \frac{Me^{-a^2/4z}}{\sqrt{(\pi z)}} < \frac{\epsilon}{4},$$

since $l(h/s)$ approaches zero with s. The functions $\omega(h)$, $\omega_i(h)$ being monotone, the required inequalities can all be satisfied; and for this choice

$$|J(z)-f(x_1, \ldots, x_n)| < \epsilon, \tag{3}$$

$$\left|\frac{\partial J}{\partial x_i}-f_{x_i}(x_1, \ldots, x_n)\right| < \epsilon \quad (i = 1 \ldots n).$$

If a set of functions f', f'', ..., $f^{(m)}$ is such that $f^{(i)}$ is continuous, together with its partial derivatives of the first order, in (D), then if the constants $M^{(i)}$, $M_k^{(i)}$ are defined for $f^{(i)}$ as M and M_k were defined for f, and the functions $\omega_\kappa^{(i)}(\delta)$ and $\omega^{(i)}(\delta)$ are defined as before, it is seen in the same manner that, if $0 < a < \frac{1}{2}$, the constants h and z can be so chosen that if ϵ is given,

$$|J^{(i)}(z)-f^{(i)}| < \epsilon, \quad \left|\frac{\partial J^{(i)}}{dx_\kappa}-f_{x_\kappa}^{(i)}\right| < \epsilon, \tag{4}$$

$$a \leqslant x_\kappa \leqslant 1-a, \quad i = 1, \ldots, m, \quad \kappa = 1, \ldots, n.$$

3. *Application to the theory of invariant integrals.*

The inequalities obtained above will be employed in a study of the solutions of the differential system

$$\frac{dx_i}{dt} = X_i(x_1, x_2, \ldots, x_n) \quad (|X_i| < M, \ i = 1, \ldots, n), \tag{1}$$

in which the functions X_i are continuous, together with their partial derivatives of the first order in the region (D). Suppose

$$\frac{dy_i}{dt} = Y_i(y_1, y_2, \ldots, y_n) \quad (|Y_i| < M', \ i = 1, \ldots, n), \tag{2}$$

a system in which the functions Y_i are analytic within the region (R): $0 < a \leqslant y_i \leqslant b < 1$, and such that in (R)

$$|X_i(x_1, \ldots, x_n)-Y_i(x_1, \ldots, x_n)| < \lambda,$$

$$\left|\frac{\partial X_i}{\partial x_\kappa}-\frac{\partial Y_i}{\partial x_\kappa}\right| < \mu \quad \left(\left|\frac{\partial X_i}{\partial x_\kappa}\right| \leqslant A_\kappa\right), \tag{3}$$

$$a \leqslant x_i \leqslant b, \quad i, \kappa = 1, \ldots, n.$$

Suppose $(a_1, a_2, \ldots, a_n)$ interior to (R), and let c be so small that the region δ: $|x_i-a_i| \leqslant c$ $(i = 1, \ldots, n)$ lies in the interior of (R). Then

solutions of (1), (2) can be defined by the method of successive approximations* in which

$$(4)\quad \begin{cases} x_i' = a_i + \int_0^t X_i(a_1, \ldots, a_n)\, dt, & y_i' = a_i + \int_0^t Y_i(a_1, \ldots, a_n)\, dt, \\ x_i^{(r+1)} = a_i + \int_0^t X_i(x_1^{(r)}, \ldots, x_n^{(r)})\, dt, & y_i^{(r+1)} = a_i + \int_0^t Y_i(y_1^{(r)}, \ldots, y_n^{(r)})\, dt \\ & (r = 1, 2, \ldots). \end{cases}$$

If

$$(5)\qquad 0 < t \leqslant T' < \frac{c}{M}, \quad 0 < t \leqslant T'' < \frac{c}{M'},$$

each sequence of approximations converges uniformly to the solution of (1) or (2), respectively, and $|x_i(t) - a_i| \leqslant c$, $|y_i(t) - a_i| \leqslant c$, $0 < t \leqslant T$, $T \leqslant T'$, $T \leqslant T''$.

From (3), (4),

$$|x_i' - y_i'| \leqslant \int_0^t [X_i(a_1, \ldots, a_n) - Y_i(a_1, \ldots, a_n)]\, dt \leqslant \lambda t.$$

Suppose $|x_i^{(r)} - y_i^{(r)}| \leqslant \rho_r(t)$ $(i = 1, \ldots, n)$. Then from (4),

$$\begin{aligned} |x_i^{(r+1)} - y_i^{(r+1)}| &\leqslant \int_0^t |Y_i(y_1^{(r)}, \ldots, y_n^{(r)}) - X_i(y_1^{(r)}, \ldots, y_n^{(r)})|\, dt \\ &\quad + \int_0^t |X_i(y_1^{(r)}, \ldots, y_n^{(r)}) - X_i(x_1^{(r)}, \ldots, x_n^{(r)})|\, dt \\ &\leqslant \lambda t + \int_0^t \Big(\sum_{s=1}^{n} A_s |y_s^{(r)} - x_s^{(r)}| \Big)\, dt \\ &\leqslant \lambda t + K \int_0^t \rho_r\, dt \quad (i = 1, \ldots, n), \end{aligned}$$

if $K = \sum_{k=1}^{n} A_k$. Choosing $\rho_1(t) = \lambda t$,

$$|x_i'' - y_i''| \leqslant \lambda t + K\lambda \frac{t^2}{2} = \frac{\lambda}{K}\left[Kt + \frac{(Kt)^2}{2}\right].$$

If we assume $\rho_r(t) = \dfrac{\lambda}{K} \sum_{s=1}^{r} \dfrac{(Kt)^s}{s!}$, we find immediately

$$|x_i^{(r+1)} - y_i^{(r+1)}| \leqslant \frac{\lambda}{K} \sum_{s=1}^{r+1} \frac{(Kt)^s}{s!} = \rho_{r+1},$$

* Picard, *Traité d'Analyse*, 2 (1905), § 343.

and consequently for every r,

$$|x_i^{(r)}-y_i^{(r)}| \leqslant \frac{\lambda}{K}\sum_{s=1}^{r}\frac{(Kt)^s}{s!} < \frac{\lambda}{K}(e^{Kt}-1). \tag{6}$$

Since $x_i^{(r)}(t)$ approaches $x_i(t)$ uniformly, and $y_i^{(r)}(t)$ approaches $y_i(t)$ uniformly, we have also,

$$|x_i(t)-y_i(t)| \leqslant \frac{\lambda}{K}(e^{Kt}-1) \quad (i=1, \ldots, n,\ 0<t\leqslant T). \tag{7}$$

Suppose α, β numbers satisfying the conditions $a<\alpha<\beta<b$, and let (S) be the region $\alpha \leqslant x_i \leqslant \beta$, $i=1, \ldots, n$, while S is the surface of this n-dimensional parallelepiped. Suppose c the smallest of the numbers $\alpha-a$, $b-\beta$; then if $(a_1, a_2, \ldots, a_n)$ is any point of (S), the region $|x_i-a_i| \leqslant c$ is interior to (R), and from the discussion above, a solution of (1) taking on the values $(a_1, \ldots, a_n)$ for $t=0$ is defined and remains in (R) for $0<t\leqslant T$.

Let $\quad x_i=\phi_i(t;\ a_1, \ldots, a_n), \quad y=\psi_i(t;\ a_1, \ldots, a_n) \quad (i=1, \ldots, n),$

be the solution of (1), (2) respectively, taking on the values $(a_1, \ldots, a_n)$ for $t=0$; the functions ϕ_i, ψ_i are continuous in all their arguments, and the partial derivatives $\partial\phi_i/\partial a_r$, $\partial\psi_i/\partial a_r$ exist and are continuous;* furthermore, considered as functions of t, these partial derivatives satisfy the equations

$$\begin{cases} \dfrac{d\xi_i^{(r)}}{dt} = \displaystyle\sum_{s=1}^{n}\frac{\partial X_i}{\partial x_s}\xi_s^{(r)}, \quad \xi_s^{(r)}=\dfrac{\partial\phi_s}{\partial a_r} \quad (i=1, \ldots, n), \\ \dfrac{d\eta_i^{(r)}}{dt} = \displaystyle\sum_{s=1}^{n}\frac{\partial Y_i}{\partial y_s}\eta_s^{(r)}, \quad \eta_s^{(r)}=\dfrac{\partial\psi_s}{\partial a_r} \quad (i=1, \ldots, n), \end{cases} \tag{8}$$

$$[\xi_s^{(r)}]_{t=0} = \begin{cases} 0, & s\neq r, \\ 1, & s=r, \end{cases} \qquad [\eta_s^{(r)}]_{t=0} = \begin{cases} 0, & s\neq r, \\ 1, & s=r, \end{cases} \qquad (r, s=1, \ldots, n).$$

Since

$$\frac{\partial X_i}{\partial x_s}-\frac{\partial Y_i}{\partial y_s} = \left[\frac{\partial X_i}{\partial x_s}\right]_{x_k=\phi_k} - \left[\frac{\partial X_i}{\partial x_s}\right]_{x_k=\psi_k} + \left[\frac{\partial X_i}{\partial x_s}\right]_{x_k=\psi_k} - \left[\frac{\partial Y_i}{\partial y_s}\right]_{y_k=\psi_k},$$

we obtain

$$\left|\frac{\partial X_i}{\partial x_s}-\frac{\partial Y_i}{\partial y_s}\right| \leqslant \nu, \quad \nu=\mu+\mathrm{Max}\left(\left[\frac{\partial X_i}{\partial x_s}\right]_{x_k=\phi_k} - \left[\frac{\partial X_i}{\partial x_s}\right]_{x_k=\psi_k}\right); \tag{9}$$

* Goursat, *Cours d'Analyse*, 3 (1915), § 460.

and, since the derivatives $\partial X_i/\partial x_s$ are continuous, ν approaches zero with λ, μ. Also

$$\left| \sum_{s=1}^{n} \left[\frac{\partial X_i}{\partial x_s}\right]_{x_k=\phi_k} u_s - \sum_{s=1}^{n} \left[\frac{\partial Y_i}{\partial y_s}\right]_{y_k=\psi_k} u_s \right| \leqslant \nu \sum_{s=1}^{n} |u_s| \leqslant \nu n A$$

if $|u_s| \leqslant A$. If equations (8) are integrated by the method of successive approximations, upper limits for the functions $|\xi_s^{(r)}|$, $|\eta_s^{(r)}|$ can easily be found, if $(a_1, \ldots, a_n)$, is in (S); if A is determined as an upper bound of these limits, we obtain the inequality* analogous to (7),

$$(7') \qquad |\xi_s^{(r)} - \eta_s^{(r)}| \leqslant \frac{\nu n A}{K}(e^{Kt}-1) \quad (r, s = 1, \ldots, n, \ 0 < t \leqslant T).$$

If each point of a point-set E lying in the interior of (S) is taken as the initial point at $t = 0$ for a solution of (1), the set E' of points defined by these solutions at a time $t' > 0$ may be termed the *consequent*† of E, $0 < t' < T$. Since the functions ϕ_i, ψ_i are continuous, the consequent of a closed set is a closed set, while the consequent of an open set is an open set. Since equations (1), (2) do not contain t explicitly, the initial value of t can be chosen arbitrarily.

Suppose (S_0) the region defined by the inequalities

$$a < a_i < \beta_i < \beta, \quad a_i \leqslant x_i \leqslant \beta_i \quad (i = 1, \ldots, n),$$

S_0 being the boundary of this region, and let S_0' be the consequent of S_0, $(S_0)'$ the consequent of (S_0) at $t = \tau < T/4$. Since the transformation is continuous, interior points are transformed into interior points, and boundary points go over into boundary points, consequently

$$(S_0)' = (S_0').$$

It is proposed to show that, if $M(x_1, x_2, \ldots, x_n)$ is any function which is continuous within (R), and the integrals are taken in the Lebesgue sense,

$$\int_{(S_0')} \cdots \int M' \, dx_1 \, dx_2 \ldots dx_n = \int_{(S_0)} \cdots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} \, da_1 \ldots da_n.$$

* The equations (8) are not of the type of (1), (2), since the coefficients in (8) contain the time explicitly; but the presence of t in the coefficients does not alter the reasoning, if the inequalities (3) are satisfied for $0 < t \leqslant T$.

† Poincaré, *Les méthodes nouvelles de la mécanique céleste*, 3 (1899), 175, 176.

For this purpose, consider an arbitrary subdivision of (R) into parallelepipeds whose faces are defined by the equations

$$x_i = x_i^{(k)}, \quad k = 1, 2, \ldots, r_i, \quad i = 1, 2, \ldots, n; \quad x_i' = a, \quad x_i^{(r_i)} = b,$$

$$0 < x_i^{(k+1)} - x_i^{(k)} < \Delta_r < \frac{c}{4}.$$

Suppose (S_r') the region consisting of the points of all the parallelepipeds of a fixed subdivision (r), which possess points in common with (S_0'), and suppose that S_r'' is such that S_r' is the consequent of S_r'' according to equations (2); corresponding points $(y_1', y_2', \ldots, y_n')$, $(y_1'', y_2'', \ldots, y_n'')$ of S_r' and S_r'', respectively, satisfy the equations* $y_i'' = \psi_i(-\tau, y_1', y_2', \ldots, y_n')$, $i = 1, \ldots, n$.

The points of (S_r'') are interior to (R) since each point $(y_1'', y_2'', \ldots, y_n'')$ or (y'') of the interior and boundary of (S_r'') is the transform of a point (y') of the interior or boundary of (S_r'), such that $|y_i'' - y_i'| \leqslant c/4$; to each point (y') can be made to correspond at least one point (x') of (S') such that $|y_i' - x_i'| \leqslant \Delta_r < c/4$; and since $|x_i' - a_i| \leqslant c/4$, (a) being a point of (S_0), we have $|y_i'' - a_i| \leqslant 3c/4 < c$.

Since the functions Y_i are analytic, we obtain

$$\int_{(S_r)} \cdots \int M' \, dy_1 \ldots dy_n = \int_{(S_r')} \cdots \int M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} \, da_1 \ldots da_n,$$

and hence

$$\int_{(S_r')} \cdots \int M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} \, da_1 \ldots da_n = \int_{(S_0)} \cdots \int M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} \, da_1 \ldots da_n + I', \tag{10}$$

$$I' = \int_{(S_r')} \cdots \int (\ldots) \, da_1 \ldots da_n - \int_{(S_0)} \cdots \int (\ldots) \, da_1 \ldots da_n.$$

* If in equations (1) or (2) the substitution $t = -\bar{t}$ is made, the method of Picard defines the solutions for positive values of $\bar{t}$, and $|x_i - a_i| \leqslant c$, $|y_i - a_i| \leqslant c$, $0 < \bar{t} \leqslant T$, $T \leqslant c/M$, $T \leqslant c/M'$, $i = 1, \ldots, n$. If the same transformation is made in equations (8), the same inequalities (7') and (7) are found to be valid for $0 > t = -\bar{t} > -T$. Hence a solution of (1) or (2) taking on the system of values $(a_1, a_2, \ldots, a_n)$ belonging to (S) at $t = 0$ must remain within (R) for $-T \leqslant t \leqslant T$, and (7), (7') with t replaced by $|t|$ are valid for this interval. In the same manner we obtain $|x_i - a_i| \leqslant c/4$, $|y_i - a_i| \leqslant c/4$, $-T/4 \leqslant t \leqslant T/4$, replacing c by $c/4$ in the discussion preceding.

From (7′) it follows that a constant B can be obtained independent of λ, μ such that $|\eta_s^{(r)}| < B$, $0 < t \leqslant T$, if $(a_1, a_2, \ldots, a_n)$ is a point of (S). Suppose

$$\left| M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} \right| < L, \quad (-T \leqslant t \leqslant T).$$

Let S'' be such that S' is the consequent of S'' according to the transformation $x_i' = \psi(\tau\,;\ \bar{y}_1'', \ldots, \bar{y}_n'')$, $\bar{y}_i'' = \psi(-\tau\,;\ x_1', \ldots, x_n')$ $(i = 1, \ldots, n)$. If (y'') is a point of (S_τ''),

$$|\bar{y}_i'' - y_i''| \leqslant \sum_{m=1}^{n} B|y_m' - x_m'| \leqslant nB\Delta_r \quad (i = 1, \ldots, n).$$

Consequently the points of S_0, S_r'' must satisfy the inequalities

$$(11) \qquad |y_i'' - a_i| \leqslant |a_i - \bar{y}_i''| + |y_i'' - \bar{y}_i''| \leqslant nB\Delta_r + \frac{\lambda}{K}(e^{K\tau} - 1) = d,$$

if the points (a) are suitably chosen.

In the difference I' the integral over the region common to (S_τ) and (S_0) drops out, leaving the integral over the region exterior to S_0 but interior to S_r'', minus the integral over the region exterior to S_r'' but interior to S_0. Hence $|I'|$ is at most equal to the sum of these integrals. From (11) it follows that S_r' lies within a parallelepiped each of whose faces extends a distance d beyond a corresponding face of S_0, and is exterior to a parallelepiped each face of which is interior to S_0 and at a distance d from a corresponding face of S_0. Hence $|I'|$ is less than the integral over the region between these two parallelepipeds, and consequently* if $d < \beta_i - a_i$ $(i = 1, \ldots, n)$,

$$(12) \qquad |I'| < L\left[\prod_{m=1}^{n}[\beta_m - a_m + d] - \prod_{m=1}^{n}[\beta_m - a_m - d]\right].$$

From the theory of linear equations and (8), (9), if

$$\theta(t) = \sum_{m=1}^{n}\left[\frac{\partial X_m}{\partial x_m}\right]_{x_i = \phi_i}, \quad \theta'(t) = \sum_{m=1}^{n}\left[\frac{\partial Y_m}{\partial y_m}\right]_{y_i = \psi_i},$$

---

* de la Vallée Poussin, *Intégrales de Lebesgue, etc.* (Paris, 1916), for if E_1, E_2 are the two regions referred to above,

$$|I'| = \left|\int_{E_1} M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)}\, dP - \int_{E_2} M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)}\, dP\right| \leqslant L\left[\int_{E_1} dP + \int_{E_2} dP\right]$$
$$\leqslant L[m(E_1) + m(E_2)],$$

by Theorem 1°, p. 41; since E_1 and E_2 have only points of S_0 in common, of measure 0, the truth of the statement follows.

then $$\frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} = e^{\int_0^{\theta(t)dt}}, \quad \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} = e^{\int_0^{\theta'(t)dt}},$$

and $$\frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} = \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} + \eta,$$

$$\int_{(S_0)} \cdots \int M \frac{D(y_1, \ldots, y_n)}{D(a_1, \ldots, a_n)} da_1 \ldots da_n = \int_{(S_0)} \cdots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} da_1 \ldots da_n + \eta',$$

where η and η' vanish with λ, μ. Combining the inequalities above,

$$(13) \quad \left| \int_{(S_r')} \cdots \int M' dy_1 \ldots dy_n - \int_{(S_0)} \cdots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} da_1 \ldots da_n \right| \leqslant |I'| + |\eta'|.$$

If $\bar{S}_r'$ is the boundary of the system of all the parallelepipeds lying entirely in the interior of S', except for points on their boundaries, $\bar{S}_r''$ the surface having $\bar{S}_r'$ for its consequent under (2), we obtain in the same manner

$$(14) \quad \left| \int_{(\bar{S}_r')} \cdots \int M' dy_1 \ldots dy_n - \int_{(S_0)} \cdots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} da_1 \ldots da_n \right| \leqslant |\bar{I}'| + |\bar{\eta}'|,$$

and consequently

$$(15) \quad \left| \int_{(S_r')} \cdots \int M' dy_1 \ldots dy_n - \int_{(\bar{S}_r')} \cdots \int M' dy_1 \ldots dy_n \right| \leqslant |I'| + |\eta'| + |\bar{I}'| + |\bar{\eta}'|.$$

If we substitute $M = 1$ in (15), and if $m(E)$ denotes the measure of a point-set E, we obtain from (15), $|m(S_r') - m(\bar{S}_r')| \leqslant |I_1'| + |\bar{I}_1'| + |\eta_1'| + |\bar{\eta}_1|$. Since (S') is included in (S_r'), and includes $(\bar{S}_r')$, $m(\bar{S}_r') \leqslant m(S') \leqslant m(S_r')$, consequently

$$|m(S') - m(S_r')| \leqslant \delta = |I_1'| + |\bar{I}_1'| + |\eta_1'| + |\bar{\eta}_1'|.$$

If $S_r' = S' + e$, $m(e) \leqslant \delta$, consequently*

$$\left| \int_{(S')} \cdots \int M' dy_1 \ldots dy_n - \int_{(S_r')} \cdots \int M' dy_1 \ldots dy_n \right| \leqslant \delta \operatorname{Max}(M);$$

* By Max (M) is meant the maximum of the absolute value of M in the region (R).

and from (13), since the integral is independent of the variables of integration,

$$(16)\quad \left| \int_{(S')} \cdots \int M' \, dx_1 \ldots dx_n - \int_{(S_0)} \cdots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} \, da_1 \ldots da_n \right| \leqslant \delta \operatorname{Max}(M) + |I'| + |\eta'|.$$

The right-hand member of (16) can be made arbitrarily small, while the left-hand member is a constant; hence it must be zero:

$$(17)\quad \int_{(S')} \cdots \int M' \, dx_1 \ldots dx_n = \int_{(S_0)} \cdots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)} \, da_1 \ldots da_n.$$

If the function M satisfies equation (2) of the introduction, along a trajectory

$$\frac{d}{dt}\left(M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\right) = 0,$$

and consequently the integral

$$I = \int_{(S')} \cdots \int M' \, dx_1 \ldots dx_n$$

is independent of τ, so that it is an invariant integral.

Generalization of the preceding results.

Suppose O an open point-set consisting only of points of (S); then it is known* that

$$O = O_1 + O_2 + \ldots + O_m + \ldots,$$

in which O_r is composed of a finite number of parallelepipeds and two sets O_r, O_s possess only boundary points in common. Hence if $\bar{O}_r$ is the set of points of O_r not in any of the sets $O_1, O_2, \ldots, O_{r-1}$,

$$(2)\quad O_1 + O_2 + \ldots + O_m = \bar{O}_1 + \bar{O}_2 + \ldots + \bar{O}_m,$$

$$O = \bar{O}_1 + \bar{O}_2 + \ldots + \bar{O}_m + \ldots.$$

Let O_r', $\bar{O}_r'$ be the consequents of O_r, $\bar{O}_r$ respectively, at $t = \tau < T/4$,

* de la Vallée Poussin, *loc. cit.*, 13.

while O' is the consequent of O; we have obviously

$$O' = O_1' + O_2' + \ldots + O_r' + \ldots = \bar{O}_1' + \bar{O}_2' + \ldots + \bar{O}_r' + \ldots . \tag{3}$$

Since $O_r = \bar{O}_r + e$, $m(e) = 0$, $O_r' = \bar{O}_r' + e'$, $m(e') = 0$;* from § 3 we obtain

$$\int_{\bar{O}_r} \ldots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP = \int_{\bar{O}_r'} \ldots \int M'\, dP'$$

$$= \int_{O_r'} \ldots \int M'\, dP' = \int_{O_r} \ldots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP,$$

and from the additive property of the Lebesgue integral

$$\int_{O'} \ldots \int M'\, dP' = \sum_{r=1}^{\infty} \int_{\bar{O}_r'} \ldots \int M'\, dP' \tag{4}$$

$$= \sum_{r=1}^{\infty} \int_{\bar{O}_r} \ldots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP = \int_{O} \ldots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP.$$

Finally, suppose E any measurable set in the interior of (S), E' its consequent at $t = \tau < T/4$; there exists an open set O in (S) and a closed set F such that $F < E < 0$, $m(O) - m(F) < \epsilon$, ϵ a given positive number. Furthermore, $q' = O' - F'$ is an open set, whence

$$m(O') - m(F') = \int_{q'} \ldots \int dP' = \int_{q} \ldots \int \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP \leqslant e^{K\tau} m(q) \leqslant \epsilon e^{K\tau},$$

since from (3), § 3, $$\left| \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i} \right| \leqslant K.$$

Since $F' < E' < O'$, and $m(O') - m(F')$ can be made arbitrarily small, the set E' is measurable, and from the theorem of the mean for the Lebesgue integral,

$$\left| \int_{O'} \ldots \int M'\, dP' - \int_{E'} \ldots \int M'\, dP' \right| \leqslant m(q') \operatorname{Max} |M'| \leqslant \epsilon e^{K\tau} \operatorname{Max} |M'|,$$

$$\left| \int_{O} \ldots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP - \int_{E} \ldots \int M \frac{D(x_1, \ldots, x_n)}{D(a_1, \ldots, a_n)}\, dP \right| \leqslant \epsilon e^{K\tau} \operatorname{Max} |M|,$$

---

* If a face of a parallelepiped F_1: $x_1 = \alpha_1$, $\alpha_i \leqslant x_i \leqslant \beta_i$ $(i = 2 \ldots n)$ is enclosed in a region (S_1): $\alpha_1 - \delta \leqslant x_1 \leqslant \alpha_1 + \delta$, $\alpha_i \leqslant x_i \leqslant \beta_i$ $(i = 2 \ldots n)$, δ being so small that (S_1) lies entirely within (S), and F_1' is the consequent of F_1, $(S_1)'$ the consequent of (S_1), we obtain from (17), § 3, with $M = 1$,

$$m(F_1') \leqslant m(S_1)' \leqslant e^{K\tau} m(S_1),$$

and since $m(S_1)$ approaches zero with δ, $m(F') = 0$. Consequently the measure of a finite or infinite number of consequents of faces of parallelepipeds is also zero.

consequently

$$\left|\int_{E'}\dots\int M'\,dP' - \int_{E}\dots\int M\,\frac{D(x_1,\,\dots,\,x_n)}{D(a_1,\,\dots,\,a_n)}\,dP\right| \leqslant 2\epsilon e^{K\tau}\,\mathrm{Max}\,|M|,$$

and finally

$$\int_{E'}\dots\int M'\,dP' = \int_{E}\dots\int M\,\frac{D(x_1,\,\dots,\,x_n)}{D(a_1,\,\dots,\,a_n)}\,dP. \tag{5}$$

The function M has been assumed continuous and bounded in (R); if M is the function satisfying equation (2) of the introduction, the function

$$M\,\frac{D(x_1,\,\dots,\,x_n)}{D(a_1,\,\dots,\,a_n)} = \phi(a_1 \dots a_n)$$

is constant along each trajectory, hence the right-hand member of (5) is independent of τ, and the left-hand member is an invariant integral.

5. *An application to the question of stability.*

Borrowing a term from hydrodynamics, we shall call the function

$$\theta = \frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \dots + \frac{\partial X_n}{\partial x_n}, \tag{1}$$

the divergence of the system of equations

$$\frac{dx_i}{dt} = X_i(x_1,\,\dots,\,x_n) \quad (i = 1,\,\dots,\,n). \tag{2}$$

Then from § 3,

$$\frac{D(x_1,\,\dots,\,x_n)}{D(a_1,\,\dots,\,a_n)} = e^{\int_0^t \theta\,dt} \leqslant e^{Kt} \quad [0 < t \leqslant T,\ (a_1,\,\dots,\,a_n) \text{ in } (S)]. \tag{3}$$

If the equations (2) are in the form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1,\,\dots,\,m), \tag{4}$$

the divergence is zero, whence it follows that the measure of E' is the same as that of E, if E is any measurable point-set contained in (S).

Suppose $(\bar{a}_1,\,\bar{a}_2,\,\dots,\,\bar{a}_n)$ a point of (S) such that $X_i(\bar{a}_1,\,\bar{a}_2,\,\dots,\,\bar{a}_n) = 0$ $(i = 1,\,\dots,\,n)$. The system of solutions of (2) may be said to be stable in the neighbourhood of $(\bar{a}_1,\,\dots,\,\bar{a}_n)$ if, given any region $(a)\ |x_i - \bar{a}_i| \leqslant a$, a region $(c)\ |x_i - \bar{a}_i| \leqslant c < a$ can be found such that the consequent of any

point of (c) remains in (a) for every $\tau > 0$. It follows immediately from (5), § 4, that if the divergence is positive at $(\bar{a}_1, \ldots, \bar{a}_n)$, the system cannot be stable for $\tau > 0$; if the divergence is negative, the system cannot be stable for $\tau < 0$.

For if $\theta(\bar{a}_1 \ldots \bar{a}_n) > 0$, a neighbourhood (A) belonging to (S) can be found such that $\theta \geqslant h > 0$ in (A); suppose (a) any region such that (a) is included in (A), and assume, if possible, that a region (c), $c < a$, exists such that the consequent of every point of (c) remains in (a) for $\tau > 0$. From (5), § 3, with the theorem of the mean, if $(c)'$ is the consequent of (c),*

$$m(c') = \int_{(c')} dP' \geqslant e^{h\tau} m(c).$$

Consequently the measure of $(c)'$ increases indefinitely with the time, hence for some $\tau > 0$, $m(c)' > m(a)$. This is impossible, since every point of $(c)'$ is a point of (a), by hypothesis. The second statement is proved in the same manner.

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* The inequality (5) is valid for $\tau > T$, if every point of $(c)'$ is known to remain within (S) for $0 < t \leqslant \tau$; for in this case a set of time-intervals $\tau_r \leqslant t \leqslant \tau_{r+1}$, $r = 0, 1, \ldots, m-1$, can be found, such that $\tau_{r+1} - \tau_r < c/4$, $\tau_{m-1} < \tau = \tau_m$,

$$\int_{(c)^{r+1}} M^{(r+1)}\, dx_1^{(r+1)} \ldots dx_n^{(r+1)} = \int_{(c)^r} M^{(r)} \frac{D(x_1^{(r+1)}, \ldots, x_n^{(r+1)})}{D(x_1^{(r)}, \ldots, x_n^{(r)})}\, dx_1^{(r)} \ldots dx_n^{(r)} \quad (r = 0, 1, \ldots, m-1),$$

$$\int_{(c)^m} M^{(m)}\, dx_1 \ldots dx_n = \int_{(c)} M^0 \prod_{r=0}^{m-1} \frac{D(x_1^{(r+1)}, \ldots, x_n^{(r+1)})}{D(x_1^{(r)}, \ldots, x_n^{(r)})}\, dx_1^0 \ldots dx_n^0 = \int_{(c)} M^0 \frac{D(x_1^{(m)}, \ldots, x_n^{(m)})}{D(x_1^{(0)}, \ldots, x_n^{(0)})}\, dx_1^{(0)} \ldots dx_n^{(0)},$$

and

$$\prod_{r=0}^{m-1} \frac{D(x_1^{(r+1)}, \ldots, x_n^{(r+1)})}{D(x_1^{(0)}, \ldots, x_n^{(0)})} = \prod_{r=0}^{m-1} e^{\int_{\tau_r}^{\tau_{r+1}} \theta\, dt} = e^{\int_0^{\tau_m} \theta\, dt}.$$

PONCELET'S PORISTIC POLYGONS

[SECOND PAPER.]

By T. W. CHAUNDY.

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Preamble.

In an earlier paper bearing this title* I discussed the conditions for a polygon to be inscribed in one circle and, at the same time, circumscribed to a second circle, and obtained, by usually laborious methods, various porisms, some of which I believe to be new. I can now replace the tentative methods of this earlier paper by a systematic treatment, which covers all porisms of prime number, as well as some of composite number, and which extends also to certain topics avoided in A, namely, the "inseparable" case of the even-sided polygons and the porisms for two unspecialized conics. In §§1 to 4 of this paper I discuss certain functions of the data of the two conics, fundamental in my theory; in §§5 to 7 I apply them to porisms of prime number. In §§8 to 11 I discuss porisms of odd composite number; in §§12 to 14 I show how to deduce porisms of even number. The discussion so far will have referred to a pair of unspecialized conics. In §15 I show how to transform to circular data. In §§16, 17 I show how to deduce from these irrational porisms their expressions in full invariant form. In the concluding §§18, 19, I touch on self-conjugate and self-polar polygons. In the supplementary § 20 I have applied the methods of the preceding paragraphs to determine, for illustration, certain porisms based on the number eleven. I have been able to make this paper self-contained and independent of A without, I think, very appreciable overlapping.

* *Proc. London Math. Soc.* (2), 22 (1923), 104–123. I shall refer to this paper as A.

1. *i_n, the common denominator of a_n, b_n, c_n.*

I start, as before, with the fundamental addition-formulae A (11).

$$(1) \qquad a_{m\pm n} = \frac{a_m b_n c_n \mp a_n b_m c_m}{\lambda_n - \lambda_m}, \quad \text{etc.},$$

where $$\lambda_n - \lambda_m = a_n^2 - a_m^2 = b_n^2 - b_m^2 = c_n^2 - c_m^2.$$

We may deduce the algebraically equivalent formulae

$$(2) \qquad a_{m\pm n} = \frac{a_m^2 a_n^2 - (a^2-b^2)(a^2-c^2)}{a_m b_n c_n \pm a_n b_m c_m}, \quad \text{etc.},$$

where $$a_1,\ b_1,\ c_1 \equiv a,\ b,\ c\,;$$

and, in particular, the duplication-formulae

$$(3) \qquad a_{2n} = \frac{-b_n^2 c_n^2 + c_n^2 a_n^2 + a_n^2 b_n^2}{2a_n b_n c_n}, \quad \text{etc.}$$

It is sometimes convenient to use conventional negative suffixes defined, consistently with these formulae, as

$$(4) \qquad a_{-n} \equiv -a_n, \quad \text{etc.}$$

The above formulae, inductively applied, establish a_n, b_n, c_n as rational, but not integral, algebraic functions of a, b, c, homogeneous of degree unity.

Since $a_n^2 - b_n^2 = a^2 - b^2$, etc., it follows that, if a, b, c are algebraically independent, a_n, b_n, c_n have a common denominator i_n (say) and numerators algebraically interprime. Hence also, from (3), we have that

$$(5) \qquad i_{2n} = 2i_n^4 a_n b_n c_n.$$

Using (1), (2) to form the product of a_{m-n}, a_{m+n}, we see that

$$(6) \qquad i_{m-n} i_{m+n} \quad \text{divides} \quad i_m^2 i_n^2 (\lambda_n - \lambda_m).$$

Hence, writing $m+n$, $m-n$, for m, n,

$$i_{2m} i_{2n} \quad \text{divides} \quad i_{m-n}^2 i_{m+n}^2 (\lambda_{m-n} - \lambda_{m+n}).$$

By (1) $$\lambda_{m-n} - \lambda_{m+n} = 4a_m b_m c_m a_n b_n c_n (\lambda_n - \lambda_m)^{-2}$$

$$= i_{2m} i_{2n} i_m^{-4} i_n^{-4} (\lambda_n - \lambda_m)^{-2}, \quad \text{by (5)}.$$

Thus $$i_m^2 i_n^2 (\lambda_n - \lambda_m) \quad \text{divides} \quad i_{m-n} i_{m+n}$$

and so, from (6), actually

$$i_{m-n}i_{m+n} = i_m^2 i_n^2(\lambda_n - \lambda_m). \tag{7}$$

I need not stipulate that $m > n$, if I make the convention

$$i_{-n} \equiv -i_n. \tag{8}$$

If μ_n is the degree of i_n, it follows from (5) and (7) that

$$\mu_{2n} = 4\mu_n + 3, \quad \mu_{m-n} + \mu_{m+n} = 2\mu_m + 2\mu_n + 2,$$

and, of course, $\mu_1 = 0$, since $i_1 = 1$.

From these equations we can prove inductively that

$$\mu_n = n^2 - 1, \tag{9}$$

and hence that

(10) *$i_n a_n$, $i_n b_n$, $i_n c_n$ are of degree n^2.*

2. *Two special transformations.*

Now we may prove inductively that

(11) *i_{2n+1}, $i_{2n+1}a_{2n+1}/a$, i_{2n}/abc, $i_{2n}a_{2n}$ are all of the form $f(a^2, b^2, c^2)$, where f is polynomial,*

i.e. that they are rational in the data of the conics

$$\Sigma x^2 = 0, \ \Sigma a^2 x^2 = 0.$$

To do this it is sufficient to use (5) and the formulae

$$i_{2n+1} = i_n^2 i_{n+1}^2 (a_n^2 - a_{n+1}^2), \quad \text{from (7)},$$

$$i_{2n}a_{2n} = i_n^4 a_n^4 - i_n^4 (a^2 - b^2)(a^2 - c^2), \quad \text{from (3), (5)},$$

$$i_{2n+1}(a - a_{2n+1}) = 2i_n^2 i_{n+1}^2 a_n b_{n+1} c_{n+1}, \quad \text{from (1), (7)}.$$

It follows that,

(12) *if we change the sign of a alone, then every a_{2n+1}, a_{2n}, b_{2n}, c_{2n}, i_{2n} merely change sign, and every b_{2n+1}, c_{2n+1}, i_{2n+1} are unaltered.*

As a useful numerical check on the work I consider the substitution

$$a = b = c = 1,$$

which I shall denote by **N**.*

Evidently it gives $a_n = b_n = c_n$ and so, from (3)

$$a_{2n} = \tfrac{1}{2}a_n.$$

From (2) we also have

$$1/a_{m+n} = 1/a_m + 1/a_n.$$

These formulae give inductively

$$\mathbf{N}a_n = 1/n. \tag{13}$$

From (5), (7) we then have

$$\mathbf{N}i_{2n} = 2n^{-3}\mathbf{N}i_n^4, \quad \mathbf{N}i_{m-n}i_{m+n} = \mathbf{N}i_m^2 i_n^2 (m^2-n^2)\, m^{-2}n^{-2},$$

giving inductively

$$\mathbf{N}i_n = n, \tag{14}$$

and so

$$\mathbf{N}i_n a_n = 1 = \mathbf{N}i_n b_n = \mathbf{N}i_n c_n. \tag{15}$$

The result of making the substitution **N** in any expression I shall call the "numerical value" of the expression.

3. *The factors of* i_{2n+1}.

Write

$$E_{m,n} \equiv a_n b_n c_n i_n^3 i_m \left(\frac{a_m}{a_n} + \frac{b_m}{b_n} + \frac{c_m}{c_n} + 1\right),$$

$$F_{m,n} \equiv a_n b_n c_n i_n^3 i_m \left(\frac{a_m}{a_n} - \frac{b_m}{b_n} - \frac{c_m}{c_n} + 1\right),$$

Then

$$\begin{aligned} E_{m,n}F_{m,n} &= i_n^6 i_m^2 \{b_n^2 c_n^2 (a_m+a_n)^2 - a_n^2 (b_m c_n + b_n c_m)^2\} \\ &= i_n^6 i_m^2 \{2a_n b_n c_n (a_m b_n c_n - a_n b_m c_m) \\ &\qquad + (\lambda_n - \lambda_m)(-b_n^2 c_n^2 + c_n^2 a_n^2 + a_n^2 b_n^2)\} \end{aligned}$$

on reduction.

* In terms of elliptic functions this substitution evaluates residues at a common pole.

Using (1), (3), (5), (7) we have

$$E_{m,n}F_{m,n} = i_{m-n}i_{m+n}i_{2n}(a_{m+n}+a_{2n}). \tag{16}$$

So writing $$G_{m,n} \equiv a_n b_n c_n i_n^3 i_m \left(-\frac{a_m}{a_n}+\frac{b_m}{b_n}-\frac{c_m}{c_n}+1\right),$$

$$H_{m,n} \equiv a_n b_n c_n i_n^3 i_m \left(-\frac{a_m}{a_n}-\frac{b_m}{b_n}+\frac{c_m}{c_n}+1\right),$$

we have similarly

$$G_{m,n}H_{m,n} = i_{m-n}i_{m+n}i_{2n}(a_{2n}-a_{m+n}), \tag{17}$$

and hence by (7)

$$E_{m,n}F_{m,n}G_{m,n}H_{m,n} = i_{m+3n}i_{m-n}^3. \tag{18}$$

Write $m = n \pm 1$ in $E_{m,n}$ and so define

$$e_{4n\pm1} \equiv a_n b_n c_n i_n^3 i_{n\pm1}\left(\Sigma\frac{a_{n\pm1}}{a_n}+1\right), \tag{19}$$

where the double signs are all "linked," *i.e.* either we take *all* upper signs, or we take *all* lower signs.

Define likewise

$$f_{4n\pm1} \equiv \pm a_n b_n c_n i_n^3 i_{n\pm1}\left(\frac{a_{n\pm1}}{a_n}-\frac{b_{n\pm1}}{b_n}-\frac{c_{n\pm1}}{c_n}+1\right), \tag{20}$$

where again the double signs are linked, and similarly define $g_{4n\pm1}$, $h_{4n\pm1}$.

It can be seen from (12) that, changing the suffix,

(21) *any* $(-)^n f_{2n+1}$, $(-)^n g_{2n+1}$, $(-)^n h_{2n+1}$ *can be derived from the corresponding* e_{2n+1} *by changing the signs of a or b or c respectively.*

We have also from (16), (17), 18)

$$\begin{cases} e_{2n+1}f_{2n+1} = i_n i_{n+1}(a_n+a_{n+1}) \\ g_{2n+1}h_{2n+1} = i_n i_{n+1}(a_n-a_{n+1}) \\ i_{2n+1} = e_{2n+1}f_{2n+1}g_{2n+1}h_{2n+1} \end{cases} \tag{22}$$

Applying the numerical substitution **N** to (19), (20) and using (13), (14) we have

$$\begin{cases} \mathbf{N}\, e_{2n+1} = 2n+1 \\ \mathbf{N}\, f_{2n+1} = 1, \quad \text{etc.} \end{cases} \tag{23}$$

From (10) we see that

(24) $e_{2n+1}, f_{2n+1}, g_{2n+1}, h_{2n+1}$ *are each of degree* $n(n+1)$.

4. *The expressions* u_n, v_n, w_n.

From the fundamental addition-formulae (1) we have

$$b_{m\pm n}+c_{m\pm n} = \frac{(a_n \mp a_m)(b_m c_n + b_n c_m)}{\lambda_n - \lambda_m} = \frac{b_m c_n + b_n c_m}{a_n \pm a_m}, \tag{25}$$

and hence

$$\frac{b_{m+n}+c_{m+n}}{b_{m-n}+c_{m-n}} = \frac{a_n - a_m}{a_n + a_m}. \tag{26}$$

In particular

$$\frac{b_{2n+1}+c_{2n+1}}{b+c} = \frac{a_n - a_{n+1}}{a_n + a_{n+1}} = \frac{g_{2n+1}h_{2n+1}}{e_{2n+1}f_{2n+1}}, \quad \text{by (22)}, \tag{27}$$

and therefore

$$\frac{\sqrt{\{(c_{2n+1}+a_{2n+1})(a_{2n+1}+b_{2n+1})\}}}{\sqrt{\{(c+a)(a+b)\}}} = \frac{f_{2n+1}}{e_{2n+1}}. \tag{28}$$

I shall write for brevity

$$u_n, v_n, w_n \equiv \sqrt{\left(\frac{b_n+c_n}{2}\right)}, \quad \sqrt{\left(\frac{c_n+a_n}{2}\right)}, \quad \sqrt{\left(\frac{a_n+b_n}{2}\right)}, \tag{29}$$

so that

$$v_{2n+1}w_{2n+1}/vw = f_{2n+1}/e_{2n+1}. \tag{30}$$

Observe, too, from (26), (27) that

$$\frac{a_{2m}+a_{2n+1}}{a_{2m}-a_{2n+1}} = \frac{e_p f_p g_q h_q}{e_q f_q g_p h_p},$$

and hence, by (7), (22),

$$i_{2m}i_{2n+1}(a_{2m}+a_{2n+1}) = e_p f_p g_q h_q, \tag{31}$$

$$i_{2m}i_{2n+1}(a_{2m}-a_{2n+1}) = e_q f_q g_p h_p,$$

where $$p,\ q \equiv 2n \pm 2m+1.$$

Again from (3)

$$b_{2n}+c_{2n} = b_n c_n/a_n, \tag{32}$$

and so

$$2v_{2n}w_{2n} = a_n. \tag{33}$$

If the definitions (19), (20) are to extend also to suffixes 1, 3 on the left we need conventions for a_0, b_0, c_0, i_0.

Consistently with the addition-formulae we must write

$$i_0 = 0, \quad i_0 a_0 = 1, \quad \text{etc.} \tag{34}$$

We then have $$e_1 = 1 = f_1, \quad \text{etc.},$$

and

$$e_3 = bc+ca+ab, \quad f_3 = -bc+ca+ab, \quad \text{etc.} \tag{35}$$

I shall make much use of these expressions (35), and it is convenient to omit their suffixes, writing

$$e_3 \equiv e, \quad f_3 \equiv f, \quad \text{etc.}$$

They evidently obey the relation

$$e = f+g+h, \tag{36}$$

whence, from (30),

$$\frac{v_3 w_3}{vw} + \frac{w_3 u_3}{wu} + \frac{u_3 v_3}{uv} = 1,$$

and so generally

$$\frac{v_{3n} w_{3n}}{v_n w_n} + \frac{w_{3n} u_{3n}}{w_n u_n} + \frac{u_{3n} v_{3n}}{u_n v_n} = 1, \tag{37}$$

a fundamental identity.

5. The porisms.

I come now to the porisms themselves. Suppose a variable n-sided polygon $A_1 A_2 \dots A_n$ inscribed in $S\,(\Sigma x^2 = 0)$ and circumscribed to $S'\,(\Sigma a^2 x^2 = 0)$. Then (as in A, pages 107, 108), when the polygon is displaced, those diagonals which cut off n sides ($n < m$) envelop the coaxal $S_m(\Sigma a_m^2 x^2 = 0)$. But these diagonals also cut off $m-n$ sides. We may thus identify S_m and S_{n-m} and so $P[m]$ gives, for every $n < m$,

$$a_n^2 = a_{m-n}^2, \quad b_n^2 = b_{m-n}^2, \quad c_n^2 = c_{m-n}^2. \tag{38}$$

From (1) we see that this gives formally

$$a_m = \infty = b_m = c_m. \tag{39}$$

This is otherwise evident, since the tangent at a vertex cuts off m sides, and so envelops the coaxal $S_m(\Sigma a_m^2 x^2 = 0)$. We may thus identify S_m and S_0 (*i.e.* S). But since $a_0^2 = b_0^2 = c_0^2$ and also $a_0^2 - b_0^2 = a^2 - b^2$, etc., we must rely once more on the convention (34) which gives

$$a_0 = \infty = b_0 = c_0.$$

Now the polygon traversed k times in succession can count as a polygon of km sides, and so $P[m]$ also gives, for every k,

$$a_{km} = \infty = b_{km} = c_{km}. \tag{40}$$

Hence, too, the diagonal cutting off n sides may also be regarded as a diagonal cutting off $m+n$, $2m+n$, ..., $km+n$ sides. We thus have as a further consequence of $P[m]$

$$a_n^2 = a_{m+n}^2, \quad b_n^2 = b_{m+n}^2, \quad c_n^2 = c_{m+n}^2, \tag{41}$$

and so on.

From (38), (41) we have

$$a_n = \pm a_{m-n} = \pm a_{m+n}, \quad \text{etc.},$$

and it is evidently important to distinguish a_{m-n}, a_{m+n} by their appropriate signs. It is clear from (2) that, taking the symmetrical case, we must have

$$a_{m+n} = a_n, \quad b_{m+n} = b_n, \quad c_{m+n} = c_n, \tag{42}$$

but

$$a_{m-n} = -a_n, \quad b_{m-n} = -b_n, \quad c_{m-n} = -c_n. \tag{43}$$

If m is odd, then $(m+n, n)$ and $(m-n, n)$ are each one even and one odd, and so, by (12), we can derive the other permissible assortments of sign by change of sign of a or of b or of c. I shall, in general, confine myself to the symmetrical arrangement as sufficiently representative.

One can summarize these results concisely by borrowing the notation (and with it the implications) of the Theory of Congruences. Thus I shall write (42), (43) as

$$m+n \equiv n \text{ (por. } m), \quad m-n \equiv -n \text{ (por. } m),$$

and so also (40) as $$km \equiv 0 \text{ (por. } m).$$

6. *Exact forms of condition.*

It is clear from (39) that $P[m]$ gives at once $i_m = 0$, and from (40) that it gives also (for every k) $i_{km} = 0$. Now, by (11), i_{2n+1} is expressible as $f(a^2, b^2, c^2)$, *i.e.* is rational in the geometric data, and so is expressible rationally in the usual invariants Δ, Θ, Θ', Δ' of S, S'.
Thus

(44) *if* $2n+1$ *is prime,* $i_{2n+1} = 0$ *is the exact expression of* $P[2n+1]$, *rational in the geometric data.*

But if $2n+1$ is composite, i_{2n+1} will be composite too, having factors i_k corresponding to every factor k of $2n+1$. To obtain the exact $P[2n+1]$ we must have first removed these irrelevant factors, usually a matter of much difficulty.

For even-sided polygons $P[2n]$ similarly requires $i_{2n} = 0$, *i.e.* by (5),

$$2i_n^4 a_n b_n c_n = 0.$$

Rejecting $i_n = 0$, which represents $P[n]$, we see that the exact $P[2n]$ factorizes into the triad of asymmetric conditions

$$a_n = 0, \quad b_n = 0, \quad c_n = 0.$$

There are, accordingly, for even-sided polygons no conditions of symmetric sign, such as (42), (43), which are thus applicable to odd-sided polygons alone. I shall distinguish these component, asymmetric conditions, when occasion arises, as $P[n, a]$, &c.

Strictly, in integral terms, I should write $P[n, a]$ as $i_n a_n = 0$.

If n is odd, then, by (11),

$$i_n a_n = af(a^2, b^2, c^2),$$

and so

(45) *if* $2n+1$ *is prime,* $i_{2n+1}a_{2n+1}/a = 0$ *is the exact expression of* $P[4n+2, a]$, *rational (but not symmetric) in the geometric data.*

If $2n+1$ is composite, then $i_{2n+1}a_{2n+1}$ will be composite too, having factors $i_k a_k$ corresponding to every factor k of $2n+1$. This is otherwise clear from (11), for writing m for $2n+1$, and multiplying every suffix by n, we find that $i_{mn}a_{mn}$ has a factor a_n. In particular, $i_{4n+2}a_{4n+2}$, whose vanishing includes $P[8n+4, a]$, has also a_2 as a factor (but not a, nor a_{2n+1} since a is not a factor of $i_2 a_2$). So $i_{8n+4}a_{8n+4}$ has a_4, but not a_2, a as factors. Hence generally

(46) *if* $2n+1$ *is prime,*

$$i_{4n+2}a_{4n+2}/i_2 a_2 = 0, \quad i_{8n+4}a_{8n+4}/i_4 a_4 = 0, \quad \ldots,$$

are the exact expressions of

$$P[8n+4, a], \quad P[16n+8, a], \quad \ldots$$

rational in the geometric data.

7. *Irrational forms of condition.*

By (22), i_{2n+1} factorizes into e_{2n+1}, together with f_{2n+1}, g_{2n+1}, h_{2n+1} derivable from e_{2n+1} by change of sign of a, b, c respectively. Hence

(47) $e_{2n+1} = 0$ *is an expression for* $P[2n+1]$, *irrational in the geometric data, but rational in* a, b, c. *It is exact, if* $2n+1$ *is prime.*

I shall go further and split up e_{2n+1} itself into four associated factors. These will no longer be rational in a, b, c, but will involve the quadratic surds

$$u, v, w \equiv \sqrt{\left(\frac{b+c}{2}\right)}, \quad \sqrt{\left(\frac{c+a}{2}\right)}, \quad \sqrt{\left(\frac{a+b}{2}\right)}.$$

For $P[4n\pm1]$ gives, by (43),

$$a_{3n} = -a_{n\pm1}, \quad b_{3n} = -b_{n\pm1}, \quad c_{3n} = -c_{n\pm1}.$$

(The symmetry of sign shows that, of the possible factors of i_{2n+1}, this corresponds to $e_{2n+1} = 0$.) We then have

$$v_{3n}w_{3n} = -v_{n\pm1}w_{n\pm1}, \quad \text{etc.}$$

Substitute these values in (37) and we have

$$\frac{v_{n\pm1}w_{n\pm1}}{v_n w_n} + \frac{w_{n\pm1}u_{n\pm1}}{w_n u_n} + \frac{u_{n\pm1}v_{n\pm1}}{u_n v_n} + 1 = 0. \tag{48}$$

If n is even and equal to $2k$, this becomes, by (30), (33)

$$\Sigma \frac{2f_{2k\pm1}vw}{e_{2k\pm1}a_k} + 1 = 0.$$

Integralize * and write

$$p(8k\pm1) \equiv i_k^3 \{\Sigma 2b_k c_k f_{2k\pm1}vw + a_k b_k c_k e_{2k\pm1}\}. \tag{49}$$

This, involving u, v, w, is irrational in a, b, c. The conjugate irrationals, obtained by change of sign of u or v or w, I denote by

$$\pm p'(8k\pm1), \quad \pm p''(8k\pm1), \quad \pm p'''(8k\pm1),$$

the double signs being still linked.

If n is odd and $n\pm1 = 2k$, then (48) similarly becomes

$$\Sigma \frac{a_k e_{2k\pm1}}{2vw f_{2k\pm1}} + 1 = 0.$$

Integralize and write

$$p(8k\pm3) \equiv i_k \{\Sigma a_k e_{2k\pm1} g_{2k\pm1} h_{2k\pm1} u + 2f_{2k\pm1} g_{2k\pm1} h_{2k\pm1} uvw\}. \tag{50}$$

The conjugate irrationals I again denote by

$$\pm p'(8k\pm3), \quad \pm p''(8k\pm3), \quad \pm p'''(8k\pm3).$$

We easily find that in every case

$$\mathbf{N}p(2n+1) = 2n+1, \quad \mathbf{N}p'(2n+1) = 1, \quad \text{etc.},$$

and that $p(2n+1)$ and its conjugates are each of degree $\frac{1}{4}n(n+1)$.

Thus the product

$$p(2n+1)\,p'(2n+1)\,p''(2n+1)\,p'''(2n+1)$$

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* I hope that I may be permitted this word on the analogy of "rationalize."

is rational and of equal degree with e_{2n+1}. But the foregoing analysis showed that $p(2n+1)=0$, rationalized, was a consequence of e_{2n+1}. Hence

$$e_{2n+1} = p(2n+1)\, p'(2n+1)\, p''(2n+1)\, p'''(2n+1), \tag{51}$$

checking the numerical factor by the substitution $\mathbf{N}$.

Since $p'(2n+1)$, etc., are irrationals conjugate to $p(2n+1)$, I can take $p(2n+1)=0$ as a sufficient expression for $e_{2n+1}=0$, and so

(52) $p(2n+1)=0$ *is an expression for* $P[2n+1]$, *irrational in* a, b, c, *and exact, if* $2n+1$ *is prime.**

For porisms whose number is prime I have failed to find any form simpler than that afforded by $p(2n+1)=0$.

For $p(3)$ we need again the conventions (34) for suffix zero; they give the special form

$$p(3) \equiv u+v+w.$$

8. *Porisms of composite number.*

For porisms of composite number I shall use, where necessary, capitals I, E, F, G, H, P to denote the expressions, now necessarily composite, defined by equations (5), (7), (19), (20), (49), (50), reserving the corresponding lower-case letters for the same expressions relieved of the irrelevant factors appropriate to divisors of the number of the porism. Thus, for instance, I shall write, from (49), putting $k=2$,

$$P(15) = p(15)\,p(5)\,p(3) = i_2^3\{\Sigma 2b_2c_2fvw + a_2b_2c_2e\}.$$

Consider now, for any n, the transformation $\mathbf{T}_n$ which replaces a, b, c by $i_n a_n, i_n b_n, i_n c_n$, or, if n is composite, by $I_n a_n, I_n b_n, I_n c_n$. By (10) the new variables are of degree n^2 in the old, and so $\mathbf{T}_n$ multiplies the degree of any expression by n^2. Moreover, the transformation is an integral one, and since $\mathbf{N} i_n a_n = 1 = \mathbf{N}a$, etc., it leaves unaltered the numerical value of any expression.

Now the effect of $\mathbf{T}_n$ is just to multiply every suffix by n, and then to integralize by a power of I_n. Hence, performed on any expression for $P[m]$, the transformation will give an expression for $mn \equiv 0$, *i.e.* for

* Comparison of (49), (50) shows that $p(8k\pm3)=0$, $p(8k\pm1)=0$ may each be obtained by equating to zero the sum of the reciprocals of the four terms of the other.

$P[mn]$, multiplied by an expression for $P[m]$, if m, n are interprime, and possibly multiplied also by an expression for $P[n]$. This latter, however, entering through I_n, would presumably indicate a redundance in the integralizing multiplier $I^{\wedge}_{n}$. We can steer our way through these possibilities by counting dimensions.

9. *Dimensions of porisms of composite number.*

I consider first the case in which the number of the porism is a pure power n^a of a prime n. Then, by the general argument,

$$P(n^2) = p(n^2)\, p(n),$$

$$P(n^3) = p(n^3)\, p(n^2)\, p(n),$$

and so on. Hence $$p(n^3) = P(n^3)/P(n^2),$$

and generally $$p(n^a) = P(n^a)/P(n^{a-1}).$$

But $P(n^a)$ is of degree $\frac{1}{16}(n^{2a}-1)$ and so $p(n^a)$ is of degree $\frac{1}{16}(n^2-1)n^{2(a-1)}$*.

Now we get an expression of just this degree by performing $\mathbf{T}(n^{a-1})$ on $p(n)$. Hence exactly

$$p(n^a) = \mathbf{T}(n^{a-1})\, p(n). \tag{53}$$

One may note that $$\mathbf{N}p(n^a) = n.$$

Suppose now that the number of the porism has distinct prime factors (other than 2), and consider the simplest case of $P[mn]$, when m, n are prime and different. Then

$$P(mn) = p(mn)\, p(m)\, p(n),$$

and the degree of $p(mn)$ is

$$\tfrac{1}{16}(m^2n^2-1)-\tfrac{1}{16}(m^2-1)-\tfrac{1}{16}(n^2-1) = \tfrac{1}{16}(m^2-1)(n^2-1).$$

But $\mathbf{T}_n p(m)$ is of degree $\frac{1}{16}(m^2-1)n^2$, which is the sum of the degrees of $p(mn)$, $p(m)$. Hence

$$\left\{\begin{array}{ll} & \mathbf{T}_n p(m) = p(mn)\, p(m), \\ \text{and so} & \mathbf{T}_m p(n) = p(mn)\, p(n), \\ \text{while} & P(mn) = p(mn)\, p(m)\, p(n). \end{array}\right. \tag{54}$$

* It follows that the order of every $p(2n+1)$, except $p(3)$, is a multiple of $\frac{3}{2}$.

Our desideratum is the common factor of these three expressions. An explicit formula for it, in the general case, is beyond my powers. In a few simple cases I have been able to determine such a formula by *ad hoc* methods.*

We may observe, in extension of (53), that

$$p(m^\alpha n^\beta) = \mathbf{T}(m^{\alpha-1} n^{\beta-1})\, p(mn). \tag{55}$$

The proof comes by the foregoing argument from dimensional considerations.

It is not difficult to perceive what will be the analogous results for many distinct prime factors.

One may add that unity is the numerical value of any p whose argument has distinct odd prime factors.

10. *Certain standard identities.*

Before I can proceed to the special cases of $p(mn)$ just mentioned, I need certain identities which are also otherwise of service.

Now we have por. $2m+2n+1$,

$$a_{2m+1} = -a_{2n}, \quad b_{2m+1} = -b_{2n}, \quad c_{2m+1} = -c_{2n}.$$

Hence, by (30), (33)

$$2f_{2m+1}vw + e_{2m+1}a_n$$

and the two similar expressions simultaneously vanish. From considerations of symmetry and irrationality a common factor is $p(2m+2n+1)$, and since $2f_{2m+1}vw+e_{2m+1}a_n$ is independent of the sign of u, it has also $p'(2m+2n+1)$ as factor. Similarly we find that

$$2f_{2m+1}vw - e_{2m+1}a_n$$

has factors $\quad p(2m+1-2n), \quad p'(2m+1-2n).$

Changing the sign of v (or w), we find that

$$2f_{2m+1}vw + e_{2m+1}a_n$$

* The forms (48), (50) of A for $P[15]$, $P[30]$ are incorrect; they have irrelevant factors $P[5]$, $P[10]$ respectively.

has also factors $p''(2m+1-2n)$, $p'''(2m+1-2n)$.

Hence, integralizing,

$$(56)\quad i_n(e_{2m+1}a_n \pm 2f_{2m+1}vw) = p(2m\pm 2n+1)\,p'(2m\pm 2n+1)\,p''(2m\mp 2n+1)\,p'''(2m\mp 2n+1),$$

since the dimensions, m^2+n^2+m, and the numerical value, $2m\pm 2n+1$, are the same on each side.

Again, by (30), (33), (37),

$$i_n^3\{\Sigma 2b_nc_nf_{2m+1}vw \pm a_nb_nc_ne_{2m+1}\} = 0, \quad \text{por. } 2m+1\pm 6n.$$

It thus has $p(2m+1\pm 6n)$ as a factor. But we may also write it as

$$i_n^3\{b_nc_n(2f_{2m+1}vw \mp a_ne_{2m+1}) + c_na_n(2g_{2m+1}wu \pm b_ne_{2m+1}) + a_nb_n(2h_{2m+1}uv \pm c_ne_{2m+1})\}.$$

Hence, by (56), it has $p'(2m+1\mp 2n)$ as a factor, and, therefore, by its symmetry in u, v, w, the further factors $p''(2m+1\mp 2n)$, $p'''(2m+1\mp 2n)$. Write always

$$(57)\quad q(2s+1) \equiv p'(2s+1)\,p''(2s+1)\,p'''(2s+1) = e_{2s+1}/p(2s+1)$$

and likewise

$$q'(2s+1) \equiv p(2s+1)\,p''(2s+1)\,p'''(2s+1) = e_{2s+1}/p'(2s+1), \quad \text{etc.}$$

Then

$$(58)\quad i_n^3\{\pm\Sigma 2b_nc_nf_{2m+1}vw + a_nb_nc_ne_{2m+1}\} = p(2m+1\pm 6n)\,q(2m+1\pm 2n),$$

for the dimensions, m^2+3n^2+m, and the numerical value, $2m\pm 6n+1$, agree on the two sides. In similar fashion we can prove that*

$$(59)\quad i_n\{\Sigma a_ne_{2m+1}g_{2m+1}h_{2m+1}u \pm 2f_{2m+1}g_{2m+1}h_{2m+1}uvw\} = p(6m\pm 2n+3)\,q(2m\mp 2n+1).$$

In particular, putting $m = 3n$, $3n-1$ in (58), we have

$$(60)\quad \pm q(8n\pm 1) = i_n^3\{-\Sigma 2b_nc_nf_{6n\pm 1}vw + a_nb_nc_ne_{6n\pm 1}\}$$

* Multitudinous formulae derivable from the foregoing results will suggest themselves. They offer, I think, more amusement than profit.

and so from (59) we have

$$(61)\quad \pm q(8n\pm 3) = i_{3n\pm 1}\{\Sigma a_{3n\pm 1}e_{2n\pm 1}g_{2n\pm 1}h_{2n\pm 1}u - 2f_{2n\pm 1}g_{2n\pm 1}h_{2n\pm 1}uvw\},$$

formulae which I subsequently need.

If any of the suffixes or any of the numerical arguments in the above formulae are composite, we must, of course, replace p, q, i, ... by the corresponding P, Q, I, ...

11. *Formulae for* $p(15)$, $p(21)$, $p(33)$, $p(35)$.

I now indicate how, in a few simple cases, we may inductively disentangle $p(mn)$ from its possible co-factors $p(m)$, $p(n)$. The first such example we meet is $p(15)$.

Now it follows from (7), (27) that

$$u_{2n+1}\sqrt{i_{2n+1}} = i_n i_{n+1}(a_n - a_{n+1})u.$$

Hence $\mathbf{T}_5$ on $p(3)$, *i.e.* on $u+v+w$, gives

$$p(15)p(3) = i_3 i_2(\Sigma a_2 u \pm 2uvw) - i_2 i_3(\Sigma a_3 u \pm 2uvw).$$

In (59) write successively (0, 2) and (0, 3) for (m, n). We then have

$$p(15)p(3) = i_3 p(7)q(3) - i_2 P(9)q(5)$$

and

$$p(15)p(3) = -i_3 q(5) + i_2 p(3)q(7),$$

according as we take the upper or the lower sign in the ambiguities.

Since $i_3 = p(3)q(3)fgh$, we may rid ourselves of the factor $p(3)$, and so have

$$(62)\quad p(15) = fgh\,p(7)q^2(3) - 2abc\,p(9)q(5)$$
$$= 2abc\,q(7) - fgh\,q(3)q(5).$$

In like fashion $p(21)p(3) = i_3 i_4 \Sigma(a_3 - a_4)u$,

whence

$$(63)\quad p(21) = fgh\,q^2(3)p(5)q(9) - i_4 q(7)$$
$$= i_4 p(9)q(5) - fgh\,q(3)p(11)q(7).$$

We shall similarly get

$$(64)\quad p(33) = 2i_3^3 a_3 b_3 c_3\,fgh\,q^2(3)p(13)q(9) - i_5 p(15)q(11)$$
$$= i_5 p(9)q(13) - 2a_3 b_3 c_3 i_3^3\,fgh\,q(3)p(7)q(11).$$

In the first of these we must substitute for $p(15)$ from (62).

The process can be continued indefinitely with increasing complexity.

These forms in which 3 is the least factor are somewhat special. The first odd number with least factor other than 3 is 35. $\mathbf{T}_7$ on $p(5)$ gives

$$p(35)p(5) = \Sigma i_7 a_7 g_7 h_7 u + 2f_7^2 g_7^2 h_7^2 uvw.$$

By (59), putting $(m, n) = (3, 2)$,

$$\Sigma a_2 i_2 e_7 g_7 h_7 u + 2i_2 f_7 g_7 h_7 uvw = P(25)\, q(3).$$

Thus
$$f_7 g_7 h_7 p(5) p(25)\, q(3) - i_2 p(5) p(35) = \Sigma i_2 i_7 (a_2 - a_7)\, g_7 h_7 u$$
$$= \Sigma e_5 f_5 G_9 H_9 g_7 h_7 u,$$

by (31).

Hence

$$2abc\, p(35) = f_7 g_7 h_7 p(25)\, q(3) - q(5)\, \Sigma f_5 g h\, g_7 h_7 g_9 h_9 u. \tag{65}$$

The limit to these discussions is evidently set less by human ingenuity than by human endurance; I proceed no further.

12. *The transformation* L.

In order to discuss the porisms for even-sided polygons I make use of another transformation* L, asymmetric in a, b, c. Choosing a as the favoured letter, I define

$$\begin{cases} \mathbf{L}a = bc/a, \\ \mathbf{L}b = a + A^2/a, \\ \mathbf{L}c = a - A^2/a; \end{cases} \tag{66}$$

where
$$A^4 \equiv (a^2 - b^2)(a^2 - c^2).$$

I shall show that, for every n, this transformation leads to the similar transformation

$$\begin{cases} \mathbf{L}a_n = b_n c_n / a_n, \\ \mathbf{L}b_n = a_n + A^2/a_n, \\ \mathbf{L}c_n = a_n - A^2/a_n. \end{cases} \tag{67}$$

* Expressed in elliptic functions it is essentially Landen's transformation.

I do this inductively, proving that, if these equations are true for suffixes m, n, then they are also true for suffix $m+n$. Now, from (25),

$$\mathbf{L}(b_{m+n}+c_{m+n}) = \mathbf{L}\,\frac{b_m c_n + b_n c_m}{a_n + a_m} = \frac{2(a_m^2 a_n^2 - A^4)}{a_m b_n c_n + a_n b_m c_m} = 2a_{m+n},$$

by (2).

But $$\mathbf{L}(b_{m+n}^2 - c_{m+n}^2) = \mathbf{L}(b^2 - c^2) = 4A^2.$$

Hence $$\mathbf{L}b_{m+n},\ \mathbf{L}c_{m+n} = a_{m+n} \pm A^2/a_{m+n}.$$

Again, by (3), (32),

$$\mathbf{L}(b_n c_n/a_n) = 2a_{2n}, \quad \mathbf{L}a_n = b_{2n} + c_{2n}.$$

Hence, by (2),

$$\begin{aligned}
\mathbf{L}a_{m+n} &= \mathbf{L}\,\frac{a_m^2 a_n^2 - (a^2-b^2)(a^2-c^2)}{a_m b_n c_n + a_n b_m c_m} \\
&= \frac{(b_{2m}+c_{2m})^2(b_{2n}+c_{2n})^2 - (b^2-c^2)^2}{2(b_{2m}+c_{2m})(b_{2n}+c_{2n})(a_{2m}+a_{2n})} \\
&= \frac{(b_{2m}+c_{2m})(b_{2n}+c_{2n}) - (b_{2m}-c_{2m})(b_{2n}-c_{2n})}{2(a_{2m}+a_{2n})} \\
&= \frac{b_{2m}c_{2n} + b_{2n}c_{2m}}{a_{2m}+a_{2n}} \\
&= b_{2m+2n} + c_{2m+2n}, \quad \text{by (25)} \\
&= b_{m+n}\,c_{m+n}/a_{m+n},
\end{aligned}$$

and the inductive proof of (67) is complete.

13. *The transformation* $\mathbf{T}_2$.

Now $\mathbf{L}$ followed by itself gives the symmetrical transformation

$$\mathbf{L}^2 a = 2a_2, \quad \mathbf{L}^2 b = 2b_2, \quad \mathbf{L}^2 c = 2c_2. \tag{68}$$

But $\mathbf{T}_2$, as defined in § 8, gives

$$\mathbf{T}_2 a = i_2 a_2, \quad \mathbf{T}_2 b = i_2 b_2, \quad \mathbf{T}_2 c = i_2 c_2. \tag{69}$$

These are identical, except for a factor abc.

By the general principles of § 8, $\mathbf{T}_2$ quadruples the dimensions, leaves the numerical value unaltered, and, applied to any expression for $P[n]$, reproduces an expression for $P[n]$, with possibly other factors.

Since $i_2a_2 = -b^2c^2+c^2a^2+a^2b^2$, etc., and since e_n is rational in a, b, c, then $\mathbf{T}_2 e_n$ is rational in a^2, b^2, c^2, and so necessarily

$$\mathbf{T}_2 e_n = i_n. \tag{70}$$

In like fashion, since $\mathbf{T}_2\, u = bc$, etc., $\mathbf{T}_2\, p(n)$ is rational in a, b, c, and so

$$\mathbf{T}_2 p(n) = e_n. \tag{71}$$

A change of sign of u relative to those of v, w is equivalent, after performing $\mathbf{T}_2$, to a change of sign of bc relative to those of ca, ab, *i.e.* can be effected by a change of sign of a alone. Checking the final sign from the numerical values, we have

$$\mathbf{T}_2 p'(n) = f_n, \quad \text{etc.} \tag{72}$$

Since $\mathbf{T}_2$ doubles every suffix and multiplies by a power of i_2 equal to the degree, we have

$$\mathbf{T}_2 a_n = i_2 a_{2n}, \quad \text{etc.}, \tag{73}$$

and so

$$\mathbf{T}_2 v_n w_n = i_2 v_{2n} w_{2n} = \tfrac{1}{2} i_2 a_n, \quad \text{by (33).} \tag{74}$$

Hence, by (30), $\mathbf{T}_2 f_n/e_n = a_n/a,$

and so by (70)

$$\mathbf{T}_2 f_n = i_n a_n/a, \quad \text{etc.} \tag{75}$$

We deduce that

$$\left\{\begin{aligned} \mathbf{L}^2\{p(n),\ p'(n),\ \ldots\} &= (abc)^{-\frac{1}{4}(n^2-1)}\{e_n,\ f_n,\ \ldots\}, \\ \mathbf{L}^2(e_n,\ f_n,\ \ldots) &= (abc)^{-\frac{1}{4}(n^2-1)}(i_n,\ i_n a_n/a,\ \ldots), \\ \mathbf{L}^2(a_n,\ \ldots) &= 2(a_{2n},\ \ldots). \end{aligned}\right. \tag{76}$$

Now, by (67), (3), we have

$$\mathbf{L} a_n b_n c_n = 2a_{2n} a_n^{-2} b_n^2 c_n^2.$$

Hence in (5)

$$\mathbf{L} i_{2n} = 4a_{2n} a_n^{-2} b_n^2 c_n^2\, \mathbf{L}\, i_n^4. \tag{77}$$

Again $$\mathbf{L}(a_n^2-a_m^2) = (a_m^2 b_n^2 c_n^2 - a_n^2 b_m^2 c_m^2) a_m^{-2} a_n^{-2}$$
$$= a_{m-n} a_{m+n} (\lambda_n-\lambda_m)^2 a_m^{-2} a_n^{-2}, \quad \text{by (1).}$$

Hence in (7)

$$\mathbf{L} i_{m-n} i_{m+n} = a_{m-n} a_{m+n} (\lambda_n-\lambda_m)^2 a_m^{-2} a_n^{-2}\, \mathbf{L} i_m^2 i_n^2. \tag{78}$$

The inductive relations (77), (78) are satisfied by

$$\mathbf{L}i_n = a_n i_n^2 a^{-n^2}, \tag{79}$$

for every n.

From (76) we have

$$\begin{aligned}\mathbf{L}^3 e_n &= \mathbf{L}(abc)^{-\frac{1}{4}(n^2-1)} i_n \\ &= a^{-1} a_n i_n^2 (2a_2 a^2 b^2 c^2)^{-\frac{1}{4}(n^2-1)} \\ &= \mathbf{L}^2 e_n f_n a^{-\frac{1}{4}(n^2-1)}.\end{aligned}$$

Thus, cancelling $\mathbf{L}^2$ from the two sides,

$$\mathbf{L}e_n = e_n f_n a^{-\frac{1}{4}(n^2-1)}; \tag{80}$$

alternatively, cancelling $\mathbf{L}$,

$$\begin{aligned}\mathbf{L}e_n f_n &= (\mathbf{L}a)^{\frac{1}{2}(n^2-1)} \mathbf{L}^2 e_n \\ &= i_n a^{-\frac{1}{2}(n^2-1)}.\end{aligned}$$

Hence

$$\mathbf{L}f_n = g_n h_n a^{-\frac{1}{4}(n^2-1)} \tag{81}$$

and, from (79),

$$\mathbf{L}g_n h_n = a_n i_n a^{-\frac{1}{2}(n^2+1)}. \tag{82}$$

Again, from (76), $\mathbf{L}^2 p(n) = e_n (abc)^{-\frac{1}{16}(n^2-1)}$

and, proceeding as for e_n, we deduce

$$\begin{cases} \mathbf{L}p(n) = p(n)\, p'(n)\, a^{-\frac{1}{16}(n^2-1)} \\ \mathbf{L}p'(n) = p''(n)\, p'''(n)\, a^{-\frac{1}{16}(n^2-1)}. \end{cases} \tag{83}$$

14. *The transformations* $\mathbf{L}'$, $\mathbf{L}''$.

I denote by $\mathbf{L}'$ the transformation in which a change of sign in c precedes $\mathbf{L}$, so that

$$\begin{cases} \mathbf{L}'a = bc/a, \\ \mathbf{L}'b = A^2/a + a, \\ \mathbf{L}'c = A^2/a - a. \end{cases} \tag{84}$$

Then $\mathbf{L}'p(n)$, being a factor of $\mathbf{L}'e_n$, is, by (21), a factor of $\mathbf{L}h_n$, and so, by (82), a factor of $a_n i_n/a$.

Thus, too, $\mathbf{LL}'p(n)$ is a factor of $\mathbf{L}a_n i_n/a$, *i.e.* by (67) of $b_n c_n$. Hence

$$(85)\qquad \begin{cases} \mathbf{L}'p(n)=0 & \text{gives} \quad P[2n,\ a], \\ \mathbf{LL}'p(n)=0 & \text{gives} \quad P[2n,\ b].P[2n,\ c]. \end{cases}$$

I denote by $\mathbf{L}''$ the transformation in which interchange of a, c precedes $\mathbf{L}$. Thus

$$(86)\qquad \begin{cases} \mathbf{L}''a = a-A^2/a, \\ \mathbf{L}''b = a+A^2/a, \\ \mathbf{L}''c = bc/a. \end{cases}$$

Hence $\mathbf{L}''a$ gives a factor of a^4-A^4, *i.e.* of $i_2 a_2$. So generally $\mathbf{L}''(a_n/a)$ gives a factor of (a_{2n}/a_2) and so

$$(87)\qquad \mathbf{L}''P[2n,\ a] \quad \textit{gives} \quad P[4n,\ a].$$

We can, of course, also derive $P[4n,\ a]$ as $\mathbf{T}_2 P[2n,\ a]$. Since $\mathbf{T}_2 a_n$ gives, not a factor of a_{2n}, but a_{2n} in full, it appears that $\mathbf{T}_2 P[2n,\ a]$ gives merely a partially rationalized form of $\mathbf{L}''P[2n,\ a]$. Which of these two forms provides the better expression for $P[4n,\ a]$, taste and experience must in each case decide.

In (85), (87) we have the machinery for determining inductively an expression for every $P[2^\alpha n]$, when once the fundamental $p(n)$ has been determined.

We, however, still lack a starting point for the family of porisms $P[2^\alpha]$, which I find more troublesome than one would expect from analogy with regular polygons. The best that I can do for them is to commence with a fictitious $p(2)$ represented by $a=0$ in either of the forms

$$\text{(i)}\quad \sqrt{(b+a)}+\sqrt{(b-a)}=0$$

or

$$\text{(ii)}\quad \sqrt{(c+a)}+\sqrt{(c-a)}=0.$$

The transformation $\mathbf{L}''$ gives from the first

$$a+A=0,$$

and from the second

$$\sqrt{\{(b+a)(c+a)\}}+\sqrt{\{(b+a)(c-a)\}} \\ +\sqrt{\{(b-a)(c+a)\}}-\sqrt{\{(b-a)(c-a)\}}=0.$$

Since a_2 is symmetrical in $-a^2$, b^2, c^2, we can derive other forms by interchange of $a\sqrt{(-1)}$, b, c. Or, again, since $i_2 a_2$ is derived from e by substitution of $-a^2$, b^2, c^2 for a, b, c, we have from $p(3)=0$ the condition

$$\sqrt{(b^2+c^2)}+\sqrt{(b^2-a^2)}+\sqrt{(c^2-a^2)}=0.$$

15. *Porisms for two circles.*

We pass from the conics to the homographically equivalent circles by the formulae A(17).

$$a^2 = R^2, \quad b^2+c^2 = R^2+r^2-d^2, \quad bc = Rr.$$

Solving them we have the substitution

$$(88) \quad \begin{cases} a = R, \\ b = \frac{1}{2}\sqrt{\{(R+r+d)(R+r-d)\}} + \frac{1}{2}\sqrt{\{(R-r+d)(R-r-d)\}}, \\ c = \frac{1}{2}\sqrt{\{(R+r+d)(R+r-d)\}} - \frac{1}{2}\sqrt{\{(R-r+d)(R-r-d)\}}. \end{cases}$$

This substitution used directly in $p(n)$ gives rather cumbersome results. Simpler formulae are obtained, if we make a preliminary application of the transformation **L**. This, followed by the substitution (88), gives the substitution

$$(89) \quad \begin{cases} a = r, \\ b = R+d, \\ c = R-d. \end{cases} \qquad \text{(I call this } \mathbf{0}\text{)}$$

So **L′** followed by (88) gives

$$(90) \quad \begin{cases} a = r, \\ b = d+R, \\ c = d-R. \end{cases} \qquad \text{(I call this } \mathbf{0}'\text{)}$$

And **L″** followed by (88) gives

$$(91) \quad \begin{cases} a = R-d, \\ b = R+d, \\ c = r. \end{cases} \qquad \text{(I call this } \mathbf{0}''\text{)}$$

Now with even-sided polygons the median diagonals, as we have seen, are concurrent at a vertex of the common self-polar triangle of the conics. When both the conics are circles, this point of concurrence may be either the point at infinity on the radical axis, which corresponds to $P[2n, a]$, or else a limiting point, which corresponds to $P[2n, b]$, $P[2n, c]$. Thus, as explained in A, § 2, the "separable" case comes from $P[2n, a]$, and the "inseparable" case from $P[2n, b]$. $P[2n, c]$. I shall distinguish them as $C[2n]$ (separable) and $K[2n]$ (inseparable), and similarly use $C[2n+1]$ for $P[2n+1]$ expressed in circular terms.

Then, by (83), (85), if n is odd,

(92) $\mathbf{0}p(n) = 0$ *gives* $C[n]$ *and* $\mathbf{0}'p(n) = 0$ *gives* $C[2n]$.

Comparison of (89), (90) shows that $C[2n]$ comes at once from $C[n]$ by simple interchange of R and d. This is the principle already established in A (20).

Again, by (85), (87), for every n,

(93) $\mathbf{0}P[2n, a]$ *gives* $K[2n]$, *and* $\mathbf{0}''P[2n, a]$ *gives* $C[4n]$.

In (92), (93) we have the machinery for expressing in circular terms every porism for a polygon of $2^r n$ sides (separable and inseparable cases included) so long as we have an expression for $p(n)$.

16. *Full invariant forms.*

For those who prefer their porisms expressed in full invariant form,* *i.e.* in terms of Δ, Θ, Θ', Δ', where

(94) $$\Delta - k\Theta + k^2\Theta' - k^3\Delta'$$

is the discriminant of $S - kS'$, it is possible to proceed fairly quickly by aid of the transformation $\mathbf{T}_2$. For, from (71),

$$\mathbf{T}_2 p(n) = 0$$

gives $e_n = 0$, the porismatic form rational in a, b, c, and, from (70), a second application of $\mathbf{T}_2$ gives $i_n = 0$, the porismatic form rational in a^2, b^2, c^2.

Now $p(n)$ itself is rational and symmetrical in u, v, w, and so is expressible in terms ρ, σ, τ, if u, v, w are the zeros of a cubic in k

(95) $$\tau - k\sigma + k^2\rho - k^3.$$

$\mathbf{T}_2$ on u, v, w gives bc, ca, ab, or, in a homogeneous expression equated to zero, sufficiently a^{-1}, b^{-1}, c^{-1}. These are the zeros of

(96) $$\delta - k\theta + k^2\theta' - k^3\delta',$$

the discriminant of $S - k\Sigma$, where

(97) $$\Sigma \equiv ax^2 + by^2 + cz^2.$$

Hence, sufficiently, $\mathbf{T}_2$ on (95) gives (96). Again $\mathbf{T}_2$ on a^{-1}, etc., gives $(-b^2c^2 + c^2a^2 + a^2b^2)^{-1}$, etc., or sufficiently $(-a^{-2} + b^{-2} + c^{-2})^{-1}$, etc. Since

* I am not of their number.

a^{-2}, b^{-2}, c^{-2} are the zeros of (94), we may say that $\mathbf{T}_2$ on a^{-1}, b^{-1}, c^{-1} sufficiently gives

$$(\Theta'-2a^{-2}\Delta')^{-1}, \quad (\Theta'-2b^{-2}\Delta')^{-1}, \quad (\Theta'-2c^{-2}\Delta')^{-1}.$$

These are the zeros of

$$1-\Theta' k+(4\Theta\Delta'-\Theta'^2)k^2-(-\Theta'^3+4\Theta\Theta'\Delta'-8\Delta\Delta'^2)k^3. \tag{98}$$

Hence, sufficiently, $\mathbf{T}_2^2$ converts (95) into (98).

As an instance consider $P[3]$. We have

$$0 = p(3) = u+v+w = \rho.$$

From (98) the full invariant form is

$$\Theta'^2 = 4\Theta\Delta',$$

the classical condition.

17. *Full invariant forms of even number.*

The full symmetrical expression of $P[2n]$ is

$$a_n b_n c_n/abc = 0.$$

Now, if n is odd, we have, by (72), (75),

$$\mathbf{T}_2^2 q(n) = \mathbf{T}_2 f_n g_n h_n = i_n^3 a_n b_n c_n/abc.$$

Hence the full invariant form for $P[2n]$ is derived from $q(n)$ exactly as that for $P[n]$ is derived from $p(n)$; we have, essentially, only to express $q(n)$ in terms of ρ, σ, τ.

Again

$$\mathbf{T}_2^2 f_n g_n h_n = i_{2n}^3 a_{2n} b_{2n} c_{2n}/i_2^3 a_2 b_2 c_2,$$

$$\mathbf{T}_2^2 i_n^3 a_n b_n c_n/abc = i_{4n}^3 a_{4n} b_{4n} c_{4n}/i_4^3 a_4 b_4 c_4.$$

Thus, to express $P[4n]$, $P[8n]$ in terms of Δ, Θ, Θ', Δ', we need essentially to express

$$f_n g_n h_n, \quad i_n^3 a_n b_n c_n/abc$$

in terms of ρ, σ, τ.

Here, again, we must consider $P[2^\alpha]$ separately. $P[4]$ is simply $i_2^3 a_2 b_2 c_2 = 0$, *i.e.*

$$\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2 = 0,*$$

* A well known result.

from the theory of the cubic (98). For $P[8]$ we must express abc in terms of ρ, σ, τ and so forth.

18. *Interpretation of* $\Sigma = 0$.

For completeness I give a brief interpretation of the conic $\Sigma = 0$, defined in (97). Now the idea of a triangle self-polar or self-conjugate with respect to a conic may be generalized for odd-sided polygons in either of two ways. We may consider a polygon $C_1 C_2 \dots C_{2n+1}$ in which every two consecutive vertices C_r, C_{r+1} are conjugate with respect to a conic Σ, calling such polygons "self-conjugate"; or we may consider a polygon $P_1 P_2 \dots P_{2n+1}$ in which the vertices are the poles of the sides with respect to a conic Σ, calling such polygons "self-polar." If, in the second case, we are to associate sides and vertices by a consistent rule, we must make any vertex P_r the pole of its opposite side $P_{n+r} P_{n+r+1}$. Hence, if we reconnect the vertices, missing always n of them, we have a polygon $P_1 P_{n+1} P_{2n+1} P_2 \dots$, self-conjugate with respect to Σ. So in the other polygon C_r is the pole of $C_{r-1} C_{r+1}$, and $C_1 C_{n+1} C_{2n+1} C_2 \dots$ is self-polar with respect to Σ.

Suppose now that the self-polar polygon $P_1 P_2 \dots P_{2n+1}$ is also inscribed in a conic S. Then, reciprocating in Σ, we see that it is also circumscribed to S', the reciprocal of S in Σ, and hence that S, S' are connected by a porism $P[2n+1]$.

The transformation of current coordinates

$$(x\sqrt{a},\ y\sqrt{b},\ z\sqrt{c}) \equiv (x',\ y',\ z')$$

converts Σ, S' into S, Σ, and hence, if $2n+1$-agons, self-polar in Σ, can be inscribed in S, then also $2n+1$-agons, self-polar in S, can be circumscribed to Σ. The porism for this simultaneous pair of possibilities I shall denote by $\pi[2n+1]$. Since $\pi[2n+1]$, as we have just seen, secures $P[2n+1]$, evidently its proper full expression is $e_{2n+1} = 0$, rational in the data of S, Σ, the alternative conditions f_{2n+1}, g_{2n+1}, $h_{2n+1} = 0$ referring to $\pm ax^2 \pm by^2 \pm cz^2 = 0$, the other reciprocating conics of S, S'.

Thus we get the full invariant expression for $\pi[2n+1]$ by substituting the coefficients of (96) for those of (95) in $p(2n+1)$, just as the full invariant expression for $P[2n+1]$ comes by substituting the coefficients of (98) for those of (95).

As a simplest instance, since $p(3)$ is ρ, $\pi[3]$ is $\theta' = 0$, the classical condition for the triangle.

19. *Interpretation of* $\pi[2n]$.

For even-sided polygons there are no sides opposite vertices: self-conjugate polygons $C_1 C_2 \dots C_{2n}$ alone exist. Reconnecting, we now have two n-agons $C_1 C_3 \dots C_{2n-1}$ and $C_2 C_4 \dots C_{2n}$ mutually reciprocal in Σ. If they are both inscribed in S, then they are both circumscribed to S', and so S, S' are connected by the porism $P[n]$. Hence by the preceding section, if n is odd, the two polygons are self-polar in one of the reciprocating conics of S, S'.

Since this cannot be Σ itself, suppose it to be

$$\Sigma' \equiv -ax^2+by^2+cz^2 = 0$$

Then (x_1, y_1, z_1) will be conjugate in Σ' to both (x_n, y_n, z_n), $(x_{n+2}, y_{n+2}, z_{n+2})$, and so $(-x_1, y_1, z_1)$ will be conjugate in Σ to both these points. This identifies $(-x_1, y_1, z_1)$ with $(x_{n+1}, y_{n+1}, z_{n+1})$, and so generally the median diagonals are concurrent at one vertex of the fundamental triangle.

The expression for $\pi[2n]$ is thus $f_n = 0$, $g_n = 0$, $h_n = 0$ according to the vertex selected, and the full invariant expression for $\pi[2n]$ is $f_n g_n h_n = 0$, *i.e.* $\mathbf{T}_2 q(n) = 0$, just as we have found the full invariant expression for $P[2n]$ to be $\mathbf{T}_2^2 q(n) = 0$.

Again, for $\pi[4n]$, we have a pair of reciprocal $2n$-agons inscribed in S, circumscribed to S', and so $\pi[4n]$ is $a_n b_n c_n / abc = 0$. and, again, $\mathbf{T}_2 \pi[4n] = P[4n]$, and so generally.

The foregoing sketch of the theory of self-conjugate polygons is meant to be concise rather than convincing. A strict theory would need to be developed *ab initio* on the basis of the conics S, Σ, and not deduced from the theory of the conics S, S'. There is much to be said for such a presentation of the Poncelet theory. It is then confessedly the theory of the rational symmetrical (2, 2) correspondence defined on S by points conjugate in Σ.

20. *An illustrative set of porisms.**

I thought I ought to take the trouble to apply the foregoing principles to calculate as many porisms as possible. It was a heartbreaking task, but I had the meagre satisfaction of scoring my century by evaluating $C[112]$. The majority of the formulae are, of course, quite unfit to print, but, as evidence of good faith, I give a set of porisms based on the number eleven, which are not covered by anything in my previous paper, or by any results I have seen elsewhere.

* This last section has been added since the paper was approved for publication.

From (50) we have $p(11)$, and hence

$$P[11]: \frac{a\sqrt{b+c}\backslash}{-bc+ca+ab}+\frac{b\sqrt{c+a}\backslash}{bc-ca+ab}$$

$$+\frac{c\sqrt{a+b}\backslash}{bc+ca-ab}+\frac{\sqrt{(b+c)(c+a)(a+b)}\backslash}{bc+ca+ab}=0.*$$

Applying $\mathbf{L}''$ we have

$$P[22]: \frac{a^2-A^2}{ef}\{\sqrt{(b+a)(c+a)}\backslash+\sqrt{(b-a)(c-a)}\backslash\}$$

$$+\frac{a^2+A^2}{gh}\{\sqrt{(b+a)(c-a)}\backslash+\sqrt{(b-a)(c+a)}\backslash\}$$

$$+\frac{2Abc}{i_2a_2+2bcA^2}-\frac{A(b\sqrt{c^2-a^2}\backslash+c\sqrt{b^2-a^2}\backslash)}{i_2a_2-2bcA^2}=0.$$

If my calculations are correct, I find for the full invariant forms

$$\pi[11]: \theta'^6\{\theta'^3-4\delta'(\theta\theta'-\delta\delta')\}^3+64\delta\delta'^3(\theta\theta'-\delta\delta')\{\theta'^3-\delta'(\theta\theta'-\delta\delta')\}$$
$$\times\{\theta'^3-4\delta'(\theta\theta'-\delta\delta')\}^2+192\delta^2\theta'^3\delta'^5(\theta\theta'-\delta\delta')\{\theta'^3-4\delta'(\theta\theta'-\delta\delta')\}$$
$$+256\theta'^3\delta^3\delta'^7(\theta\theta'-\delta\delta')=0,$$

$$P[11]: (\Theta'^2-4\Theta\Delta')^6\{(\Theta'^2-4\Theta\Delta')^3-32\Delta\Delta'^2(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)\}^3$$
$$-512\Delta\Delta'^2(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)^3$$
$$\times\{(\Theta'^2-4\Theta\Delta')^3-32\Delta\Delta'^2(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)\}^2$$
$$\times\{(\Theta'^2-4\Theta\Delta')^3-8\Delta\Delta'^2(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)\}$$
$$+1536\Delta\Delta'^2(\Theta'^2-4\Theta\Delta')^3(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)^5$$
$$\times\{(\Theta'^2-4\Theta\Delta')^3-32\Delta\Delta'^2(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)\}$$
$$-2048\Delta\Delta'^2(\Theta'^2-4\Theta\Delta')^3(\Theta'^3-4\Theta\Theta'\Delta'+8\Delta\Delta'^2)^7=0.$$

In circular terms I obtain, by $\mathbf{0}P[11]$,

$$C[11]: \frac{-r\sqrt{2R}\backslash}{R^2-d^2-2Rr}+\frac{(R+d)\sqrt{R+r-d}\backslash}{R^2-d^2+2Rd}$$

$$+\frac{(R-d)\sqrt{R+r+d}\backslash}{R^2-d^2-2Rd}+\frac{\sqrt{2R(R+r+d)(R+r-d)}\backslash}{R^2-d^2+2Rr}=0.$$

* I venture to introduce the notation $\sqrt{x+y}\backslash$ for $\sqrt{(x+y)}$ and so $\sqrt{(x+y)(y+z)}\backslash$ for $\sqrt{\{(x+y)(y+z)\}}$ as more economical of brackets in these elaborate formulae.

By interchange of R and d we get

$$C[22]: \frac{r\sqrt{2d\,\backslash}}{R^2-d^2+2rd} - \frac{(R+d)\sqrt{r-R+d\,\backslash}}{R^2-d^2-2Rd}$$

$$+\frac{(R-d)\sqrt{R+r+d\,\backslash}}{R^2-d^2+2Rd} - \frac{\sqrt{2d(R+r+d)(r-R+d)\,\backslash}}{R^2-d^2-2rd} = 0.$$

By $\mathbf{0}P[22]$ we get

$$K[22]: -\frac{r^2-\Delta}{(R^2-d^2)^2-4R^2r^2}\{\sqrt{(R+r)^2-d^2\,\backslash}+\sqrt{(R-r)^2-d^2\,\backslash}\}$$

$$+\frac{r^2+\Delta}{(R^2-d^2)^2-4R^2d^2}\{\sqrt{R^2-(r+d)^2\,\backslash}+\sqrt{R^2-(r-d)^2\,\backslash}\}$$

$$+\frac{2(R^2-d^2)\sqrt{\Delta}}{r^4-\Delta^2+2(R^2-d^2)\Delta}$$

$$-\frac{\{(R+d)\sqrt{(R-d)^2-r^2\,\backslash}+(R-d)\sqrt{(R+d)^2-r^2\,\backslash}\}\sqrt{\Delta}}{r^4-\Delta^2-2r(R^2-d^2)\Delta} = 0,$$

where $$\Delta^2 \equiv R^4+r^4+d^4-2R^2r^2-2R^2d^2-2r^2d^2.$$

Lastly, by $\mathbf{0}''P[22]$, comes

$$C[44]: \frac{(R-d)^2-2\Omega}{(R^2-d^2+2Rr)(R^2-d^2-2rd)}\{\sqrt{R(r+R-d)\,\backslash}+\sqrt{d(r-R+d)\,\backslash}\}$$

$$-\frac{(R-d)^2-2\Omega}{(R^2-d^2+2rd)(R^2-d^2-2Rr)}\{\sqrt{R(r-R+d)\,\backslash}+\sqrt{d(r+R-d)\,\backslash}\}$$

$$+\frac{2r(R+d)\sqrt{\Omega}}{(R^2-d^2)^2-4r^2Rd+4r(R+d)\Omega}$$

$$-\frac{\{(R+d)\sqrt{r^2-(R-d)^2\,\backslash}+2r\sqrt{Rd\,\backslash}\}\sqrt{\Omega}}{(R^2-d^2)^2-4r^2Rd-4r(R+d)\Omega} = 0,$$

where $$\Omega^2 \equiv 2Rd(r+R-d)(r-R+d).$$

I have calculated $K[44]$, but it is unprintable.

EXTENSION TO OTHER SERIES OF ABEL'S AND TAUBER'S THEOREMS ON POWER SERIES

By A. KIENAST.

[Received 3 March, 1924.—Read 13 March, 1924.]

1. The well known theorem of Abel on the continuity of power series has been generalized in various ways. Hardy* and C. N. Moore,† for example, have proved theorems for a series

$$\Sigma a_n f_n(x)$$

(the f_n's satisfying certain conditions), which are directly analogous to Abel's theorem.

Abel considers only the case where x approaches the limit 1 along the radius. Stolz introduced paths ("Stolz-paths") which lie entirely within a region ("Stolz-region") formed by two chords of the unit circle, meeting at $x = 1$. Throughout this paper I suppose that x approaches 1 along a Stolz-path.

THEOREM I.—*If* $f_0(x)$, $f_1(x)$, $f_2(x)$ ... *is a series of functions of the complex variable* x *such that, for every* x *within an assigned Stolz-region*

$$\sum_m^n |f_\kappa(x) - f_{\kappa+1}(x)| < K, \tag{1}$$

where K *is a constant, and* Σa_n *is any convergent series, then the series* $\Sigma a_n f_n(x)$ *is uniformly convergent throughout the Stolz-region.*

This theorem is proved by Hardy.‡ I prove here that the assumptions of the theorem are satisfied in certain cases for which, as far as I am aware, this has not yet been effected.

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* *Proc. London Math. Soc.* (2), 4 (1907), 247–265; *Math. Annalen*, 64 (1907), 77–94.

† *Trans. Amer. Math. Soc.*, 8 (1907), 299–330.

‡ *L.c. supra.*

It is natural to suppose that, since Abel's theorem admits of this extension, the converse theorems of Tauber,* and their extensions by Landau,† may be generalized in the same way. This is, in fact, possible, as I prove in Theorems II and III below.

Lastly I apply Theorem III to prove some theorems in the Analytic Theory of Numbers.

2. Suppose that the functions $f_n(x)$ satisfy in addition to (1) the conditions

$$|1-f_n(x)| < n|1-x| \qquad (0<|x|<1), \tag{2}$$

$$\sum_{n+1}^{\infty} \frac{1}{\kappa}|f_\kappa(x)| < K, \tag{3}$$

$$|x| = 1-1/n, \quad n > n_1,$$

where K is a constant.

THEOREM II.—*If* (i) $\kappa a_\kappa \to 0$, (ii) $\lim_{x\to 1} \Sigma a_\kappa f_\kappa(x) = l$ *when* x *approaches* 1 *along a Stolz-path* S, *then the series* Σa_κ *is convergent to sum* l.

We have

$$D = \left|\sum_1^n a_\kappa - \sum_1^\infty a_\kappa f_\kappa(x)\right| \leqslant \left|\sum_1^n a_\kappa\{1-f_\kappa(x)\}\right| + \left|\sum_{n+1}^\infty a_\kappa f_\kappa(x)\right|$$

$$\leqslant |1-x|\sum_1^n \kappa|a_\kappa| + \epsilon_n \sum_{n+1}^\infty \frac{1}{\kappa}|f_\kappa(x)|,$$

where ϵ_n is the upper limit of the moduli of $(n+1)a_{n+1}$, $(n+2)a_{n+2}$, ... Taking x on the Stolz-path S in such a way that $|x| = 1-1/n$, and remembering that

$$\frac{|1-x|}{1-|x|} < \gamma,$$

where γ is a constant for a definite Stolz-path S,‡ we obtain

$$D \leqslant \frac{\gamma}{n}\sum_1^n \kappa|a_\kappa| + \epsilon_n K.$$

The right-hand side tends to 0 for $n\to\infty$, in virtue of (i), which proves the theorem.

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* *Monatshefte für Math. und Physik.*, 8 (1897), 273–277.

† *Monatshefte für Math. und Physik.*, 18 (1907), 8–28.

‡ See Pringsheim, *Acta Math.*, 28 (1904), 1–30, 4.

This proof is the same as that given by Tauber and Landau* in the case of power series. The conditions (2), (3) express the properties of $f_n(x) = x^n$ used in that proof by Landau. I found that the same inequalities are satisfied in the cases of the series considered in §§ 3 and 4 below, but I do not propose to discuss the possibility of replacing them by others of a more primitive nature.

THEOREM III.—*If*

$$\text{(i)}\ \frac{1}{n}\sum_1^n \kappa a_\kappa \to 0, \qquad \text{(ii)}\ \lim_{x\to 1}\Sigma a_n f_n(x) = l,$$

when x approaches 1 along a Stolz-path S, then the series Σa_κ is convergent to the sum l.

To prove this, put $\sum_1^n \kappa a_\kappa = r_n, \quad r_0 = 0.$

Then
$$a_n = \frac{r_n}{n} - \frac{r_{n-1}}{n-1} + \frac{r_{n-1}}{n(n-1)},$$

(4)
$$\sum_1^n a_\kappa = \frac{r_n}{n} + \sum_1^{n-1} \frac{r_\kappa}{\kappa(\kappa+1)},$$

$$\sum_1^\infty a_n f_n(x) = \sum_1^\infty \left(\frac{r_n}{n} - \frac{r_{n-1}}{n-1}\right) f_n(x) + \sum_1^\infty \frac{r_n}{n(n+1)} f_{n+1}(x).$$

Since
$$\sum_1^m \left(\frac{r_n}{n} - \frac{r_{n-1}}{n-1}\right) = \frac{r_m}{m},$$

condition (i) and Theorem I give

$$\lim_{x\to 1} \sum_1^\infty \left(\frac{r_n}{n} - \frac{r_{n-1}}{n-1}\right) f_n(x) = 0.$$

Hence
$$\lim_{x\to 1} \sum_1^\infty \frac{r_n}{n(n+1)} f_{n+1}(x) = l.$$

This series satisfies the conditions of Theorem II, and therefore

$$\sum_1^\infty \frac{r_n}{n(n+1)} = l,$$

* Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, § 8.

which, after (4), is equivalent to

$$\sum_1^\infty a_n = l.$$

3. The conditions (1), (2), (3) are satisfied if

$$f_n(x) = \frac{nx^n(1-x)}{1-x^n} = \frac{nx^n}{p_n},$$

where $$p_n = 1+x+x^2+\ldots+x^{n-1}.$$

We find first

$$S = \Sigma \left| \frac{\lambda x^\lambda}{p_\lambda} - \frac{(\lambda+1)x^{\lambda+1}}{p_{\lambda+1}} \right| = |1-x| \Sigma |x|^\lambda \left| \frac{p_1+p_2+\ldots+p_\lambda}{p_\lambda p_{\lambda+1}} \right|.$$

If $|x| = r < 1$, then

$$\frac{1}{|p_\lambda(x)|} = \frac{|1-x|}{1-r} \left(\frac{|1-x^\lambda|}{1-r^\lambda} \right)^{-1} \frac{1}{p_\lambda(r)}.$$

We have $$1 \leqslant \frac{|1-x|}{1-r} \leqslant \gamma$$

for every x in an assigned Stolz-region, and therefore

$$|p_\lambda(x)|^{-1} < \gamma |p_\lambda(r)|^{-1},$$

so that $$S \leqslant \gamma^3(1-r) \Sigma r^\lambda \frac{p_1+p_2(r)+\ldots+p_\lambda(r)}{p_\lambda(r)\, p_{\lambda+1}(r)}$$

$$\leqslant \gamma^3 \sum_m^n \left(\frac{\lambda r^\lambda}{p_\lambda(r)} - \frac{(\lambda+1)r^{\lambda+1}}{p_{\lambda+1}(r)} \right) < 2\gamma^3 \frac{mr^m}{p_m(r)},$$

since $$\lambda r^\lambda / p_\lambda(r) \geqslant (\lambda+1)\, r^{\lambda+1} / p_{\lambda+1}(r).$$

Therefore condition (1) is satisfied. Again

$$|1-f_n(x)| = \left| \frac{1-x}{p_n} \right| |p_n + x p_{n-1} + \ldots + x^{n-1} p_1| \leqslant n|1-x|,$$

which is condition (2). Finally

$$\sum_{n+1}^\infty \frac{1}{\kappa} |f_\kappa(x)| \leqslant \frac{r^{n+1}(1-r)}{1-r^{n+1}} \sum_1^\infty r^\lambda = \frac{r^{n+1}}{1-r^{n+1}} < (e-1)^{-1},$$

since $(1-1/n)^n < e^{-1}$; and so condition (3) is satisfied.

Theorems I, II, III are therefore valid for "Lambert's" series

$$(1-x)\,\Sigma a_\kappa \frac{\kappa x^\kappa}{1-x^\kappa}.$$

Theorem II has been proved for this series, where x is restricted to real values, by K. Ananda Rau.*

4. Consider conditions (1), (2), (3) for

$$f_n(x) = \frac{1-x^n}{n(1-x)} = \frac{1}{n}\,p_n(x).$$

We have, if $|x| = r < 1$,

$$\begin{aligned} T &= \Sigma\left|\frac{1}{\lambda}\,p_\lambda(x) - \frac{1}{\lambda+1}\,p_{\lambda+1}(x)\right| \\ &= |1-x|\,\Sigma\frac{1}{\lambda(\lambda+1)}|p_\lambda + xp_{\lambda-1} + \ldots + x^{\lambda-1}p_1| \\ &\leqslant \frac{|1-x|}{1-r}\,\Sigma\frac{1}{\lambda(\lambda+1)}\,[p_\lambda(r) - \lambda r^\lambda] \\ &= \frac{|1-x|}{1-r}\sum_m^n \left\{\frac{1}{\lambda}\,p_\lambda(r) - \frac{1}{\lambda+1}\,p_{\lambda+1}(r)\right\} \\ &\leqslant 2\gamma\,\frac{1}{m}\,p_m(r) \leqslant 2\gamma, \end{aligned}$$

since $p_\lambda(r)/\lambda > p_{\lambda+1}(r)/(\lambda+1)$. Thus condition (1) is satisfied. Again

$$|1-f_n(x)| = \frac{|1-x|}{n}|p_1 + p_2 + \ldots + p_{n-1}| < n\,|1-x|,$$

which proves condition (2). Finally

$$\begin{aligned} \sum_{n+1}^{\infty}\frac{1}{\lambda^2}|p_\lambda(x)| &\leqslant \sum_{n+1}^{\infty}\frac{1}{(\lambda-1)\,\lambda}\,p_\lambda(r) \leqslant (1-r)^{-1}\sum_{n+1}^{\infty}(1-r^\lambda)\left(\frac{1}{\lambda-1} - \frac{1}{\lambda}\right) \\ &\leqslant (1-r)^{-1}n^{-1} = 1, \end{aligned}$$

since $r = 1-1/n$. Thus condition (3) is satisfied. Hence Theorems I, II, III hold good for the series

$$(5) \qquad (1-x)^{-1}\Sigma a_\kappa \kappa^{-1}(1-x^\kappa).$$

* *Proc. London Math. Soc.* (2), 19 (1920), 1–20.

Theorems of the same kind have been proved by Hardy and Littlewood* with respect to the series (5) without restricting x to approach 1 along Stolz-paths. But the admission of tangential paths necessitates restrictions on the a_n's.

5. In this last section I draw some inferences with regard to expressions depending on the functions $\mu(n)$ and $\Lambda(n)$ used in the Analytic Theory of Numbers. $\mu(n)$ is defined to be zero if n contains any squared factor, to be $(-1)^\rho$ if n is the product of ρ different primes, and to be 1 for $n = 1$. $\Lambda(n)$ denotes the arithmetical function which is zero unless n is a power of a prime p, and is then equal to $\log p$. Up to date the theorems of the Analytic Theory of Numbers have been proved only with the help of the Theory of Functions of a complex variable (*i.e.* not in an elementary manner). But it is possible to deduce some of these theorems from others without recurring anew to the function theory. So we say that an assertion A is "equivalent" to another assertion B if either can be deduced from the other in an elementary manner. The theorems which follow are of this character.

(i) THEOREM IV.—*The two equations*

$$\frac{1}{n}\sum_1^n \mu(\kappa) \to 0\,; \quad \Sigma\frac{\mu(\kappa)}{\kappa} = 0$$

are equivalent.

The first follows from the second by partial summation. The assumption that the first be true, and Theorem III, applied to Lambert's series

$$\sum_1^\infty \frac{\mu(n)x^n}{1-x^n} = x,$$

lead to the second equation. This was proved for the first time by Axer.†

(ii) Suppose $$a_n = \Lambda(n)-1,$$

then we have $$\Sigma\frac{a_\kappa x^\kappa}{1-x^\kappa} = \Sigma b_\kappa x^\kappa,$$

where $$b_\kappa = \sum_{\lambda\,|\,\kappa}[\Lambda(\lambda)-1] = \log\kappa - d(\kappa),$$

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* *Proc. London Math. Soc.* (2), 18 (1919), 205–235.

† *Sitz. Ber. Ak. Wiss. Wien*, 120 Abt. IIa (1911), 1253–1298.

$d(\kappa)$ being the number of divisors of κ. We have

$$\sum_1^n b_\kappa \sim -2Cn,$$

C being Euler's constant, so that

$$\lim_{x\to 1}(1-x)\,\Sigma\frac{a_\kappa x^\kappa}{1-x^\kappa} = -2C.$$

THEOREM V.—*The two equations*

$$\frac{1}{n}\sum_1^n[\Lambda(\kappa)-1]\to 0, \quad \Sigma\frac{1}{\kappa}[\Lambda(\kappa)-1] = -2C$$

are equivalent.

The first follows from the second by partial summation. The assumption that the first be true, and Theorem III, applied to the preceding Lambert's series, lead to the second equation.

(iii) It is easily verified that

$$\Sigma\frac{\mu(\kappa)\log\kappa\, x^\kappa}{1-x^\kappa} = -\Sigma\Lambda(\kappa)x^\kappa,$$

and, if we suppose it known that $\sum_{\kappa\leqslant n}\Lambda(\kappa)\sim n$, we obtain

$$\lim_{x\to 1}(1-x)\,\Sigma\frac{\mu(\kappa)\log\kappa\, x^\kappa}{1-x^\kappa} = -1. \tag{6}$$

THEOREM VI.—*The two equations*

$$\frac{1}{n}\sum_1^n\mu(\kappa)\log\kappa\to 0, \quad \Sigma\frac{\mu(\kappa)\log\kappa}{\kappa} = -1$$

are equivalent.

Partial summation applied to the sum in the second equation leads to the first. If we suppose that the first sum tends to zero, we obtain

$$\frac{1}{n}\sum_1^n\mu(\kappa)\log\kappa = \frac{1}{n}\sum_1^n\mu(\kappa)\left(1+\tfrac{1}{2}+\ldots+\frac{1}{\kappa}+C+\frac{\theta_\kappa}{\kappa}\right)$$

$$= \frac{1}{n}\sum_1^n\mu(\kappa)\left(1+\tfrac{1}{2}+\ldots+\frac{1}{n}\right)-\frac{1}{n}\sum_2^n\frac{1}{\kappa}\sum_1^{\kappa-1}\mu(\lambda)+\frac{C}{n}\sum_1^n\mu(\kappa)+\frac{1}{n}\sum_1^n\frac{\theta_\kappa\mu(\kappa)}{\kappa},$$

where C is Euler's constant, and $\theta_\kappa \to \frac{1}{2}$; therefore

$$\frac{\log n}{n} \sum_1^n \mu(\kappa) + a \to 0, \quad (|a| < 1).$$

Hence $\frac{1}{n} \sum_1^n \mu(\kappa) \to 0$, which is known to be equivalent to $\sum_{\kappa \leqq n} \Lambda(\kappa) \sim n$. So the assumption involves the existence of (6). The application of Theorem III to (6) proves the second equation.

(iv) Axer* proved in an elementary way

THEOREM VII.—*The two equations*

$$\log n \sum_1^n \frac{\mu(\kappa)}{\kappa} \to 0, \quad \sum \frac{\mu(\kappa) \log \kappa}{\kappa} = -1$$

are equivalent.

(v) THEOREM VIII.—*Each two of the equations*

$$(\alpha) \quad \sum \frac{\mu(\kappa) \log \kappa}{\kappa} = -1, \qquad (\beta) \quad \log n \sum_1^n \frac{\mu(\kappa)}{\kappa} \to 0,$$

$$(\gamma) \quad \frac{1}{n} \sum_1^n \mu(\kappa) \log \kappa \to 0, \qquad (\delta) \quad \frac{\log n}{n} \sum_1^n \mu(\kappa) \to 0$$

are equivalent.

The passages from (α) to (γ), (δ); from (β) to (γ), (δ); and from (δ) to (γ) follow by partial summation. From (γ) we find (α) by Theorem III; and the equivalence of (α) and (β) is stated by Axer's Theorem VII. The other passages are obtained by a succession of those mentioned.

The equivalence of (β) and (δ), as well as that of (α) and (γ), has not been proved before.

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* *Loc. cit.*, 120 Abt. IIa (1911), §4, §5.

THE REAL ZEROS OF SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION WITH PERIODIC COEFFICIENTS

By E. L. INCE.

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1. *Introduction.*

In a paper recently published by this Society* it was shown that, when p is a positive integer, the equation

$$\frac{d^2y}{dx^2} + \xi \sin 2x \frac{dy}{dx} + (\eta - p\xi \cos 2x)\, y = 0 \tag{1.1}$$

possesses, for certain discrete values of η, a solution of period 2π expressible in a finite form. The equation arises out of the discussion of the vibrations of an elliptic membrane whose density diminishes radially from the centre outwards according to a quadratic law.† The investigation of this physical problem calls for a determination of the number and the distribution of the real zeros of the periodic solutions of the equation. The present paper contains a study of the real zeros of those periodic solutions which are of a finite form.

2. *Notation.*

The following notation has been found useful, and will be adopted in the present paper. The periodic solution of (1.1) which reduces to $\cos mx$ when $\xi = 0$ will be denoted by $C_p^m(x, \xi)$, and the corresponding value of η by $a_p^m(\xi)$. Similarly, that solution which reduces to $\sin mx$ will be denoted by $S_p^m(x, \xi)$, and the corresponding value of η by $b_p^m(\xi)$. It is known (§ 8 of the paper cited) that, when $\xi > 0$ and m is a positive integer,

$$a_p^m(\xi) > b_p^m(\xi) > a_p^{m-1}(\xi), \tag{2.1}$$

* *Proc. London Math. Soc.*, (2), 23 (1923), 56.

† Ince, *Proc. Royal Soc., Edinburgh*, 45 (1925), 102.

and when $\xi < 0$ and m is an even positive integer,

$$(2.2) \qquad a_p^m(\xi) > b_p^m(\xi) > b_p^{m-1}(\xi) > a_p^{m-1}(\xi) > a_p^{m-2}(\xi).$$

The periodic solution has a finite form when m and p are either both even or both odd and $m \leqslant p$.

One result which, although not mentioned explicitly in the previous paper, follows immediately from the discussion there placed on record, is that the value of η which corresponds to a periodic solution of finite form varies continuously with ξ. It is also to be remembered that η, though a many-valued function of ξ, has no real branch-points. Since the coefficients of the equation which admits a solution C or S in a finite form are continuous functions of ξ for all real values of ξ, it follows that C (or S) is then a continuous function of the real variable ξ, and moreover that the zeros of C (or S) are continuous functions of ξ.

3. *The Number of Zeros.*

As ξ, remaining real, varies, the number of zeros of a solution $C_p^m(x, \xi)$ or $S_p^m(x, \xi)$ in the interval $0 \leqslant x < \pi$ remains fixed. For if an additional zero were to arise, it would first appear as a double zero arising through the coalescence of two complex zeros. But a double zero at $x = c$ is equivalent to the initial conditions $y(c) = y'(c) = 0$, which are inconsistent with a solution not identically zero, continuous for all real values of x and ξ. Since the number of zeros in the interval $0 \leqslant x < \pi$ is fixed, it equals the number when ξ is zero, that is, m. Consequently, when $C_p^m(x, \xi)$ or $S_p^m(x, \xi)$ is of a finite form, it has exactly m simple zeros in the interval $0 \leqslant x < \pi$.

Since $$a_p^{m+1}(\xi) > a_p^m(\xi) > a_p^{m-1}(\xi),$$

and when p and m are of unlike parity, $a_p^{m+1}(\xi)$ and $a_p^{m-1}(\xi)$ are finite, $a_p^m(\xi)$ is finite for all values of ξ. In this case $\eta = a_p^m(\xi)$ is a zero of an infinite determinant $\Delta(\xi, \eta)$ which converges uniformly for all values of ξ, η. Thus $a_p^m(\xi)$, defined implicitly by a determinantal equation, is continuous for all values of ξ if $m < p$. The same is true with regard to $b_p^m(\xi)$. It follows, by repeating the argument given above when m and p are positive integers, or zero, and $m \leqslant p$, $C_p^m(x, \xi)$ and $S_p^m(x, \xi)$ have exactly m simple zeros in the interval $0 \leqslant x < \pi$.

4. *Upper and Lower Bounds of* η.

On reduction to its normal form (1 . 1) becomes

$$\frac{d^2u}{dx^2}+\{\eta-\tfrac{1}{8}\xi^2-(p+1)\,\xi\cos 2x+\tfrac{1}{8}\xi^2\cos 4x\}\,u=0, \tag{4 . 1}$$

where $u = ye^{-\frac{1}{4}\xi\cos 2x}$. The zeros of u are exactly those of y.

Now, for real values of x and positive values of ξ,

$$\max\{\eta-\tfrac{1}{8}\xi^2-(p+1)\,\xi\cos 2x+\tfrac{1}{8}\xi^2\cos 4x\}=\eta+(p+1)\,\xi,$$

$$\min\{\ldots\}=\eta-(p+1)\,\xi \quad \text{if} \quad \xi<2(p+1),$$

$$=\eta-\tfrac{1}{4}\xi^2-(p+1)^2 \quad \text{if} \quad \xi>2(p+1).$$

The case $\xi<2(p+1)$ will be considered.

In the interval $0\leqslant x<r\pi$, to be called I_r, where r is a positive integer, $C_p^m(x,\xi)$ has mr zeros. Now consider the comparison equation

$$\frac{d^2v}{dx^2}+\{\eta+(p+1)\,\xi\}\,v=0; \tag{4 . 2}$$

its solutions will have at least mr zeros in I_r provided that

$$\eta+(p+1)\,\xi>\left(m-\frac{1}{2r}\right)^2.$$

But solutions of (4 . 1) cannot have, in I_r, zeros greater in number than those of solutions of (4 . 2), and therefore, in order that $C_p^m(x,\xi)$ may have its mr zeros it is necessary that

$$\eta>\left(m-\frac{1}{2r}\right)^2-(p+1)\,\xi.$$

Consider also the comparison equation

$$\frac{d^2w}{dx^2}+\{\eta-(p+1)\,\xi\}\,w=0; \tag{4 . 3}$$

its solutions will have at most mr zeros in I_r provided that

$$\eta-(p+1)\,\xi<\left(m+\frac{1}{2r}\right)^2.$$

Since solutions of (4.1) have at least as many zeros in I_r as those of (4.3) it follows that, for $C_p^m(x, \xi)$ to have its mr zeros in I_r it is necessary that

$$\eta < \left(m+\frac{1}{2r}\right)^2+(p+1)\xi.$$

But r is arbitrary and may be made indefinitely great. Consequently for $0 < \xi < 2(p+1)$,

$$m^2-(p+1)\xi \leqslant a_p^m \leqslant m^2+(p+1)\xi.$$

Similarly, for $\xi > 2(p+1)$,

$$m^2-(p+1)\xi \leqslant a_p^m \leqslant m^2+\tfrac{1}{4}\xi^2+(p+1)^2.$$

The same inequalities are satisfied by b_p^m; the inequalities which hold when $\xi < 0$ may easily be supplied.

5. *Asymptotic Forms.*

For large values of ξ and η, and any value of x, equation (1.1) is satisfied by

$$\begin{aligned}(5.1)\quad y = \cos^s x \sin^{p-s} x &+\tfrac{1}{4}\{(p-s)(p-s-1)\cos^s x \sin^{p-s-2}x \\ &\qquad -s(s-1)\cos^{s-2}x \sin^{p-s}x\}\,\xi^{-1} \\ &+\tfrac{1}{32}\{(p-s)(p-s-1)(p-s-2)(p-s-3)\cos^s x \sin^{p-s-4}x \\ &\qquad -2(p-s)(p-s-1)[s^2-5s-2]\cos^s x \sin^{p-s-2}x \\ &\qquad -2s(s-1)[(p-s)^2-5(p-s)-2]\cos^{s-2}x \sin^{p-s}x \\ &\qquad +s(s-1)(s-2)(s-3)\cos^{s-4}x \sin^{p-s}x\}\,\xi^{-2}+\ldots,\end{aligned}$$

when

$$(5.2)\quad \eta = (2s-p)\xi+2s^2-p+(2s-p)\{s(p-s)+\tfrac{1}{2}(p-1)\}\,\xi^{-1}+\ldots.$$

The argument of the previous paper shows that this solution is equivalent to $C_p^s(x, \xi)$ when $p-s$ is even, and to $S_p^{s+1}(x, \xi)$ when $p-s$ is odd.

When ξ is very large, and $\sin^2 x$ is of the order of ξ^{-1}, the above

expansion may be replaced by

$$(5.3)\quad \cos^s x\left\{\sin^{p-s}x+\frac{(p-s)(p-s-1)}{1!\,2^2\xi}\sin^{p-s-2}x\right.$$
$$\left.+\frac{(p-s)(p-s-1)(p-s-2)(p-s-3)}{2!\,2^4\xi^2}\sin^{p-s-4}x+\ldots\right\},$$

and when $\cos^2 x$ is of the order of ξ^{-1}, it reads

$$(5.4)\quad \sin^{p-s}x\left\{\cos^s x-\frac{s(s-1)}{1!\,2^2\xi}\cos^{s-2}x\right.$$
$$\left.+\frac{s(s-1)(s-2)(s-3)}{2!\,2^4\xi^2}\cos^{s-4}x-\ldots\right\}.$$

Since, when $p>s$ and $\xi>0$, all the coefficients in (5.3) are positive, there can be no zero near $x=0$* except (when $p-s$ is odd) a simple zero at $x=0$ itself. As to (5.4) it has, when $\xi>0$, a simple zero at $x=\frac{1}{2}\pi$ when s is odd, and in addition

no other zeros near $x=\frac{1}{2}\pi$ when $s=0$ or 1,
two „ „ „ $s=2$ or 3,
four „ „ „ $s=4$ or 5,

and so on.

The corresponding statements when $\xi<0$ can easily be supplied.

6. *Summary.*

The results thus obtained, coupled with the classical Sturmian theory, lead to the following oscillation theorem:—

The functions $C_p^m(x,\xi)$ and $S_p^m(x,\xi)$ have, for $m\leqslant p$, m distinct zeros in the interval $0\leqslant x<\pi$. When $\xi=0$ the zeros are equally spaced in the interval; as $\xi\to+\infty$ they tend to cluster in a small interval $(\frac{1}{2}\pi-\delta,\frac{1}{2}\pi+\delta)$, and as $\xi\to-\infty$ they tend to cluster in the two intervals $(0,\delta)$ and $(\pi-\delta,\pi)$, where δ is a positive number depending upon ξ, and tending to zero as ξ tends to infinity.

One zero may be exceptional, namely $x=\frac{1}{2}\pi$ is a fixed zero of $C_p^m(x)$ when m is odd and a fixed zero of $S_p^m(x)$ when m is even, and $x=0$ is a fixed zero of $S_p^m(x)$ for all values of m.

* That is, in the interval $(0, K\xi^{-\frac{1}{2}})$ when K is a finite positive number.

7. *The Mathieu Equation.*

If $p \to \infty$ and $\xi \to 0$ in such manner that $p\xi \to 2\theta$, equation (1.1) becomes the Mathieu equation

$$\frac{d^2y}{dx^2} + (\eta - 2\theta \cos 2x)\, y = 0, \tag{7.1}$$

and effectively

$$\lim C_p^m(x, \xi) = ce_m(x, \theta), \quad \lim S_p^m(x, \xi) = se_m(x, \theta).$$

In this case (on account of the infinite form involved) the continuity of η with respect to θ is not evident. But, *assuming this continuity*, the foregoing arguments prove that $ce_m(x, \theta)$ and $se_m(x, \theta)$ have each m zeros in the interval $0 \leqslant x < \pi$. The corresponding values of η satisfy the inequality

$$m^2 - 2\theta \leqslant \eta \leqslant m^2 + 2\theta \tag{7.2}$$

when $\theta > 0$, and the reverse inequality when $\theta < 0$.

ON PSEUDO-ORTHOGONAL SYSTEMS OF FUNCTIONS ARISING AS SOLUTIONS OF AN INTEGRAL EQUATION

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1. IN a paper recently presented to the London Mathematical Society, Prof. Proudman has discussed the representation of an "arbitrary" function by a series of the type $\Sigma A_n v_n(x)$, where the functions $v(x)$ are defined by means of the differential equation

$$\frac{d}{dx}\left(\eta(x)\frac{dv(x)}{dx}\right)+\lambda v(x) = -1 \quad (0 \leqslant x \leqslant 1), \tag{1.1}$$

together with the boundary conditions

$$v'(0) = v(1) = 0, \tag{1.2}$$

and the characteristic constants of the problem,

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,$$

are determined by the condition that

$$\int_0^1 v_n(x)\,dx = a\lambda_n, \tag{1.3}$$

where a is a given real constant.* From equations (1.1), (1.2), and (1.3) it readily follows that

$$\int_0^1 v_m(x)\,v_n(x)\,dx = a \quad (m \neq n). \tag{1.4}$$

* J. Proudman, *Proc. London Math. Soc.* (2), 24 (1925), 131–139. I wish to express my thanks to Prof. Proudman for allowing me to read his paper while still in manuscript.

Proudman shows, by an application of Cauchy's Theorem, that an "arbitrary" function can be represented, in an infinite number of ways, by a series of the prescribed form; he also studies, by graphical arguments, the distribution of the real characteristic values of λ, defined by (1 . 3), and the possible occurrence of complex characteristic values.

It is well known that equations (1 . 1) and (1 . 2), as also the more general equations

$$\frac{d}{dx}\left(\eta(x)\frac{dv(x)}{dx}\right)+\{q(x)+\lambda\}\,v(x)+r(x)=0 \quad (0\leqslant x\leqslant 1),$$

$$h_1v(0)+H_1v'(0)=h_2v(1)+H_2v'(1)=0$$

are together equivalent to an integral equation of the form

$$v(x)=\int_0^1 k(x,y)\big(r(y)+\lambda v(y)\big)\,dy, \tag{1 . 5}$$

where $k(x, y)$ is a symmetric function of x and y. In the present paper a family of functions $v(x)$ is defined by means of a non-homogeneous integral equation with symmetric nucleus, together with a supplementary condition analogous to (1 . 3).

If every pair of functions of a system of functions $v(x)$ satisfies (1 . 4), the two systems of functions

$$v_n(x), \quad v_n(x)-v_1(x) \quad (n=2,\,3,\,4,\,\ldots) \tag{1 . 6}$$

are biorthogonal, and may, therefore, be expected to appear, in the present theory, as the solutions of two adjoint homogeneous integral equations. But it is found that, with the more general definition of the functions $v(x)$ that is here given, a system of functions $v(x)$ such that (1 . 4) is satisfied by every pair is not, unless certain rather special conditions* are fulfilled, closed†; the functions which have to be added to the original system $v(x)$ to make it closed present themselves, however, naturally in the course of our investigation, and it is shown that any pair of functions of the composite system so formed satisfies either (1 . 4) or the orthogonal relation

$$\int_0^1 v_m(x)\,v_n(x)\,dx=0 \quad (m\neq n). \tag{1 . 7}$$

* Assumptions (A) and (B) of § 3.

† And therefore, in particular, not adequate for the representation of an "arbitrary" function.

A system of functions $v(x)$ which is such that every pair of its members satisfies *either* (1 . 4) *or* (1 . 7) may be called a *pseudo-orthogonal* system; thus both an orthogonal system and Proudman's system satisfying (1 . 4) are special cases of pseudo-orthogonal systems. In the present paper it is shown that, with suitable assumptions concerning the given functions, a non-homogeneous integral equation with a symmetric nucleus gives rise, for a given real value of α, to a continuous infinity of pseudo-orthogonal systems of functions; the functions of such a system are obtained as the solutions of a homogeneous integral equation with a symmetrizable nucleus. The pseudo-orthogonal system may then be studied with the aid of general theorems in the theory of integral equations with symmetrizable nuclei.*

2. Let $k(s, t)$ be a real symmetric function of s and t, defined in the square $0 \leqslant s \leqslant 1$, $0 \leqslant t \leqslant 1$, and such that the integral

$$\int_0^1 \Big(k(s, t)\Big)^2 dt$$

is a bounded function of s in $0 \leqslant s \leqslant 1$; the discontinuities of $k(s, t)$ in the unit square are supposed to be "regularly distributed."† Lastly, we suppose that $k(s, t)$ is a closed function, *i.e.* that, if $h(t)$ is a function whose square is integrable in $(0, 1)$, the identity

$$\int_0^1 k(s, t)\, h(t)\, dt \equiv 0 \quad (0 \leqslant s \leqslant 1)$$

implies that

$$\int_0^1 \Big(h(t)\Big)^2 dt = 0.‡$$

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* In addition to Proudman's paper quoted above, the two following memoirs by H. Bateman are closely related to the present investigation: "The Application of Integral Equations to the Determination of Expansions . . .", *Trans. Camb. Phil. Soc.*, 20 (1908), 281–290; and "The Reality of the Roots of certain Transcendental Equations . . .", *loc. cit.*, 371–382.

† See E. W. Hobson, *Proc. London Math. Soc.* (2), 13 (1914), 307–340. It follows from our assumptions that, if $h(s)$ is a function whose square is integrable in (0, 1), the function

$$h^{(2)}(s) = \int_0^1 k(s, t)\, h(t)\, dt$$

is continuous in (0, 1). Thus, in particular, the iterated nucleus $k^{(2)}(s, t)$ is continuous in the unit square.

‡ Henceforward we shall not distinguish between functions which differ only in a set of measure zero, and so should here say that $h(t)$ is identically zero.

Also let $f(s)$ be a bounded function in (0, 1), and consider the integral equation

$$\phi(s) = f(s) + \lambda \int_0^1 k(s, t)\, \phi(t)\, dt. \tag{2.1}$$

If $\phi(s)$ denotes the solution of this equation for any value of λ, the function*

$$w(\lambda) = \int_0^1 f(s)\, \phi(s)\, ds \tag{2.2}$$

is a meromorphic function of λ, whose poles are included among the characteristic constants,

$$\mu_1,\ \mu_2,\ \mu_3,\ \ldots,$$

of the nucleus $k(s, t)$. In order to obtain values of λ,

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots,$$

such that, for the corresponding solutions of (2.1),

$$\phi_1(s),\ \phi_2(s),\ \phi_3(s),\ \ldots,$$

$$\int_0^1 \phi_m(s)\, \phi_n(s)\, ds = \alpha \quad (m \neq n), \tag{2.4}$$

where α is a given real constant, we note that, on account of the symmetry of $k(s, t)$,

$$(\lambda_m - \lambda_n) \int_0^1 \phi_m(s)\, \phi_n(s)\, ds = \lambda_m w(\lambda_m) - \lambda_n w(\lambda_n).$$

Hence, for the truth of (2.4), it is necessary and sufficient that the selected values of λ should satisfy the equation

$$\lambda w(\lambda) = \alpha\lambda + \beta, \tag{2.5}$$

where β is any constant. This is, therefore, the supplementary condition which corresponds to (1.3). In the further discussion of equations (2.1), (2.5), two cases have to be distinguished, according as β is or is not zero. The value of β will always be supposed real.

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* Bateman (*loc. cit.*, 371 *et seq.*) introduced the function $w(\lambda)$ and pointed out its importance in connexion with the equation (2.1). Following his physical analogy we may call $w(\lambda)$ the "energy-function" of (2.1).

3. Suppose, first, that β is zero. Equation (2 . 5) now has at least one zero root in λ, and we may therefore write

$$\lambda_1 = 0, \quad \phi_1(s) \equiv f(s).$$

The solutions of our problem other than this one are given by equation (2 . 1) combined with

$$w(\lambda) = \int_0^1 f(s)\, \phi(s)\, ds = a. \tag{3 . 1}$$

Write now

$$\int_0^1 \big(f(s)\big)^2\, ds = a^2 > 0, \tag{3 . 15}$$

$$\int_0^1 k(s, t) f(t)\, dt = f^{(2)}(s), \tag{3 . 2}$$

so that $f^{(2)}(s)$ is a continuous function in $(0, 1)$;* then from equations (2 . 1) and (3 . 1) we find

$$a = a^2 + \lambda \int_0^1 \int_0^1 k(s, t) f(s)\, \phi(t)\, ds\, dt. \tag{3 . 25}$$

Consequently, if we suppose that $\alpha \neq a^2$, and write

$$K(s, t) = k(s, t) + \frac{f(s) f^{(2)}(t)}{a - a^2}, \tag{3 . 3}$$

we have

$$\phi_n(s) = \lambda_n \int_0^1 K(s, t)\, \phi_n(t)\, dt. \quad (n \geqslant 2) \tag{3 . 4}$$

But, conversely, we can show that, if the function $w(\lambda)$ is defined by equations (2 . 1) and (2 . 2) [thus being determined when $f(s)$ and $k(s, t)$ are given], equation (3 . 4) does not admit a solution $\phi_n(s)$, such that

$$\int_0^1 f(s)\, \phi_n(s)\, ds \neq 0$$

unless λ_n is a root of the equation

$$w(\lambda) = \alpha; \tag{3 . 5}$$

also that the solution $\phi_n(s)$, corresponding to a root λ_n of (3 . 5), when

* See footnote † of p. 61.

normalized, so as to make

$$\int_0^1 f(s)\,\phi_n(s)\,ds = a, \tag{3.55}$$

satisfies (2.1) for $\lambda = \lambda_n$. For, on multiplying (3.4) by $f(s)ds$ and integrating from 0 to 1, we find, remembering that (3.55) is to be satisfied,

$$\frac{\lambda_n}{a-a^2}\int_0^1 f^{(2)}(t)\,\phi_n(t)\,dt = 1. \quad (n \geqslant 1)$$

On substituting into (3.4) we find that $\phi_n(s)$ satisfies (2.1) for $\lambda = \lambda_n$.

But (3.4) may very well be soluble for values of λ other than the roots of (3.5), and the solutions $\phi(s)$ corresponding to the additional values of λ will not satisfy either (2.1) or (3.55), but will satisfy the homogeneous relations

$$\int_0^1 f(s)\,\phi(s)\,ds = 0, \tag{3.6}$$

$$\phi(s) = \lambda\int_0^1 k(s,t)\,\phi(t)\,dt.* \tag{3.7}$$

For if (3.4) possesses a solution $\phi(s)$ distinct from the functions which satisfy (2.1) and (3.55), $\phi(s)$ must satisfy (3.6); on treating (3.4) as before we therefore now find

$$\int_0^1 f^{(2)}(t)\,\phi(t)\,dt = 0,$$

and, on substituting this into (3.4), we see that $\phi(s)$ satisfies (3.7).

The proof that, conversely, any solution of (3.6) and (3.7) satisfies (3.4) is immediate.

In the earlier part of this paper it is convenient to assume, for the sake of simplicity of notation, that equation (3.4) possesses no solutions which (after multiplication by a suitable constant) do not satisfy (2.1)

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* The existence of solutions of (3.4) which do not, after multiplication by a suitable constant, satisfy (2.1) and (3.1) seems to have been overlooked by Bateman. He studies the solutions of (2.1) for which $\lambda w(\lambda) = \beta \neq 0$, but seems to regard the equation he obtains [corresponding to (3.4)] as *equivalent* to the pair (2.1) and (3.1); see Bateman, *loc. cit.*, 374 *et seq.*

and (3 . 1). A necessary and sufficient condition for this is that (3 . 6) and (3 . 7) should be inconsistent for all values of λ. We therefore assume that

(A) The nucleus $k(s, t)$ has only one characteristic function $\chi_r(s)$ for each characteristic constant μ_r ; and

(B) The function $f(s)$ is not orthogonal to any characteristic function $\chi_r(s)$.

The above two assumptions are made throughout §§ 5–7. When the special case so arising has been dealt with, it is easy to discuss the more general case in which the assumptions are not made ; this is done in § 8.

With assumptions (A) and (B) the pair of equations (3 . 4) and (3 . 55) is exactly equivalent to the pair (2 . 1) and (3 . 1). Further, it is easily verified that the functions $\phi_n(s)-f(s)$ $(n \geqslant 2)$ are, as was anticipated in § 1, solutions of the equation adjoint to (3 . 4) ; *i.e.* we have

$$\phi_n(s)-f(s) = \lambda_n \int_0^1 K(t, s)\big(\phi_n(t)-f(t)\big)\,dt \quad (n \geqslant 2). \tag{3 . 8}$$

An immediate consequence of assumption (B) is that $f(s)$ cannot itself be a characteristic function for $k(s, t)$; hence $f(s)$ and $f^{(2)}(s)$ are independent functions, so that $K(s, t)$ is not symmetric.

4. We must now recall some definitions and known theorems and also establish some additional results belonging to the theory of integral equations with symmetrizable nuclei.* All nuclei $P(s, t)$ which occur in the present section are supposed to be such that the integrals

$$\int_0^1 \big(P(s, t)\big)^2\,dt, \quad \int_0^1 \big(P(s, t)\big)^2\,ds \tag{4 . 1}$$

are, in the unit square, bounded functions of s and of t respectively.

* These nuclei were first introduced by J. Marty, *Comptes rendus*, 28 February, 7 March, 25 April, and 6 June, 1910 (with an essential correction in the last note). For the general theory reference may also be made to the text-books on Integral Equations by Lalesco (75–85) and by Vivanti. It should, however, be explicitly pointed out that the proof of an expansion-theorem in terms of the functions of the biorthogonal systems arising from a symmetrizable nucleus which is given by Lalesco (84–85) is wrong; for this theorem, Theorems 2 and 3 below are to be substituted.

The nucleus $K(s,t)$ is said to be symmetrizable (from before) if a positive-definite symmetric nucleus $G(s,t)$* exists, such that

$$N(s,t) = \int_0^1 G(s,x)\,K(x,t)\,dx \tag{4.2}$$

is symmetric. It is known that a symmetrizable nucleus $K(s,t)$ admits at least one characteristic constant, that all its characteristic constants are real, and that its solving nucleus has only simple poles, so that the aggregate of its characteristic functions coincides with the aggregate of fundamental solutions of the homogeneous integral equation having $K(s,t)$ as nucleus. It is also easily proved that if $u(s)$ is any characteristic function, so that

$$u(s) = \lambda \int_0^1 K(s,t)\,u(t)\,dt, \tag{4.3}$$

the adjoint equation

$$v(s) = \lambda \int_0^1 K(t,s)\,v(t)\,dt \tag{4.35}$$

is satisfied for the same value of λ by

$$v(s) = \int_0^1 G(s,t)\,u(t)\,dt, \tag{4.4}$$

where $G(s,t)$ is the symmetrizing nucleus of equation (4.2).† The biorthogonal sets of functions $u_n(s)$, $v_n(s)$ are therefore normalized by taking $u_n(s)$ so that

$$\int_0^1\int_0^1 G(s,t)\,u_n(s)\,u_n(t)\,ds\,dt = 1. \tag{4.5}$$

This normalization is supposed to have been effected (and the notation used above is conserved) in the enunciation of the following theorems.

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* $G(s,t)$ is positive-definite if

$$\int_0^1\int_0^1 G(s,t)\,h(s)\,h(t)\,ds\,dt > 0$$

for every function $h(s)$ for which $\int_0^1 \big(h(s)\big)^2 ds = 1.$

† Proofs of all the above statements will be found in the text-books of Lalesco and of Vivanti already referred to.

THEOREM I (Bessel's Inequality).*—*If $h(s)$ is any function whose square is integrable, and we write*

$$h_n = \int_0^1 h(s)\, v_n(s)\, ds,$$

the series Σh_n^2 is convergent, and

$$\sum_{n=1}^{\infty} h_n^2 \leqslant \int_0^1 \int_0^1 G(s, t)\, h(s)\, h(t)\, ds\, dt. \tag{4.55}$$

The theorem follows at once from the inequality

$$\int_0^1 \int_0^1 G(s, t)\, R_n(s)\, R_n(t)\, ds\, dt \geqslant 0,$$

where

$$R_n(s) = h(s) - \sum_{r=1}^{n} h_r\, u_r(s).$$

THEOREM 2 (Carleman's Expansion Theorem).—*If $H(s)$ is any function of the form*

$$H(s) = \int_0^1 N(s, t)\, h(t)\, dt \tag{4.6}$$

where $h(s)$ is a function whose square is integrable, and we write

$$A_n = \int_0^1 H(s)\, u_n(s)\, ds = \frac{1}{\lambda_n} \int_0^1 h(s)\, v_n(s)\, ds,$$

then

$$H(s) = \sum_{n=1}^{\infty} A_n\, v_n(s),$$

and the series on the right is absolutely and uniformly convergent.

For a proof of this theorem we refer to Carleman's dissertation.†

The condition (4.6) imposed, in Theorem 2, on the function to be expanded is clearly very restrictive, and it is natural to suppose that,

* Lalesco, *loc. cit.*, 83.

† T. Carleman, *Über das Neumann-Poincarésche Problem für ein Gebiet mit Ecken*, Upsala (1916), 147–166 (152); the proof given depends upon the theory of quadratic forms in an infinite number of variables. Carleman supposes $K(s, t)$ continuous, but the boundedness of the integrals (4.1) suffices for the validity of his reasoning.

with suitable restrictions on one or more of the nuclei $K(s, t)$, $G(s, t)$, and $N(s, t)$, expansion-theorems of wider applicability (in so far as concerns the function to be expanded) can be obtained. We proceed to an investigation of these theorems in one important case.*

Suppose that $G(s, t)$ is a continuous function of s and t; then, in virtue of (4 . 4), the functions $v_n(s)$ are continuous functions of s, and hence, if we write

$$G_n(s, t) = G(s, t) - \sum_{r=1}^{n} v_r(s)\, v_r(t),$$

$G_n(s, t)$ is a continuous symmetric nucleus. But, if $h(s)$ is any function whose square is integrable in (0, 1), inequality (4 . 55) of Theorem 1 may be written in the form

$$\int_0^1 \int_0^1 G_n(s, t)\, h(s)\, h(t)\, ds\, dt \geqslant 0;$$

that is to say $G_n(s, t)$ is, for all values of n, a continuous symmetric nucleus of positive type.† Hence, in virtue of a theorem due to Mercer,

$$G_n(s, s) = G(s, s) - \sum_{r=1}^{n} \left(v_r(s)\right)^2 \geqslant 0 \ddagger$$

for all values of n and s; thus the series $\sum_{r=1}^{\infty} \left(v_r(s)\right)^2$ converges to a bounded function of s in (0, 1). From this it follows that, for any fixed value of s, the series

$$V(s, t) = \sum_{r=1}^{\infty} v_r(s)\, v_r(t) \tag{4 . 7}$$

---

* My thanks are due to a referee who pointed out that such an investigation had already been carried through by J. Mercer in his memoir "Symmetrizable Functions and their Expansion . . .", *Proc. Royal Soc.* (A), 97 (1920), 401–413. Thus Theorem 3 below is a particular case of the expansion-theorem there (409) obtained. As the proof of the special case which we require is simpler and shorter than that of the general theorem given by Mercer, we have, however, in spite of this anticipation, thought it better to leave our account in its original form. It should be noted that Mercer's "completely symmetrizable" nucleus (*loc. cit.*, 402) is not, in our sense, symmetrizable. This paper will, in what follows, be referred to as Mercer II.

† See J. Mercer, *Trans. Royal Soc.* (A), 209 (1909), 415–446 (416). We shall refer to this paper as Mercer I.

‡ Mercer I, 422.

converges absolutely and uniformly with respect to t;* since the systems of functions $u_n(s)$, $v_n(s)$ are biorthogonal we therefore have, in virtue of (4 . 4),

$$\int_0^1 \Big(G(s, t) - V(s, t) \Big) u_n(t)\, dt = 0$$

for all values of n. We have thus proved the following theorem :—

THEOREM 3 (Mercer's Expansion Theorem).—*If $G(s,t)$ is continuous, the series* (4 . 7) *converges uniformly in t for any fixed value of s, and defines a function $V(s,t)$ whose difference from $G(s,t)$ is orthogonal to all the functions $u_n(t)$; thus, in particular, if the system of functions $u_n(s)$ is closed,*

$$V(s,t) = G(s,t). \tag{4 . 75}$$

We now note that, when (4 . 75) holds, the series of positive terms $\Sigma \Big(v_r(s) \Big)^2$ converges to the continuous function $G(s,s)$; consequently, in virtue of Dini's Theorem, it must do so uniformly in s. From this we readily deduce† the following

ADDENDUM TO THEOREM 3.—*If $G(s,t)$ is continuous, and if the system of functions $u_n(s)$ is closed, the series* (4 . 7) *converges to $G(s,t)$ uniformly in the unit square.*‡

In consequence of Theorem 3 and its addendum it becomes of interest to discover when the functions $u_n(s)$ form a closed system. A necessary condition for closure is furnished by

THEOREM 4.—*If the system of functions $u_n(s)$ is closed, the system $v_n(s)$ is also closed.*

The proof is immediate on using (4 . 4) and remembering that $G(s,t)$, being positive-definite, is certainly a closed nucleus.

For the closure of the system $v_n(s)$, on the other hand, it is easy to give necessary and sufficient conditions; we have, in fact,

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* The argument to be used is that employed by Kneser in his proof of Mercer's Theorem, see A. Kneser, *Rend. di Palermo*, 37 (1914), 169–197 (195–196).

† For the argument compare E. W. Hobson, *Proc. London Math. Soc.* (2), 14 (1915), 5–30 (11).

‡ This addendum is also covered by the theorem of Mercer quoted in foot-note * of p. 68, [Mercer II (409)]. It should, however, be mentioned that Mercer throughout postulates more concerning the continuity of $K(s,t)$ than we have done.

THEOREM 5.—*Of the three identities*

$$\int_0^1 h(s)\, v_n(s)\, ds = 0 \quad (n = 1, 2, 3, \ldots),$$

$$\int_0^1 K(s, t)\, h(t)\, dt \equiv 0,$$

$$\int_0^1 N(s, t)\, h(t)\, dt \equiv 0,$$

any one involves the other two.

Theorem 2 shows (use being made of the form of the coefficients A_n) that the first and third of these identities are equivalent; the equivalence of the second and third follows by writing

$$\int_0^1 N(s, t)\, h(t)\, dt = \int_0^1 \int_0^1 G(s, x)\, K(x, t)\, h(t)\, dx\, dt,$$

and remembering that $G(s, t)$ is a closed nucleus.

Finally, we shall have occasion to use the following sufficient condition for the closure of the system of functions $u_n(s)$:

THEOREM 6.—*If the system of functions $v_n(s)$ is closed, and if there exists a closed symmetric nucleus $H(s, t)$, such that the nucleus*

$$M(s, t) = \int_0^1 K(s, x)\, H(x, t)\, dx \tag{4.8}$$

is symmetric, then the system $u_n(s)$ is also closed.

For we show at once that

$$u_n(s) = C_n \int_0^1 H(s, t)\, v_n(t)\, dt, \tag{4.85}$$

where the numbers C_n are constants.

When the conditions of this section are satisfied, equation (4.75) is therefore true; the deduced expansion-theorem for the expression of an "arbitrary" function in terms of the functions $v_n(s)$ is obvious. Theorems of expansion in terms of the functions $u_n(s)$ may now be deduced from (4.85), but need not be obtained here.

It is perhaps worthy of remark that, when $G(s, t)$ is continuous [and without any assumptions concerning the closure of the systems of

functions $u_n(s)$, $v_n(s)$], Carleman's Expansion Theorem may be replaced by the absolutely and uniformly convergent development

$$N(s, t) = \sum_{n=1}^{\infty} \frac{v_n(s)\, v_n(t)}{\lambda_n}.$$

5. Taking now the integral equation (3 . 4), we denote by

$$k^{(2)}(s, t), \quad k^{(3)}(s, t), \quad \ldots$$

the iterated nuclei of $k(s, t)$, and we write

$$(5 . 1) \qquad G(s, t) = \int_0^1 k(s, x)\, K(x, t)\, dx = k^{(2)}(s, t) + \frac{f^{(2)}(s)\, f^{(2)}(t)}{\alpha - a^2},$$

a symmetric nucleus. Also, if $h(s)$ is any function whose square is integrable,

$$\int_0^1 \int_0^1 G(s, t)\, h(s)\, h(t)\, ds\, dt$$

$$= \int_0^1 \left\{ \int_0^1 k(s, x)\, h(s)\, ds \right\}^2 dx + \frac{1}{\alpha - a^2} \left\{ \int_0^1 f^{(2)}(s)\, h(s)\, ds \right\}^2,$$

so that $G(s, t)$ is certainly positive-definite if $\alpha > a^2$. But it is so also when α is negative, for we have

$$\left\{ \int_0^1 f^{(2)}(s)\, h(s)\, ds \right\}^2$$

$$= \left\{ \int_0^1 \int_0^1 k(s, x)\, h(s) f(x)\, ds\, dx \right\}^2 \leqslant a^2 \int_0^1 \left\{ \int_0^1 k(s, x)\, h(s)\, ds \right\}^2 dx,$$

by Schwarz's inequality. Hence, if $\alpha < a^2$,

$$\int_0^1 \int_0^1 G(s, t)\, h(s)\, h(t)\, ds\, dt \geqslant \frac{\alpha}{\alpha - a^2} \int_0^1 \left\{ \int_0^1 k(s, x)\, h(s)\, ds \right\}^2 dx > 0,$$

provided that $\alpha < 0$. In either of these cases we may therefore use $G(s, t)$ as symmetrizing nucleus, and write

$$(5 . 15) \qquad N(s, t) = \int_0^1 G(s, x)\, K(x, t)\, dx$$

$$= k^{(3)}(s, t) + \rho \{ f^{(2)}(s) f^{(3)}(t) + f^{(3)}(s) f^{(2)}(t) \} + \rho^2 \omega f^{(2)}(s) f^{(2)}(t),$$

where

$$\rho = \frac{1}{a-a^2}, \tag{5.2}$$

$$f^{(3)}(s) = \int_0^1 k^{(2)}(s, t) f(t)\, dt = \int_0^1 k(s, t) f^{(2)}(t)\, dt, \tag{5.25}$$

$$\omega = \int_0 f(s) f^{(2)}(s)\, ds = \int_0^1 \int_0^1 k(s, t) f(s) f(t)\, ds\, dt. \tag{5.3}$$

Thus, if either $a > a^2$ or $a < 0$, the nucleus $K(s, t)$ is symmetrizable from before, and hence, in either of these cases, the numbers

$$\lambda_2, \lambda_3, \lambda_4, \ldots \quad (\lambda_1 = 0)$$

are all real. Also we readily find, by use of equations (2.1), (3.1), and (5.1),

$$\int_0^1 \int_0^1 G(s, t)\, \phi_n(s)\, \phi_n(t)\, ds\, dt = \frac{\gamma_n^2 - a}{\lambda_n^2}, \tag{5.35}$$

where

$$\gamma_n^2 = \int_0^1 \left(\phi_n(s)\right)^2 ds\,; \tag{5.4}$$

hence, if we write $$c_n^2 = \gamma_n^2 - a$$

[so that c_n^2 is positive, in virtue of (5.35)], we must, in order to normalize the functions as in § 4, take

$$u_n(s) = \frac{\lambda_n}{c_n}\, \phi_n(s).$$

The associated functions are given by the equation

$$v_n(s) = \int_0^1 G(s, t)\, u_n(t)\, dt = \frac{\phi_n(s) - f(s)}{c_n \lambda_n}.$$

The system of functions $v_n(s)$ is closed. For the identity

$$\int_0^1 K(s, t)\, h(t)\, dt \equiv 0$$

may be written

$$\int_0^1 k(s,t)\,h(t)\,dt + Af(s) \equiv 0, \tag{5.5}$$

where
$$A = \rho \int_0^1 h(t) f^{(2)}(t)\,dt$$

$$= \rho \int_0^1 \int_0^1 k(s,t)\,h(t) f(s)\,ds\,dt = -\rho a^2 A.$$

Hence, since
$$1 + \rho a^2 = \frac{a}{a - a^2} \neq 0,$$

A is zero; in virtue of (5.5) and of the closure of $k(s,t)$, $h(s)$ must be identically zero, and the result enunciated follows from Theorem 5 of § 4.

The system $u_n(s)$ is also closed; for if, in the enunciation of Theorem 6 of § 4, we take

$$H(s,t) = k(s,t) + \rho\{f(s)f^{(2)}(t) + f^{(2)}(s)f(t)\} + \rho^2 \omega f(s) f(t), \tag{5.6}$$

we readily verify that the function $M(s,t)$ of that theorem is symmetric. Further, from the identity

$$\int_0^1 H(s,t)\,h(t)\,dt \equiv 0,$$

we should deduce

$$\int_0^1 k(s,t)\,h(t)\,dt + Af(s) + Bf^{(2)}(s) \equiv 0, \tag{5.65}$$

where

$$A = \rho \int_0^1 f^{(2)}(t)\,h(t)\,dt + \rho^2 \omega \int_0^1 f(t)\,h(t)\,dt, \tag{5.66}$$

$$B = \rho \int_0^1 f(t)\,h(t)\,dt. \tag{5.68}$$

Equation (5.65) shows that there must exist a function $f^{(-1)}(s)$, such that

$$f(s) = \int_0^1 k(s,t) f^{(-1)}(t)\,dt, \tag{5.7}$$

and that then
$$h(s) \equiv -Af^{(-1)}(s) - Bf(s).$$

On substituting this expression for $h(s)$ into equations (5 . 66) and (5 . 68), we find that the constants A, B satisfy the equations

$$A(1+a^2\rho+\rho^2\omega\sigma)+B\omega\rho(1+a^2\rho) = 0,$$

$$A\rho\sigma+B(1+a^2\rho) = 0,$$

where ρ, ω are the constants defined in (5 . 2), (5 . 3), and

$$\sigma = \int_0^1\int_0^1 k(s,t)f^{(-1)}(s)f^{(-1)}(t)\,ds\,dt. \tag{5 . 8}$$

The determinant of this pair of linear equations is

$$(1+a^2\rho)^2 = \left(\frac{a}{a-a^2}\right)^2 \neq 0,$$

so that A and B are zero. Equation (5 . 65) now shows that $h(s)$ is identically zero, and the closure of the system of functions $u_n(s)$ follows in virtue of Theorem 6.

Noting now that the function $G(s,t)$ of (5 . 1) is a continuous function of s and t^*, we see that we have, in virtue of Theorem 3 and its addendum, established the following development, which is absolutely and uniformly convergent in the unit square

$$k^{(2)}(s,t)+\frac{f^{(2)}(s)f^{(2)}(t)}{a-a^2} = \sum_{n=2}^{\infty}\frac{\big(\phi_n(s)-\phi_1(s)\big)\big(\phi_n(t)-\phi_1(t)\big)}{\lambda_n^2(\gamma_n^2-a)}. \tag{5 . 9}$$

$(a > a^2 \text{ or } a < 0)$

6. If the nucleus $k(s,t)$ is not only closed but is positive-definite, we may take $k(s,t)$ itself as symmetrizing nucleus (from before) of $K(s,t)$, and so have, in the notation of § 4,

$$N(s,t) = k^{(2)}(s,t)+\rho f^{(2)}(s)f^{(2)}(t).\dagger$$

There is now no need to exclude from our theory values of a lying between 0 and a^2; only the values 0 and a^2 themselves need be excluded.

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* See foot-note † of p. 61.

† See equation (5 . 1). It may be noted that, even in this simpler case, $K(s,t)$ is not of either of the two common types of symmetrizable nuclei described by Mercer (Mercer II, 402) as due to Hilbert and to Marty respectively. The component $\rho f(s)f^{(2)}(t)$ is, however, a symmetrizable nucleus of Marty's type, and the remaining term, $k(s,t)$, is symmetrizable by means of the same symmetrizing nucleus $k(s,t)$.

In the case now considered

$$G(s,t) = k(s,t),$$

$$\int_0^1\int_0^1 G(s,t)\,\phi_n(s)\,\phi_n(t)\,ds\,dt = \frac{\gamma_n^2-\alpha}{\lambda_n} > 0,$$

$$u_n(s) = \sqrt{\left(\frac{\lambda_n}{\gamma_n^2-\alpha}\right)}\,\phi_n(s),$$

$$v_n(s) = \sqrt{\left(\frac{1}{\lambda_n(\gamma_n^2-\alpha)}\right)}\left(\phi_n(s)-f(s)\right).$$

Consequently, if $k(s,t)$ is a continuous positive-definite nucleus, (5 . 9) may be replaced by the absolutely and uniformly convergent development

$$k(s,t) = \sum_{n=2}^{\infty} \frac{\left(\phi_n(s)-\phi_1(s)\right)\left(\phi_n(t)-\phi_1(t)\right)}{\lambda_n(\gamma_n^2-\alpha)} \quad (\alpha \neq a^2;\ \alpha \neq 0). \tag{6 . 3}$$

It is also worth remarking that, in this case, the corresponding development in terms of the functions $u_n(s)$ can be obtained with great ease. For, if $H(s,t)$ is the function defined by (5 . 6), it is seen that

(6 . 4)

$$\int_0^1\int_0^1 H(s,t)\,h(s)\,h(t)\,ds\,dt = \int_0^1\int_0^1 k(s,t)\left(h(s)+Af(s)\right)\left(h(t)+Af(t)\right)ds\,dt$$

where

$$A = \rho\int_0^1 f(s)\,h(s)\,ds.$$

The left-hand member of (6 . 4) is therefore positive, unless

$$h(s)+Af(s) \equiv 0\,;$$

if this identity is supposed satisfied we find, by multiplying by $f(s)ds$ and integrating, that

$$(1+\rho a^2)\int_0^1 f(s)\,h(s)\,ds = 0,$$

i.e. since $1+\rho a^2 \neq 0$, that $h(s)$ must be orthogonal to $f(s)$, and that therefore $A = 0$. But when $A = 0$, (6 . 4) shows that the double integral is positive, unless $h(s)$ vanishes identically. Thus $H(s,t)$ has been shown to be positive-definite, so that, since the function $M(s,t)$ of (4 . 8) is symmetric, the nucleus $K(s,t)$ is also symmetrizable from behind. The theory of § 4 can therefore be applied to equation (3 . 4), but interchanging throughout the functions $u_n(s)$, $v_n(s)$ and replacing $G(s,t)$ by

$H(s, t)$. In this way we find, after normalizing the functions in accordance with (4 . 5), that, *if* $k(s, t)$ *is continuous and positive-definite, and if* $f(s)$ *is continuous,*

$$H(s, t) = \sum_{n=2}^{\infty} \frac{\phi_n(s)\,\phi_n(t)}{\lambda_n(\gamma_n^2 - a)}, \tag{6 . 6}$$

where the convergence is absolute and uniform in the unit square.

On account of the importance of the case when $k(s, t)$ is itself positive-definite it is worth while to note that the assumption that $f(s)$ is continuous is unnecessary for the truth of (6 . 6).* For from the uniform convergence of (6 . 3) we at once deduce the two equations

$$\rho f^{(2)}(s) = \rho \int_0^1 k(s, t) f(t)\, dt = \sum_{n=2}^{\infty} \frac{\phi_n(s) - \phi_1(s)}{\lambda_n(\gamma_n^2 - a)}, \tag{6 . 7}$$

$$\rho^2 \omega = \rho^2 \int_0^1 \int_0^1 k(s, t) f(s) f(t)\, ds\, dt = \sum_{n=2}^{\infty} \frac{1}{\lambda_n(\gamma_n^2 - a)}. \tag{6 . 8}$$

On using (5 . 6), (6 . 6) now follows by combining (6 . 3), (6 . 7), and (6 . 8); for a discontinuous $f(s)$ the convergence of (6 . 6) is still uniform in the unit square, since, by using (6 . 3) and (6 . 8), (6 . 7) is readily shown to converge uniformly in s.

7. The distribution of the characteristic values of λ may be discussed by a consideration of the graphs of $w(\lambda)$ and of $\lambda w(\lambda)$ for real values of λ. If $\phi(s)$ denotes the solution of (2 . 1) for any value of λ, we readily prove the three equations

$$w(\lambda) = \int_0^1 \big(\phi(s)\big)^2\, ds - \lambda \int_0^1 \int_0^1 k(s, t)\, \phi(s)\, \phi(t)\, ds\, dt, \tag{7 . 1}$$

$$\frac{d}{d\lambda} w(\lambda) = \int_0^1 \int_0^1 k(s, t)\, \phi(s)\, \phi(t)\, ds\, dt, \tag{7 . 2}$$

$$\frac{d}{d\lambda} \big(\lambda w(\lambda)\big) = \int_0^1 \big(\phi(s)\big)^2\, ds.\dagger \tag{7 . 3}$$

Also, in virtue of assumption (B) of § 3, all the characteristic constants μ_n of $k(s, t)$ are poles of $w(\lambda)$.

If $k(s, t)$ is a positive-definite nucleus, so that the numbers μ_n are all positive, equation (7 . 2) shows that $w(\lambda)$ is a steadily increasing function of the real variable λ for all values of λ for which it is continuous. Also,

* When $f(s)$ is discontinuous each function $\phi_n(s)$ has the same discontinuities as $f(s)$.

† See Bateman, *loc. cit.*, 373.

as λ increases from $-\infty$ to μ_1, $w(\lambda)$ increases from 0 to $+\infty$, passing through the value a^2 when $\lambda = 0$; and as λ increases from μ_n to μ_{n+1} $(n \geqq 1)$, $w(\lambda)$ increases from $-\infty$ to $+\infty$. Consequently the numbers λ_n $(n \geqq 2)$, *i.e.* the intersections of the graphs of $w(\lambda)$ and of α, when arranged in ascending order, fall as follows :—

(1) If $\alpha > a^2$,

$$0 < \lambda_2 < \mu_1 < \lambda_3 < \mu_2 < \lambda_4 < \mu_3 < \dots;$$

(2) If $a^2 > \alpha > 0$,

$$\lambda_2 < 0 < \mu_1 < \lambda_3 < \mu_2 < \lambda_4 < \dots;$$

(3) If $\alpha < 0$,

$$0 < \mu_1 < \lambda_2 < \mu_2 < \lambda_3 < \mu_3 < \dots \quad (\lambda_1 = 0).$$

If it is supposed only that $k(s, t)$ is closed and has an infinite number both of positive and of negative characteristic constants, and that α does not lie between 0 and a^2, a consideration of the graphs of $\lambda w(\lambda)$ and of $\alpha\lambda$ shows, on using (7 . 3) and remembering that $\gamma_n^2 > \alpha$,* that —

(1) If $\alpha > a^2$ there is one and only one characteristic value of λ between each pair of consecutive positive values of μ and between each pair of consecutive negative values of μ; and, in addition, two characteristic values of λ (not counting $\lambda_1 = 0$), one of which is positive and one negative, lying between the least positive and the numerically least negative values of μ;

(2) If $\alpha < 0$ there is one and only one characteristic value of λ between each pair of consecutive positive values of μ and between each pair of consecutive negative values of μ, and no values of λ (other than λ_1) between the least positive and the numerically least negative values of μ.

Further, in the special case when $k(s, t)$, though not positive-definite, has only a finite number of negative characteristic constants, it is seen that the results just given still hold if $\alpha > a^2$; but that if $\alpha < 0$ there is an additional characteristic value of λ which is less than the numerically greatest negative value of μ. A similar result is found if $k(s, t)$ has only a finite number of positive characteristic constants.

* In virtue of (5 . 35).

8. We will now consider the more general case in which assumptions (A) and (B) of § 3 are not made. Let then

$$\chi_{n1}(s),\ \chi_{n2}(s),\ \ldots,\ \chi_{nk}(s)$$

denote a complete set of normalized characteristic functions of the nucleus $k(s,t)$ for the characteristic value μ_n,* and divide the numbers μ_n into two classes μ' and μ'', as follows :—

(1) For a characteristic number μ', $f(s)$ is not orthogonal to all the corresponding functions $\chi(s)$; whereas

(2) For a number μ'', $f(s)$ is orthogonal to all the corresponding functions $\chi(s)$.

For a value μ' let the functions $\chi(s)$ be so arranged that

$$\chi_{n1}(s),\ \chi_{n2}(s),\ \ldots,\ \chi_{nm}(s) \quad (1 \leqslant m \leqslant k)$$

denote those of the functions $\chi(s)$, corresponding to μ', to which $f(s)$ is not orthogonal. Then the numbers μ' are the poles of $w(\lambda)$; for a number μ'', $\phi(s)$ becomes indeterminate, but $w(\lambda)$ remains finite and determinate. Let the roots of (3 . 1) now be denoted by λ'_n $(n \geqslant 2)$; then we have seen that equation (3 . 4) may admit, in addition to the numbers λ', some of the numbers μ as characteristic constants, but that it admits no characteristic constant distinct both from the numbers λ' and from the numbers μ. We know also that a number μ will be a characteristic constant for (3 . 4) if and only if the homogeneous integral equation with $k(s,t)$ as nucleus admits, for this value μ of the parameter, a solution which is orthogonal to $f(s)$, and that the complete set of such independent solutions of the equation with $k(s,t)$ as nucleus is also the complete set of solutions of (3 . 4) for the value μ of λ.†

From the above it follows immediately that every number μ''_n is a characteristic number for equation (3 . 4), and that the corresponding functions

$$\chi''_{n1}(s),\ \chi''_{n2}(s),\ \ldots,\ \chi''_{nk}(s)$$

form a complete set‡ of normalized independent solutions of (3 . 4) for the value μ''_n.

Similarly, for a number μ'_n, we have at once the following $k-m$

* When necessary we shall show the dependence of k on n by writing it as k_n; similarly for the other numbers (and dashes) which occur in this section.

† Unless the value μ considered happens also to be a root λ' of (3 . 1), being then necessarily a number μ''; in this case there will be an additional solution of (3 . 4), which will satisfy, not (3 . 6) and (3 . 7), but (2 . 1) and (3 . 1).

‡ Subject, of course, to the case of exception referred to in the preceding foot-note.

independent solutions of (3 . 4) :—

$$\chi'_{nr}(s) \quad (r = m+1,\ m+2,\ \ldots,\ k).$$

But, in addition to these, we can form $m-1$ (and only $m-1$) independent linear combinations of the m functions

$$\chi'_{nr}(s) \quad (r = 1,\ 2,\ 3,\ \ldots,\ m),$$

which are orthogonal to $f(s)$.* Since they are also solutions of the homogeneous equation with $k(s,t)$ as nucleus they are solutions of (3 . 4), and they are clearly independent of the solutions formerly obtained for this value μ'. Form now $m-1$ independent normal orthogonal linear combinations of these $(m-1)$ solutions obtained ;† then, by adding these solutions to the $k-m$ already obtained, we have all the independent solutions of (3 . 4) for the value μ'. These solutions are orthogonal to one another, and they have been normalized.

Thus for a value μ'_n there are in all k'_n-1 independent solutions of (3 . 4), and for a value μ''_n there are in all k''_n solutions of (3 . 4). Let all these solutions for all values μ' and μ'' be typified by $\phi''(s)$. The functions $\phi''(s)$ all satisfy (3 . 6) and (3 . 7), and they form a normal orthogonal system.

The remaining solutions of (3 . 4) satisfy (2 . 1) and (3 . 1), and correspond to the numbers λ' ; these solutions may be typified by $\phi'(s)$. There is one and only one solution $\phi'(s)$ for each root λ' of (3 . 1).‡

After this analysis of the solutions of (3 . 4) it is easy to see how the theory of §§ 5–7 is affected by the removal of assumptions (A) and (B). It is readily verified that the biorthogonal pair of systems of functions arising from (3 . 4) and its adjoint (3 . 8) is obtained by associating $\phi'_n(s)-f(s)$ with $\phi'_n(s)$ $(n \geqq 2)$, and $\phi''_n(s)$ with $\phi''_n(s)$.§ If the number of functions $\phi''_n(s)$ is finite, the alterations to be made in the preceding sections are obvious and trivial ; if the number of functions $\phi'_n(s)$ is finite, the whole problem is at once reducible to that of the homogeneous integral equation with symmetric nucleus $k(s,t)$, and the expansion-theorems of Hilbert or of Mercer will supplant the expansions found in § 5 or § 6. If, however, there are an infinite number both of functions $\phi'_n(s)$ and of functions $\phi''_n(s)$, the formulae given in §§ 5–6 are to be regarded as

* We can, for example, take these combinations to be of the form $\chi'_{n1}(s) + d_r \chi'_{nr}(s)$ $(r = 2,\ 3,\ \ldots,\ m)$, where the constants d_r are uniquely determined.

† By Schmidt's process.

‡ Again unless the number λ' considered coincides with a number μ'' ; in this case $\phi'(s)$ becomes, for this value λ', indeterminate. This exceptional case is dealt with later and the ambiguity removed ; see foot-note † of p. 80.

§ See, however, foot-note † of p. 80.

applying to the functions $\phi'_n(s)$ only, and a parallel formula is, in each case, required for the functions $\phi''_n(s)$.* These parallel formulae can be supplied very easily, and the whole theory thus generalized to the case now considered. The integral relations between the functions ϕ', ϕ'' of the extended system are

$$(8.1) \qquad \int_0^1 \phi'_m(s)\,\phi'_n(s)\,ds = a \quad (m \neq n),$$

$$(8.2) \qquad \int_0^1 \phi''_m(s)\,\phi''_n(s)\,ds = 0 \quad (m \neq n),$$

$$(8.3) \qquad \int_0^1 \phi'_m(s)\,\phi''_n(s)\,ds = 0\,\dagger \quad (m > = < n),$$

so that the complete system ϕ', ϕ'' is pseudo-orthogonal.

The theory of § 7 now determines the disposition of the numbers λ' relative to the poles μ' of $w(\lambda)$; the remaining characteristic constants (λ'', say) for (3 . 4) are all numbers μ''_n, together with all numbers μ'_n for which $k'_n \geqq 2$.

* Thus, for a function $\phi''(s)$ equations (5 . 35) and (5 . 4) are replaced by the equations

$$\int_0^1\int_0^1 G(s,\,t)\,\phi_n(s)\,\phi_n(t)\,ds\,dt = \frac{1}{\lambda_n^2}, \quad \int_0^1 \big(\phi_n(s)\big)^2\,ds = 1$$

respectively. Also we now have $u_n(s) = \lambda_n \phi_n(s)$, $v_n(s) = \lambda_n^{-1}\phi_n(s)$. Thus instead of (5 . 9) we obtain the expansion

$$(8.05) \qquad G(s,\,t) = \sum_{n=2}^{\infty} \frac{\big(\phi'_n(s) - f(s)\big)\big(\phi'_n(t) - f(t)\big)}{\lambda'^2_n(\gamma_n^2 - a)} + \sum_{n=1}^{\infty} \frac{\phi''_n(s)\,\phi''_n(t)}{\lambda''^2_n}.$$

The equations of § 6 are dealt with similarly when functions $\phi''(s)$ are present. It may be advisable to avoid the confusion that might possibly be occasioned by the formal similarity of an equation such as (8 . 05) of this foot-note to the expansion of the symmetrizing nucleus obtained by Mercer (Mercer II, 409) by pointing out explicitly that our functions $\phi''(s)$ have no connexion whatever with Mercer's functions $\eta(s)$. In fact, since $K(s, t)$ is closed in t, Mercer's functions $\eta(s)$ do not exist for this nucleus, and both the functions $\phi'(s) - f(s)$ and the functions $\phi''(s)$ are, in the notation used by Mercer, functions $\psi(s)$; for both sets of functions satisfy the homogeneous equation (3 . 8).

† The first two of these relations follow immediately from what precedes; the third follows from the theory of adjoint integral equations (and is, in addition, readily proved directly) unless the functions $\phi'(s)$ and $\phi''(s)$ involved correspond to a coincident λ' and μ''. In this case, however, the function $\phi'(s)$ is indeterminate to the extent of arbitrary additive multiples of the functions $\chi''(s)$ corresponding to this value μ''; the functions $\phi''(s)$ are these functions $\chi''(s)$ themselves. The constant multipliers of the functions $\chi''(s)$ in the expression for $\phi'(s)$ can obviously be chosen in one and only one way so that the third relation in the text shall be satisfied. When this is done, the complete system $\phi'(s)$, $\phi''(s)$ has been made pseudo-orthogonal, and, simultaneously, the possible indeterminateness of some functions $\phi'(s)$, mentioned in preceding foot-notes, has been removed.

9. The case when $\alpha = a^2$ has been explicitly excluded from all that precedes; this case presents some points of interest, and will now be dealt with. When $\alpha = a^2$, zero is a repeated root of the equation

$$\lambda w(\lambda) = a\lambda,$$

and we therefore write

$$\lambda_1' = \lambda_2' = 0, \quad \phi_1'(s) = \phi_2'(s) = f(s).$$

Still denoting by $\phi'(s)$ any solution of (2 . 1) and (3 . 1) we now have, from equation (3 . 25), that, for $n \geqslant 3$,

$$\int_0^1 f^{(2)}(s)\, \phi_n'(s)\, ds = 0. \tag{9 . 1}$$

We will suppose that (9 . 1) is not satisfied when $n = 2$, *i.e.* that

$$\int_0^1 f(s) f^{(2)}(s)\, ds = \int_0^1 \int_0^1 k(s, t) f(s) f(t)\, ds\, dt = \omega \neq 0;^*$$

equation (2 . 1) now gives, on multiplication by $f^{(2)}(s)\, ds$ and integration from 0 to 1, that, for $n \geqslant 3$,

$$0 = \omega + \lambda_n' \int_0^1 \int_0^1 k(s, t) f^{(2)}(s)\, \phi_n'(t)\, ds\, dt,$$

so that, combining this equation with (2 . 1), we find that the functions $\phi_n'(s)$ satisfy, for $n \geqslant 3$,

$$\phi(s) = \lambda \int_0^1 \bar{k}(s, t)\, \phi(t)\, dt, \tag{9 . 2}$$

where

$$\bar{k}(s, t) = k(s, t) - \frac{f(s) f^{(3)}(t)}{\omega}. \tag{9 . 25}$$

Also, if we write

$$\int_0^1 f(s) f^{(3)}(s)\, ds = \int_0^1 \left(f^{(2)}(s)\right)^2 ds = b^2 > 0, \tag{9 . 3}$$

$$\int_0^1 \bar{k}(s, t) f(t)\, dt = f^{(2)}(s) - \frac{b^2}{\omega} f(s);$$

hence, in consequence of (9 . 1) and (9 . 2), we see that the functions

* The hypothesis $\alpha - a^2 = \omega = 0$ would make $\lambda = 0$ a triple root of the characteristic equation.

$\phi_n'(s)$ satisfy, for *all* values of n,

$$\phi(s) = \bar{\lambda} \int_0^1 \bar{K}(s, t)\, \phi(t)\, dt, \tag{9.4}$$

where

$$\bar{K}(s, t) = \bar{k}(s, t) - \frac{f^{(2)}(s) f^{(2)}(t)}{\omega}, \tag{9.42}$$

$$\bar{\lambda}_2' = -\omega/b^2, \tag{9.44}$$

$$\bar{\lambda}_n' = \lambda_n' \quad (n \geqslant 3). \tag{9.45}$$

But, conversely, it is easy to show that every solution of (9.2) is orthogonal to $f^{(2)}(s)$, and that the same is true of (9.4), unless $\bar{\lambda} = \bar{\lambda}_2'$; and from this it follows, just as in § 3, that the aggregate of functions which satisfy (9.2) [or (9.4)], but are distinct from all the functions $\phi_n'(s)$ $(n \geqslant 3)$ [or $(n \geqslant 2)$], coincides with the aggregate of functions determined by the simultaneous conditions (3.6) and (3.7). These additional solutions of (9.2) and (9.4) are the functions $\phi''(s)$ of § 8.

Let us now suppose, for simplicity, that $k(s, t)$ is positive-definite. If we write

$$\bar{N}(s, t) = \int_0^1 k(s, x)\, \bar{K}(x, t)\, dx = k^{(2)}(s, t) - \frac{f^{(2)}(s) f^{(3)}(t) + f^{(3)}(s) f^{(2)}(t)}{\omega},$$

$$\bar{H}(s, t) = k(s, t) - \frac{f^{(2)}(s) f^{(2)}(t) + f(s) f^{(3)}(t) + f^{(3)}(s) f(t)}{\omega} + \frac{b^2}{\omega^2}\left(f(s) f^{(2)}(t) + f^{(2)}(s) f(t)\right),$$

$$\bar{M}(s, t) = \int_0^1 \bar{K}(s, x)\, \bar{H}(x, t)\, dx,$$

the nuclei $\bar{N}(s, t)$, $\bar{M}(s, t)$ are symmetric. Thus the nucleus $\bar{K}(s, t)$ is symmetrizable from before, and in order to be able to apply the full theory of § 4 to the solutions of (9.4) we have only to show $\bar{K}(s, t)$ is closed in t* and that $\bar{H}(s, t)$ is closed.† We will outline the proof of closure for $\bar{H}(s, t)$ only, as that for $\bar{K}(s, t)$ is similar but simpler. The

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* *I.e.* that if $\int_0^1 \bar{K}(s, t)\, h(t)\, dt \equiv 0$, $h(s) = 0$. See Theorem 5 of § 4.

† See Theorem 6 of § 4.

assumption that $$\int_0^1 \bar{H}(s, t)\, h(t)\, dt \equiv 0$$

would lead to an identity

$$\int_0^1 k(s, t)\, h(t)\, dt \equiv Af(s)+Bf^{(2)}(s)+Cf^{(3)}(s), \tag{9.6}$$

where

$$A = \frac{1}{\omega}\int_0^1 f^{(3)}(s)\, h(s)\, ds - \frac{b^2}{\omega^2}\int_0^1 f^{(2)}(s)\, h(s)\, ds, \tag{9.62}$$

and there are two similar expressions for B and C. But it follows from (9.6) that

$$h(s) \equiv Af^{(-1)}(s)+Bf(s)+Cf^{(2)}(s), \tag{9.64}$$

where $f^{(-1)}(s)$, the existence of which is assured by (9.6), is the function defined in (5.7). If we substitute this expression for $h(s)$ into (9.62) and into the two similar equations for B and C, we obtain three simultaneous equations, homogeneous and linear in A, B, and C. The determinant of this system of equations is found to be

$$\Delta = a^4\omega(b^4-\omega\eta),$$

where we have written

$$\eta = \int_0^1\int_0^1 k(s, t) f^{(2)}(s) f^{(2)}(t)\, ds\, dt.$$

Now, since $f(s)$, $f^{(2)}(s)$ are independent functions,

$$\int_0^1\int_0^1 k(s, t)\left(f(s)+xf^{(2)}(s)\right)\left(f(t)+xf^{(2)}(t)\right) ds\, dt = \omega+2b^2x+\eta x^2 > 0$$

for all values of x; consequently

$$b^4-\eta\omega < 0.$$

Since Δ is not zero, A, B, and C must all vanish, and hence, in virtue of (9.64), $h(s)$ vanishes identically. Thus $H(s, t)$ is closed.

If we note that, in the biorthogonal pair of systems of functions ϕ', $\phi'-f$, with which we are now concerned, the function associated with $\phi_2 \equiv f(s)$ is exceptional [being, in fact, $f^{(2)}(s)$], we see, after normalization of the functions in accordance with (4.5), and still retaining the notation of § 8, that the expansion of $k(s, t)$ to which we are led is

now as follows: *if $k(s,t)$ is continuous and positive-definite, and if $\alpha = a^2$ but $\omega \neq 0$, then*

$$k(s,t) = \frac{f^{(2)}(s)\,f^{(2)}(t)}{\omega} + \sum_{n=3}^{\infty} \frac{\left(\phi_n'(s) - f(s)\right)\left(\phi_n'(t) - f(t)\right)}{\lambda_n'(\gamma_n^2 - \alpha)} + \sum_{n=1}^{\infty} \frac{\phi_n''(s)\,\phi_n''(t)}{\lambda_n''},$$

where both series on the right converge absolutely and uniformly in the unit square.

We omit, for brevity, further discussion of the case in which $\alpha = a^2$ but $k(s,t)$ is supposed to be merely closed, and will only remark that it is better, in this case, to regard (9 . 2) as the equation defining the functions $\phi'(s)$; the symmetrizability of the nucleus of this equation is then established by the equation

$$\int_0^1 k^{(2)}(s,x)\,\bar{k}(x,t)\,dx = k^{(3)}(s,t) - \frac{f^{(3)}(s)\,f^{(3)}(t)}{\omega}.$$

Secondly, in order to make the system of functions $v_n(s)$ arising from (9 . 2) a closed system, we must add to the aggregate of solutions $\phi(s)$ of the equation the function $f(s)$; and, thirdly, that, in order to prove that the function $\bar{H}(s,t)$ [and therefore also the system $u_n(s)$] is closed, it would appear that we must introduce the further restriction that $b^4 \neq \omega\eta$.

10. A further (but trivial) special case occurs when $\alpha = 0$. The equation for the functions $\phi(s)$ may now be written in the form

$$\phi(s) = \lambda \int_0^1 \left\{ k(s,t) - \frac{f(s)\,f^{(2)}(t) + f^{(2)}(s)\,f(t)}{a^2} \right\} \phi(t)\,dt. \tag{10 . 1}$$

The solutions of (10 . 1) are (*a*) the functions $\phi_n'(s)$ for $\lambda = \lambda_n'$ $(n \geqslant 2)$, (*b*) the function $f(s)$ for $\lambda = -a^2/\omega$, and (*c*) the functions $\phi_n''(s)$ for $\lambda = \lambda_n''$. Since the equation has a symmetric nucleus it need delay us no longer.

11. Leaving behind these special cases, we wish now to take up a discussion of the case when, in the characteristic equation (2 . 5), β is given a real value which is different from zero. Since, in this case, $\lambda = 0$ is not a root of (2 . 5), we may write this equation in the form

$$w(\lambda) = \alpha + \beta/\lambda \quad (\beta \neq 0). \tag{11 . 1}$$

We suppose the function $w(\lambda)$ to have at least two distinct poles, so that,

from the form of the graph of $\lambda w(\lambda)$,* equation (11 . 1) has at least one real root, and there is at least one real function $\psi(s)$ satisfying (2 . 1) and (11 . 1). We will suppose further, in the first place, that this selected solution $\psi(s)$ is such that

$$\int_0^1 \big(\psi(s)\big)^2 ds = \gamma^2 \neq a\,; \tag{11 . 2}$$

our assumptions concerning $f(s)$ and $k(s, t)$ remain the same as in the preceding sections of this paper.

Let now $\phi(s)$ be any solution, distinct from $\psi(s)$, of the pair of equations (2 . 1) and (11 . 1); then it follows from § 2 that

$$W(\lambda) = \int_0^1 \phi(s)\,\psi(s)\,ds = a\,; \tag{11 . 25}$$

consequently, if λ is the value of the parameter which corresponds to the solution $\phi(s)$,

$$\int_0^1 \big(f(s)-\psi(s)\big)\,\phi(s)\,ds = \frac{\beta}{\lambda}. \tag{11 . 3}$$

From (2 . 1) and (11 . 3) we obtain the equations

$$\phi(s) = \lambda \int_0^1 \left\{k(s, t)+\frac{f(s)}{\beta}\,\big(f(t)-\psi(t)\big)\right\}\phi(t)\,dt, \tag{11 . 4}$$

$$\phi(s) = \psi(s)+\lambda \int_0^1 \kappa(s, t)\,\phi(t)\,dt, \tag{11 . 5}$$

where

$$\kappa(s, t) = k(s, t)+\frac{\big(f(s)-\psi(s)\big)\big(f(t)-\psi(t)\big)}{\beta} \tag{11 . 55}$$

is symmetric in s and t. Conversely, if $\phi(s)$ is any function which satisfies both the equations (11 . 5) and (11 . 25), we easily show, by multiplying (11 . 5) by $\psi(s)ds$ and integrating from 0 to 1 [remembering that $\psi(s)$ satisfies (2 . 1) and (11 . 1) for $\lambda = \lambda_1$, say], that

$$(a-\gamma^2)\int_0^1 f(s)\,\phi(s)\,ds = (a-\gamma^2)\Big(a+\frac{\beta}{\lambda}\Big),$$

and hence, since (11 . 2) is supposed satisfied, that $\phi(s)$ satisfies (11 . 1). On substituting from (11 . 1) and (11 . 25) into (11 . 5) we now see that $\phi(s)$ is a solution of (2 . 1) and (11 . 1).

* See § 7.

When $k(s,t)$ is a closed nucleus, $\kappa(s,t)$ is also closed, provided that the inequality (11 . 2) holds; this is readily proved by an application of the method that has been used several times already in the course of this paper in order to prove similar results. *

Equations (11 . 5), (11 . 25) are of the same form as (2 . 1), (3 . 1) respectively; thus the theory given in §§ 3–8, in which, in the characteristic equation, β is zero, becomes immediately applicable to the case now under discussion; we have merely to replace $k(s,t)$ by $\kappa(s,t)$, $f(s)$ by $\psi(s)$, and a^2 by γ^2. Thus, if either

(1) $k(s,t)$ is closed and $\alpha < 0$, or

(2) $k(s,t)$ is closed and $\alpha > \gamma^2$,

the theory of the pair of equations is covered by the developments given in § 5; whereas, if

(3) $k(s,t)$ is positive-definite and $\beta > 0$ [or, more generally, if $\kappa(s,t)$ is positive-definite], it is covered by § 6. Thus, in any of the above three cases, the reality of all the characteristic values of λ is assured, and we can at once write down uniformly convergent developments of $k^{(2)}(s,t)$ or of $k(s,t)$† itself in terms of the functions $\phi'(s)$, $\phi''(s)$, and known functions.

If any of the sets of conditions (1), (2), (3) is satisfied, the theory of § 7 would determine the distribution of the numbers λ' with reference to the poles of $W(\lambda)$. It is, however, more convenient to obtain this distribution with reference to the poles of $w(\lambda)$, and this can easily be done by a direct consideration of (11 . 1). Thus, if either (1) or (3) is satisfied, equations (7 . 3) and (7 . 2) show that the roots λ' of (11 . 1) are the points of intersection of a steadily increasing graph with a steadily decreasing graph. From this we see at once that, in either of these cases, there is one and only one root λ' between each pair of consecutive poles of $w(\lambda)$, together with other roots λ' lying to the left of a smallest pole or to the right of a greatest pole; in case (3) these additional roots λ' can be enumerated precisely, according to the sign of α. The distribution in case (2) is slightly different from the above, but will follow at once from the considerations of the next section, which is devoted to a further analysis of conditions (2).

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* See, for example, § 5 or § 9.

† For $\kappa^{(2)}(s,t)$ or $\kappa(s,t)$ are, of course, immediately expressible in terms of $k^{(2)}(s,t)$ or of $k(s,t)$ and of known functions.

The homogeneous equation for $\phi(s)$ arising from (11 . 5) and (11 . 25) is (11 . 4); thus, in any of the three cases covered by this section, this equation admits only real characteristic values of λ.

12. It has been seen in § 11 that our present problem is reducible to that treated in § 5, if it is possible to choose the special solution $\psi(s)$, so as to make

$$\gamma^2 < \alpha \quad (\alpha > 0). \tag{12 . 1}$$

Now, if $\phi_m(s)$, $\phi_n(s)$ are any two solutions of the pair of equations (2 . 1), (2 . 5) for different values of λ, they are independent functions; consequently, if γ_n^2 is defined by (5 . 4), we have, by Schwarz's inequality,

$$\alpha^2 = \left(\int_0^1 \phi_m(s)\,\phi_n(s)\,ds\right)^2 < \gamma_m^2\,\gamma_n^2.$$

It follows that there cannot exist more than one solution $\psi(s)$ for which

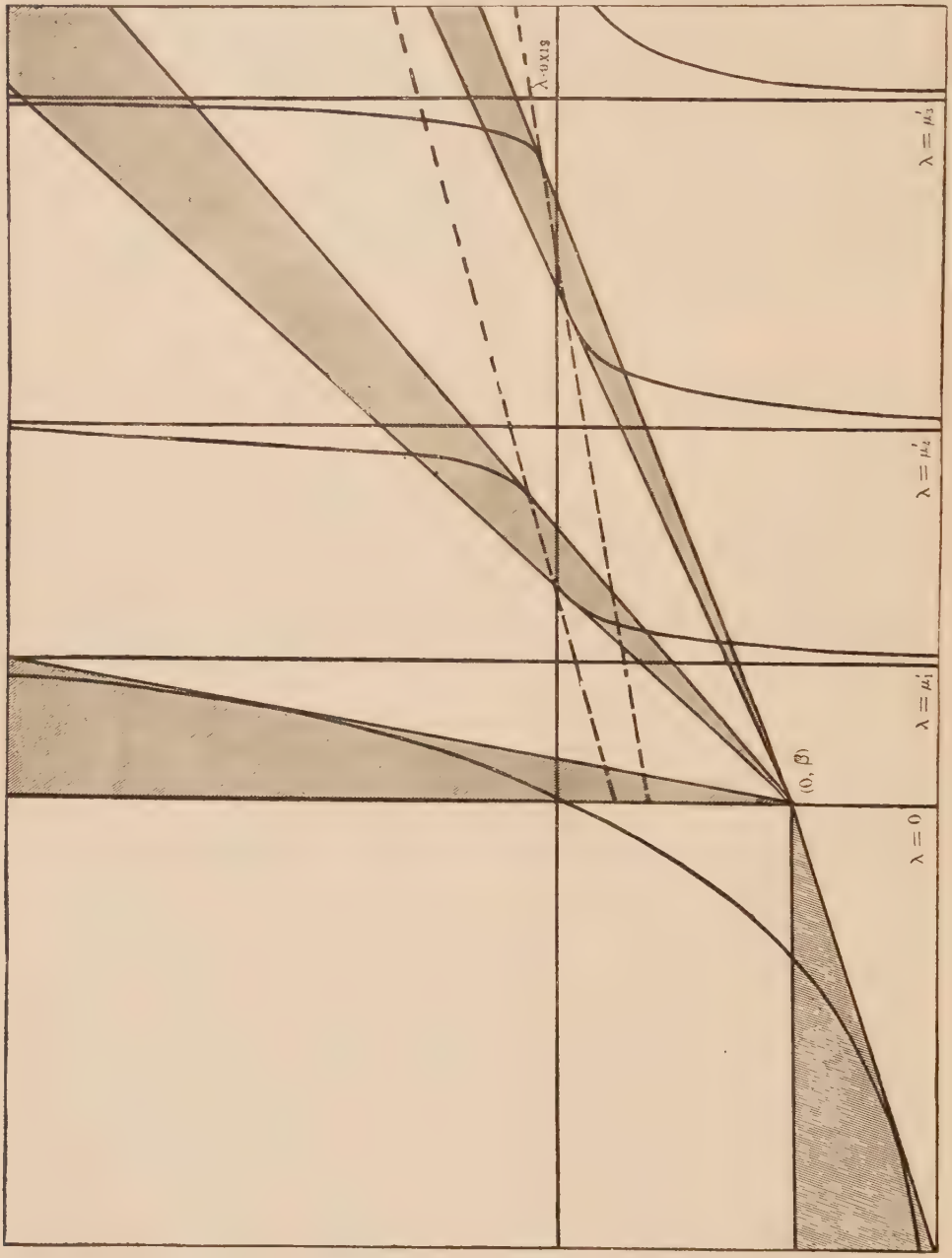
$$\gamma^2 \leqslant \alpha. \tag{12 . 2}$$

On the other hand we can show that, if α and β have suitable values, a solution $\psi(s)$ which satisfies (12 . 1) will exist. For equation (7 . 3) shows that the inequality (12 . 1) is equivalent to the statement that the corresponding intersection of the graphs of $\lambda w(\lambda)$ and of $\alpha\lambda+\beta$ is a descending node for the curve with the straight line; and it is easy to see that a necessary and sufficient condition for the existence of a descending node is that one branch of the curve (either between two consecutive asymptotes $\lambda = \mu$, or to the right of a last asymptote or to the left of a first) should be intersected more than once by the straight line. Also it is easily seen that each branch of the curve lying between two consecutive asymptotes contains at least one point of inflexion, and that, in consequence, points exist on the line $\lambda = 0$, from which at least two tangents can be drawn to this branch.* These tangents are shown in the accompanying figure, in the drawing of which it was supposed, for simplicity, that $k(s, t)$ is positive-definite. The two tangents to a branch from a point of $\lambda = 0$, if they exist,† bound at angular region R, which

* Actually it can be shown, by making use of the uniqueness of a solution $\psi(s)$ which satisfies (12 . 2), that each such branch of the curve has exactly one point of inflexion, and that, in consequence, there are never more than the two tangents mentioned in the text. But this geometrical theorem is, as a referee kindly pointed out to me, irrelevant for the further development of the analytical argument, and proof of it, together with other similar matter, has therefore been omitted.

† For the existence of these tangents to a branch of the curve lying between two asymptotes it is obviously necessary and sufficient that the point $(0, \beta)$ should lie below the point of intersection of the inflexional tangent to the branch with the axis $\lambda = 0$. Similar conditions are readily obtained for the other branches also.

has the property that, if the line $\alpha\lambda+\beta$ lies inside R, the second intersection of the straight line with that branch of the curve has, as abscissa,



Graph of $\lambda w(\lambda)$, $k(s, t)$ being positive-definite.

a value of λ, for which the corresponding solution $\psi(s)$ satisfies (12 . 1). There are also regions R, giving rise to solutions $\psi(s)$ which satisfy

(12 . 1), attached to the branch of the curve which passes through the origin, and to the branches which lie to the left of a first asymptote or to the right of a last asymptote. In the figure these angular regions R are shown shaded, and the inflexional tangents by dotted lines.

The distribution, in case (2) of § 11, of the roots λ' of (11 . 1) is now obvious. For it follows from the uniqueness of a solution $\psi(s)$ which satisfies (12 . 2) that there cannot be more than three roots λ' between any two consecutive poles of $w(\lambda)$, or more than one root λ' between more than one pair of consecutive poles of $w(\lambda)$. Intersections with other branches of the curve are treated similarly.

The remaining theory for this case follows, as was stated in § 11, from § 5.

13. We must now give a brief discussion of the case in which the selected solution $\psi(s)$ satisfies, instead of (11 . 2), the equation

$$\gamma^2 = \alpha; \tag{13 . 1}$$

for all other solutions $\phi_n(s)$ we shall then have

$$\gamma_n^2 > \alpha.$$

The theory of §§ 5–6 can no longer be applied to the pair of equations (11 . 5), (11 . 25) since now

$$\int_0^1 \kappa(s, t)\, \psi(t)\, dt \equiv 0.$$

If we still denote by λ_1 the value of λ for the solution $\psi(s)$, and if we write

$$\left(\frac{\partial \phi(s)}{\partial \lambda}\right)_{\lambda=\lambda_1} = \frac{\partial \psi(s)}{\partial \lambda_1}$$

we easily prove, by using (7 . 2), that

$$\int_0^1 f(s)\, \frac{\partial \psi(s)}{\partial \lambda_1}\, ds = \frac{d}{d\lambda_1}\, w(\lambda_1) = -\frac{\beta}{\lambda_1^2};$$

also, from (2 . 1), if $\phi(s)$ is any solution of (2. 1) and (11 . 1) which is distinct from $\psi(s)$,

$$\frac{\partial \psi(s)}{\partial \lambda_1} = \frac{\psi(s) - f(s)}{\lambda_1} + \lambda_1 \int_0^1 k(s, t)\, \frac{\partial \psi(t)}{\partial \lambda_1}\, dt,$$

$$\int_0^1 \phi(s)\, \frac{\partial \psi(s)}{\partial \lambda_1}\, ds = \frac{1}{\lambda_1} \int_0^1 \big(\psi(s) - f(s)\big)\, \phi(s)\, ds + \frac{\lambda_1}{\lambda} \int_0^1 \big(\phi(s) - f(s)\big)\, \frac{\partial \psi(s)}{\partial \lambda_1}\, ds$$

$$= \frac{\lambda_1}{\lambda} \int_0^1 \phi(s)\, \frac{\partial \psi(s)}{\partial \lambda_1}\, ds,$$

in virtue of (11 . 1), (11 . 25), and (13 . 2). Thus, since $\lambda \neq \lambda_1$,

$$\int_0^1 \phi(s) \frac{\partial \psi(s)}{\partial \lambda_1} ds = 0.$$

If we now suppose, on the other hand, that

$$\int_0^1 \psi(s) \frac{\partial \psi(s)}{\partial \lambda_1} ds = \tfrac{1}{2} \frac{d^2}{d\lambda_1^2} \left(\lambda_1 w(\lambda_1)\right) \neq 0,^* \tag{13 . 4}$$

the equation which takes the place of (11 . 5) is now

$$\phi(s) = \psi(s) + \lambda \int_0^1 \left(\kappa(s, t) + \frac{\partial \psi(s)}{\partial \lambda_1} \frac{\partial \psi(t)}{\partial \lambda_1}\right) \phi(t)\, dt; \tag{13 . 5}$$

the nucleus of (13 . 5) is readily shown, by use of (13 . 4), to be closed; it is also positive-definite when $\kappa(s, t)$ is positive-definite. Also equations (13 . 5) and (11 . 25) are satisfied by all functions $\phi(s)$ and by the function $\psi(s)$ itself; the solution $\psi(s)$ now corresponds to the value $\lambda = 0$. The converse theorem can be proved as in § 11, and the theory of §§ 3–10 is now immediately applicable to this case also.

14. It has been shown in §§ 11–13 that, if $\alpha > 0$ and $\beta \neq 0$, the existence of a real $\psi(s)$, satisfying (2 . 1) and (11 . 1), for which (12 . 2) holds, is sufficient to ensure the reality of all the roots of (11 . 1). It will now be shown that the condition is also necessary. The proof of this can be given most concisely in the language of the graphical theory of equation (11 . 1) which was introduced in § 12. Let α_1, β_1 be real numbers such that the line $\alpha_1\lambda + \beta_1$ is a tangent (but, we will suppose, not an inflexional tangent)† to some branch Λ of the curve $\lambda w(\lambda)$. The line $\alpha_1\lambda + \beta_1$ is thus one of the bounding lines of an angular region R belonging to the branch Λ and to the value β_1 of β; let us suppose it to be that one of the two bounding lines of R which has the lesser gradient, and let λ_1 denote the abscissa of its point of contact with Λ. Thus there exists a positive number ϵ such that if

$$\alpha_1 + \epsilon \geqslant \alpha \geqslant \alpha_1, \quad \beta = \beta_1,$$

---

* If $\psi(s)$ were orthogonal to $\partial\psi(s)/\partial\lambda_1$ the line $\alpha\lambda + \beta$ would be an inflexional tangent to $\lambda w(\lambda)$ at $\lambda = \lambda_1$; $\lambda = \lambda_1$ which is, in virtue of (13 . 1), a double root of (11 . 1) would then be a triple root of the equation. Thus the case which is, by (13 . 4), being ruled out of discussion is analogous to that mentioned, and also ruled out in § 9, in which we had $\beta = \alpha - a^2 = \omega = 0$, so that $\lambda = 0$ was a triple root of the characteristic equation.

† The changes to be made in the following proof when $\alpha_1\lambda + \beta_1$ is an inflexional tangent are obvious.

equation (11 . 1) has all its roots real. Keeping β equal to the constant β_1, consider $\alpha = \alpha(\lambda)$ as a function of the complex variable λ defined by (11 . 1); we find that, when $\lambda = \lambda_1$,

$$\alpha(\lambda_1) = \alpha_1, \quad \alpha'(\lambda_1) = 0, \quad \alpha''(\lambda_1) = \frac{1}{\lambda_1}\frac{d^2}{d\lambda_1^2}\left(\lambda_1 w(\lambda_1)\right).$$

But it is easily proved that

$$\lambda_1 \frac{d^2}{d\lambda_1^2}\left(\lambda_1 w(\lambda_1)\right) > 0;^*$$

hence in the neighbourhood of the regular point λ_1 of $\alpha(\lambda)$ we have

$$\alpha(\lambda) = \alpha_1 + m^2(\lambda-\lambda_1)^2 + \ldots, \tag{14 . 2}$$

where m is a real number. The reversion of the series (14 . 2) to obtain a branch of the inverse function $\lambda(\alpha)$ then establishes the existence of a positive number δ such that if

$$\alpha_1 - \delta \leqslant \alpha < \alpha_1, \quad \beta = \beta_1,$$

equation (11 . 1) has two non-real roots in the neighbourhood of $\lambda = \lambda_1$. But, on the other hand, as α continues to decrease, (11 . 1) can neither gain nor lose in the number of its non-real roots, except when $\alpha\lambda+\beta_1$ passes over an arm of some other region R. For it can so gain or lose only by the coalescence of a pair of real or non-real roots in a multiple real root, and (11 . 1) has a multiple real root if and only if $\alpha\lambda+\beta$ coincides with an arm of some region R.

The case of an increasing α may be treated similarly, and it follows that, if

$$\gamma^2 > \alpha > 0 \tag{14 . 4}$$

for every real solution of (2 . 1) and (11 . 1), (11 . 1) has at least two non-real roots.

15. Let α, β have real values such that (14 . 4) is satisfied by every real solution of (2 . 1) and (11 . 1), and let

$$\lambda_1 = \lambda_0 + i\lambda_0', \quad \bar{\lambda}_1 = \lambda_0 - i\lambda_0' \quad (\lambda_0' \neq 0)\dagger$$

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* If, for example, Λ lies between two asymptotes of $\lambda w(\lambda)$ and to the right of $\lambda = 0$, all that has to be proved is that λ_1 is greater than the abscissa of the point of inflexion of Λ associated with R. But this is obviously true; similarly with the other cases to be considered.

† Throughout § 15 we shall continually use the notation $\bar{x}$ to denote the number conjugate to x; similarly for conjugate functions of a real variable.

be a pair of conjugate complex roots of (11 . 1). Also let

$$\psi(s) = \rho(s)+i\sigma(s), \quad \bar{\psi}(s) = \rho(s)-i\sigma(s)$$

be the solutions of (2 . 1) for the values λ_1, $\bar{\lambda}_1$ of λ. Then, if $\phi(s)$ is any (real or complex) solution of (2 . 1) and (11 . 1) which is distinct from both $\psi(s)$ and $\bar{\psi}(s)$, the equations

$$\int_0^1 \phi(s)\,\psi(s)\,ds = \int_0^1 \phi(s)\,\bar{\psi}(s)\,ds = a$$

show at once that

$$\int_0^1 \rho(s)\,\phi(s)\,ds = a, \quad \int_0^1 \sigma(s)\,\phi(s)\,ds = 0;$$

hence we readily prove, by the same method as that by which (11 . 5) was obtained, that $\phi(s)$ satisfies, for the corresponding value of λ, each of the equations

$$\phi(s) = \rho(s)+\lambda\int_0^1 N_1(s, t)\,\phi(t)\,dt, \tag{15 . 1}$$

$$\phi(s) = \rho(s)+\lambda\int_0^1 N_2(s, t)\,\phi(t)\,dt, \tag{15 . 15}$$

where each of the functions

$$N_1(s, t) = k(s, t)+\frac{\big(f(s)-\rho(s)\big)\big(f(t)-\rho(t)\big)}{\beta}, \tag{15 . 2}$$

$$N_2(s, t) = k(s, t)+\frac{\big(f(s)-\psi(s)\big)\big(f(t)-\bar{\psi}(t)\big)}{2\beta} + \frac{\big(f(s)-\bar{\psi}(s)\big)\big(f(t)-\psi(t)\big)}{2\beta} \tag{15 . 25}$$

is a real symmetric nucleus. But the function $\phi(s)$ has already been seen to satisfy also the supplementary condition

$$U(\lambda) = \int_0^1 \rho(s)\,\phi(s)\,ds = a; \tag{15 . 3}$$

and, finally, since

$$\int_0^1 \Big\{\big(\rho(s)\big)^2+\big(\sigma(s)\big)^2\Big\}\,ds = \int_0^1 \psi(s)\,\bar{\psi}(s)\,ds = a, \tag{15 . 32}$$

$$\int_0^1 \big(\rho(s)\big)^2\,ds = \gamma_0^2 < a. \tag{15 . 34}$$

Thus the theory of §§ 2–5* will be applicable to the pair of equations (15 . 1), (15 . 3), or to the pair (15 . 15), (15 . 3), provided that the nucleus $N_1(s,t)$ or the nucleus $N_2(s,t)$ is closed, and in either case it will immediately follow that λ_1 and $\bar{\lambda}_1$ are the only non-real roots of (11 . 1).

The problem of the closure of the two nuclei $N_1(s,t)$, $N_2(s,t)$ is one of essentially greater difficulty than any of the apparently similar problems that have occurred earlier in these investigations. The following two theorems are, however, easily proved, and are amply sufficient for our purpose :

THEOREM A.—*If we write*

$$\int_0^1 \big(\psi(s)\big)^2\, ds = \gamma^2, \tag{15.35}$$

a necessary and sufficient condition that $N_1(s,t)$ should be closed is that

$$\mathfrak{R}\left(\lambda_1(\alpha-\gamma^2)\right) \neq 0.^{\dagger} \tag{15 . 4}$$

THEOREM B.—*A necessary and sufficient condition that $N_2(s,t)$ should be closed is that*

$$|\beta+\lambda_1(\alpha-\gamma^2)| \neq |\beta|. \tag{15 . 45}$$

Deferring for the moment the proof of these theorems, we note at once that it will follow from them that at least one of the two nuclei concerned is always closed; for, if (15 . 4) is not satisfied, $\lambda_1(\alpha-\gamma^2)$ is a non-zero,‡ purely imaginary number, and, therefore, since β is real,

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* The theory of § 6 becomes inapplicable to the present investigations since, when $k(s,t)$ is positive definite, neither $N_1(s,t)$ nor $N_2(s,t)$ will, in general, be positive-definite unless $\beta > 0$; and, in that case, all the roots of (11 . 1) are real (see § 11). The case in which $k(s,t)$ is positive-definite and $\beta < 0$ presents points of interest, and obviously needs more detailed and more specific investigation than can be given to it in this paper. In order to obtain satisfactory expansion-theorems [*i.e.* theorems for the expansion of $k(s,t)$ itself, and not merely of $k^{(2)}(s,t)$], it would be necessary to generalize the theory of § 4 by supposing the symmetrizing nucleus $G(s,t)$ to be closed, but to have a finite number (for the immediate application, one) of negative characteristic numbers. The same step would enable us to deal with the case, ruled out in § 5, in which $k(s,t)$ is closed and $0 < \alpha < a^2$. The required generalization is not, at any rate, immediate, and will not be attempted in the present paper; thus, for present purposes, when $k(s,t)$ is positive-definite and $\beta < 0$, we have to regard $k(s,t)$ as merely closed.

† $\mathfrak{R}(x)$ denotes the real part of x.

‡ Since $\mathfrak{R}(\gamma^2) = \int_0^1 \Big(\big(\rho(s)\big)^2 - \big(\sigma(s)\big)^2\Big)\, ds = \alpha - 2\int_0^1 \big(\sigma(s)\big)^2 ds, \quad \alpha-\gamma^2 \neq 0,$

(15 . 45) will be satisfied. Thus Theorems A and B together enable us in all cases to make the desired reduction of the problem now considered to one which was answered in § 5.

To prove Theorem A we note that, since

$$\rho(s) = f(s)+\lambda_0 \int_0^1 k(s, t)\,\rho(t)\,dt-\lambda_0' \int_0^1 k(s, t)\,\sigma(t)\,dt, \tag{15 . 5}$$

and since $k(s, t)$ is itself closed, a function $h(t)$, for which

$$\int_0^1 N_1(s, t)\,h(t)\,dt \equiv 0$$

is necessarily a constant multiple of $\lambda_0\rho(t)-\lambda_0'\sigma(t)$. On substituting this expression for $h(t)$, and making use of the equations

$$\int_0^1 f(s)\,\rho(s)\,ds = \alpha+\frac{\beta\lambda_0}{|\lambda_1|^2}, \tag{15 . 55}$$

$$\int_0^1 f(s)\,\sigma(s)\,ds = -\frac{\beta\lambda_0'}{|\lambda_1|^2}, \tag{15 . 6}$$

and of equation (15 . 5), we find, as a necessary and sufficient condition that $N_1(s, t)$ should *not* be closed, the equation

$$\lambda_0 \int_0^1 \big(\sigma(s)\big)^2\,ds+\lambda_0' \int_0^1 \rho(s)\,\sigma(s)\,ds = 0. \tag{15 . 62}$$

But, in virtue of (15 . 32) and (15 . 35),

$$\alpha-\gamma^2 = 2\int_0^1 \big(\sigma(s)\big)^2\,ds-2i\int_0^1 \rho(s)\,\sigma(s)\,ds, \tag{15 . 65}$$

so that equation (15 . 62) may be written in the form

$$\Re\left(\lambda_1(\alpha-\gamma^2)\right) = 0.$$

Thus the theorem is proved as enunciated.

Theorem B is proved in a similar way by means of the equations

$$\psi(s) = f(s)+\lambda_1 \int_0^1 k(s, t)\,\psi(t)\,dt,$$

$$\bar{\psi}(s) = f(s)+\bar{\lambda}_1 \int_0^1 k(s, t)\,\bar{\psi}(t)\,dt.$$

From the assumed identity

$$\int_0^1 N_2(s, t)\, h(t)\, dt \equiv 0$$

it would follow that $h(t)$ is of the form $A\psi(t)+\bar{A}\bar{\psi}(t)$, where $A, \bar{A}$ are conjugate constants, and where, as we find on substitution,

$$A\beta = \bar{A}\lambda_1\left(\alpha+\frac{\beta}{\bar{\lambda}_1}-\bar{\gamma}^2\right)$$

and, consequently,

$$A\beta^2 = A\lambda_1\bar{\lambda}_1\left(\alpha+\frac{\beta}{\lambda_1}-\gamma^2\right)\left(\alpha+\frac{\beta}{\bar{\lambda}_1}-\bar{\gamma}^2\right).$$

Since $A \neq 0$ the theorem follows at once.

If $k(s, t)$ is positive-definite, the position is simplified by the fact that condition (15 . 4) of Theorem A is always satisfied, so that, in this case, Theorem B is unnecessary, and the theory of the solutions and theorems of expansion in terms of the solutions are to be derived exclusively from the pair of equations (15 . 1), (15 . 3). For, from (15 . 65) and from the equation

$$\sigma(s) = \lambda_0\int_0^1 k(s, t)\,\sigma(t)\, dt+\lambda_0'\int_0^1 k(s, t)\,\rho(t)\, dt, \tag{15 . 8}$$

we find that

$$\begin{aligned}\tfrac{1}{2}\mathfrak{R}\left(\lambda_1(\alpha-\gamma^2)\right) &= \int_0^1 \sigma(s)\left(\lambda_0\sigma(s)+\lambda_0'\rho(s)\right) ds \\ &= \int_0^1\int_0^1 k(s, t)\left(\lambda_0\sigma(s)+\lambda_0'\rho(s)\right)\left(\lambda_0\sigma(t)+\lambda_0'\rho(t)\right) ds\, dt \\ &> 0,\end{aligned}$$

in virtue of our hypothesis concerning $k(s, t)$.

16. Before we can deduce from the general theory of §§ 2–5 expansion-theorems appropriate to the case now considered, we must discover what functions (if any) other than all real solutions $\phi(s)$ of the pair of equations (2 . 1), (11 . 1) satisfy either the pair of equations (15 . 1), (15 . 3), or the pair (15 . 15), (15 . 3). A detailed discussion with this end in view will be given only for the pair (15 . 1), (15 . 3); the method to be used in treating the second pair is similar, and will therefore be dismissed in a few lines.

Since we are concerned with the solutions of (15 . 1) and (15 . 3) we suppose condition (15 . 4) to be satisfied. From equations (15 . 5), (15 . 8) we derive the equations

$$\lambda_0\rho(s)+\lambda_0'\sigma(s) = \lambda_0 f(s)+|\lambda_1|^2\int_0^1 k(s,t)\,\rho(t)\,dt, \tag{16 . 1}$$

$$\lambda_0'\rho(s)-\lambda_0\sigma(s) = \lambda_0' f(s)-|\lambda_1|^2\int_0^1 k(s,t)\,\sigma(t)\,dt. \tag{16 . 15}$$

Hence, if $\phi(s)$ now denotes any function which satisfies (15 . 1) and (15 . 3), we find, on multiplying (15 . 1) by $\rho(s)ds$, integrating from 0 to 1, and making use of (16 . 1), that

$$\alpha = \gamma_0^2+\frac{\lambda}{|\lambda_1|^2}\int_0^1\left(\lambda_0\rho(s)+\lambda_0'\sigma(s)-\lambda_0 f(s)\right)\phi(s)\,ds \cdot$$

$$+\frac{\lambda}{\beta}\int_0^1\left(f(s)-\rho(s)\right)\rho(s)\,ds\int_0^1\left(f(s)-\rho(s)\right)\phi(s)\,ds,$$

where γ_0^2 is defined by (15 . 34); this equation involves only two integrals whose values are unknown, and, on substituting the values of the remaining integrals as given by (15 . 3), (15 . 34), and (15 . 55), it is found to reduce to

$$\int_0^1 f(s)\,\phi(s)\,ds-\alpha-\frac{\beta}{\lambda} = -\frac{\beta\lambda_0'}{(\alpha-\gamma_0^2)|\lambda_1|^2}\int_0^1\phi(s)\,\sigma(s)\,ds. \tag{16 . 2}$$

Equation (16 . 2) shows that, if

$$\int_0^1\phi(s)\,\sigma(s)\,ds = 0, \tag{16 . 25}$$

$\phi(s)$ satisfies (11 . 1); hence, by substitution from (11 . 1) and (15 . 3) into (15 . 1), $\phi(s)$ satisfies (2 . 1). The additional solutions of (15 . 1) and (15 . 3) which are now being sought are thus those solutions (if any such exist) which do not satisfy (16 . 25).

In a similar way, by multiplying (15 . 1) by $\sigma(s)ds$, integrating from 0 to 1, and making use of (16 . 15) and other known results, we obtain, after some reduction, the equation

$$\left\{\int_0^1 f(s)\,\phi(s)\,ds-\alpha-\frac{\beta}{\lambda}\right\}\int_0^1\rho(s)\,\sigma(s)\,ds$$

$$=\beta\left(\frac{\lambda_0}{|\lambda_1|^2}-\frac{1}{\lambda}\right)\int_0^1\phi(s)\,\sigma(s)\,ds. \tag{16 . 3}$$

Equations (16 . 2) and (16 . 3) are satisfied by all solutions $\phi(s)$ of (15 . 1)

and (15 . 3); but, if $\phi(s)$ is a solution which does not satisfy (16 . 25), we at once deduce from (16 . 2) and (16 . 3) that, if λ_c denotes the value of λ corresponding to this additional solution,

$$\frac{1}{\lambda_c} = \frac{1}{|\lambda_1|^2}\left(\lambda_0 + \frac{\lambda_0'}{\alpha-\gamma_0^2}\int_0^1 \rho(s)\,\sigma(s)\,ds\right) = \frac{\mathfrak{R}\left(\lambda_1(\alpha-\gamma^2)\right)}{2\,|\lambda_1|^2(\alpha-\gamma_0^2)}, \tag{16 . 35}$$

in virtue of (15 . 32), (15 . 34), and (15 . 65). Thus equation (15 . 3), which is known to be satisfied by all the real roots of (11 . 1), can have at most one root in addition to these, namely the number λ_c. Since (15 . 4) is supposed satisfied, λ_c is finite.

Knowing now that the pair of equations (15 . 1), (15 . 3) has, at most, only one solution, $\phi_c(s)$ say, in addition to the real solutions of the pair (2 . 1), (11 . 1), we can easily find this additional solution, when it exists. For suppose, in the first instance, that

$$\int_0^1 \rho(s)\,\sigma(s)\,ds \neq 0, \tag{16 . 4}$$

and define a constant A by means of the equation

$$\int_0^1 \left(\sigma(s)\right)^2 ds = A\int_0^1 \rho(s)\,\sigma(s)\,ds;$$

then we readily verify, by direct integration, that

$$\int_0^1 N_1(s,t)\left(\rho(t)+A\sigma(t)\right)dt = \frac{\lambda_0'+\lambda_0 A}{|\lambda_1|^2}\,\sigma(s) = \frac{A}{\lambda_c}\,\sigma(s)$$

in virtue of (16 . 35). Thus $\rho(s)+A\sigma(s)$ satisfies (15 . 1) for the value λ_c of λ; but this function obviously satisfies (15 . 3) also, and consequently, when (16 . 4) holds, the additional solution $\phi_c(s)$ exists and is given by

$$\phi_c(s) = \rho(s)+A\sigma(s).$$

Suppose, in the second place, that

$$\int_0^1 \rho(s)\,\sigma(s)\,ds = 0; \tag{16 . 5}$$

it is easily verified that now

$$\lambda_c = \frac{|\lambda_1|^2}{\lambda_0},$$

$$\int_0^1 N_1(s,t)\,\sigma(t)\,dt = \frac{\lambda_0}{|\lambda_1|^2}\,\sigma(s),$$

i.e. that λ_c is a characteristic constant for the nucleus $N_1(s,t)$, and $\sigma(s)$ is one of the corresponding characteristic functions for $N_1(s,t)$. In virtue of (16 . 5) $\sigma(s)$ is, in the notation of § 8, a function $\phi''(s)$ for the pair of equations (15 . 1), (15 . 3). There are now three possibilities: (1) equation (15 . 1) may be insoluble for the value λ_c of λ; (2) equation (15 . 1) may be soluble for this value of λ, but its solution may not satisfy (15 . 3); or (3) equations (15 . 1), (15 . 3) may possess a solution for the value λ_c of λ. Of the first two of these three possibilities no more need be said. But in the third case we note that any solution of (15 . 1) for $\lambda = \lambda_c$ may be written

$$\phi(s) = \Phi(s) + B\sigma(s),$$

where B is a constant, and where

$$\int_0^1 \Phi(s)\,\sigma(s)\,ds = 0.$$

Then, by multiplying (15 . 1) by $\rho(s)ds$, integrating from 0 to 1 and making use of (16 . 1), (15 . 3), (15 . 34), (15 . 55) and the values of other known integrals, we find that $\Phi(s)$ satisfies (11 . 1) for the value λ_c of λ; and, on making use of this result, we find that $\Phi(s)$ also satisfies (2 . 1) for the same value of λ, so that λ_c is a root of (2 . 1).

Thus we have proved that in all three cases, whenever (16 . 5) holds, the roots of (15 . 3) are all the real roots of (2 . 1), and *only these roots*.

17. The expansion theorems to be deduced will involve also the functions $\phi''(s)$ of our problem. It is therefore necessary to connect the functions $\phi''(s)$ of equation (15 . 1) with the functions $\phi''(s)$ of (2 . 1). For this purpose the relevant theorem is as follows: *all functions* $\phi''(s)$ *for* (2 . 1) *are also functions* $\phi''(s)$ *for* (15 . 1); *and all functions* $\phi''(s)$ *for* (15 . 1), *except the function* $\sigma(s)$ [*if* $\sigma(s)$ *is a function* $\phi''(s)$ *for* (15 . 1)], *are functions* $\phi''(s)$ *for* (2 . 1).

To prove the first part we recall that functions $\phi''(s)$ of (2 . 1) are defined by the pair of equations (3 . 6), (3 . 7); these functions are all real, and are known, from § 8, to satisfy the equation

$$\int_0^1 \phi(s)\,\psi(s)\,ds = 0.$$

It therefore follows that, for all these functions,

$$\int_0^1 \phi(s)\,\rho(s)\,ds = 0. \tag{17 . 1}$$

In consequence of (3 . 6), (3 . 7), and (17 . 1) these functions $\phi(s)$ satisfy, for the same value of λ, the equation

$$\phi(s) = \lambda \int_0^1 N_1(s, t)\, \phi(t)\, dt. \tag{17 . 2}$$

Since they satisfy both (17 . 1) and (17 . 2), the functions are functions $\phi''(s)$ for (15 . 1).

In the second place, the functions $\phi''(s)$ of (15 . 1) are characterized by the pair of relations (17 . 1), (17 . 2); on multiplying (17 . 2) in turn by $\rho(s)ds$, $\sigma(s)ds$ and integrating from 0 to 1, we find, by using (16 . 1), (16 . 15), and after some reduction, that, for these functions,

$$(\alpha-\gamma_0^2) \int_0^1 f(s)\, \phi(s)\, ds = -\frac{\beta\lambda_0'}{|\lambda_1|^2} \int_0^1 \sigma(s)\, \phi(s)\, ds, \tag{17 . 3}$$

$$\int_0^1 \rho(s)\, \sigma(s)\, ds \int_0^1 f(s)\, \phi(s)\, ds = \beta \left(\frac{\lambda_0}{|\lambda_1|^2} - \frac{1}{\lambda}\right) \int_0^1 \sigma(s)\, \phi(s)\, ds. \tag{17 . 35}$$

By elimination of $\int_0^1 f(s)\, \phi(s)\, ds$ from these two equations, we find at once that either

$$\int_0^1 \sigma(s)\, \phi(s)\, ds = 0, \tag{17 . 4}$$

or

$$\frac{1}{\lambda} = \frac{1}{|\lambda_1|^2} \left(\lambda_0 + \frac{\lambda_0'}{\alpha-\gamma_0^2} \int_0^1 \rho(s)\, \sigma(s)\, ds\right) = \frac{1}{\lambda_c}. \tag{17 . 5}$$

If (17 . 4) is satisfied, $\phi(s)$ is readily shown to be a function $\phi''(s)$ for (2 . 1). If, on the other hand, λ_c is a characteristic constant for $N_1(s, t)$, for which there exists a function $\phi''(s)$ of (15 . 1), there are two cases to be considered, namely according as (16 . 4) or (16 . 5) is satisfied. If (16 . 4) is satisfied there exists, as has been proved in § 16, a finite constant A, such that $\rho(s)+A\sigma(s)$ is a solution of the pair of equations (15 . 1), (15 . 3); consequently, in virtue of equations (17 . 1), (8 . 3),

$$\int_0^1 \sigma(s)\, \phi(s)\, ds = \frac{1}{A} \int_0^1 \left(\rho(s)+A\sigma(s)\right) \phi(s)\, ds = 0.$$

Thus, since (17 . 4) is satisfied,

$$\int_0^1 f(s)\, \phi(s)\, ds = 0$$

in virtue of (17 . 3), and $\phi(s)$ is immediately shown to be a function $\phi''(s)$ for (2 . 1). If, on the other hand, (16 . 5) is satisfied, it has been

shown in § 16 that $\sigma(s)$ is a function $\phi''(s)$ for (15 . 1); but (15 . 6) shows that $\sigma(s)$ cannot be a function $\phi''(s)$ for (2 . 1), since β and λ_0' are both different from zero. Furthermore, if, for the value λ_c of λ, (15 . 1) possesses any function $\phi''(s)$ which is distinct from $\sigma(s)$, it is known from (8 . 2) that this function satisfies (17 . 4), and is therefore a function $\phi''(s)$ for (2 . 1) also.

18. It follows from §§ 15–17 that, if we denote by $\phi_c(s)$ the first or the second of the two functions

$$\rho(s)+A\sigma(s), \quad \sigma(s)$$

according as (16 . 4) or (16 . 5) is satisfied, the complete system of functions $\phi'(s)$, $\phi''(s)$ for the equations (15 . 1), (15 . 3) is composed of (1) the function $\rho(s)$ corresponding to $\lambda = 0$, (2) the function $\phi_c(s)$, (3) all real solutions $\phi'(s)$ of the pair of equations (2 . 1), (11 . 1), and (4) all functions $\phi''(s)$ of (2 . 1). If we now write

$$P(s) = \int_0^1 N_1(s, t)\, \rho(t)\, dt,$$

the theory of § 5 gives an absolutely and uniformly convergent expansion of the function

$$N_1^{(2)}(s, t)+\frac{P(s)\ P(t)}{\alpha-\gamma_0^2}$$

in a series of the form

$$(18 . 2) \quad p\phi_c(s)\ \phi_c(t)+\Sigma q_n\Big(\phi_n'(s)-\rho(s)\Big)\Big(\phi_n'(t)-\rho(t)\Big)+\Sigma r_n\phi_n''(s)\ \phi_n''(t),$$

where the functions ϕ', ϕ'' are all real and refer to the equations (2 . 1), (11 . 1), and where the values of the constant coefficients p, q, r are readily obtained from § 5. On evaluating the function $N_1^{(2)}(s, t)$ we find that this leads to the expression of the repeated nucleus $k^{(2)}(s, t)$ as the sum of the series (18 . 2) and a finite expression involving only the functions f, $f^{(2)}$, ρ, $\rho^{(2)}$.

19. To treat the pair of equations (15 . 15), (15 . 3) from the same point of view as that which has been taken in §§ 16–18 for the pair (15 . 1), (15 . 3) we should, by multiplying (15 . 15) by $\psi(s)\, ds$ and integrating from 0 to 1, find that, for every solution $\phi(s)$ of (15 . 15) and (15 . 3),

$$\frac{\alpha-\gamma^2}{2}\left(\int_0^1 f(s)\ \phi(s)\ ds-\alpha-\frac{\beta}{\lambda}\right) = \left(\frac{\beta}{\lambda}-\frac{\beta}{\lambda_1}-\frac{\alpha-\gamma^2}{2}\right) i\int_0^1 \sigma(s)\ \phi(s)\ ds.$$

By combining this equation with its conjugate, we find that, if

$$\int_0^1 f(s)\,\phi(s)\,ds \neq a+\frac{\beta}{\lambda},$$

$$(19.2)\qquad \frac{4}{\lambda}\int_0^1 \big(\sigma(s)\big)^2\,ds = \frac{(a-\gamma^2)(a-\bar{\gamma}^2)}{\beta}+\frac{a-\gamma^2}{\bar{\lambda}_1}+\frac{a-\bar{\gamma}^2}{\lambda_1}$$

$$= \frac{|\beta+\lambda_1(a-\gamma^2)|^2-\beta^2}{\beta\,|\lambda_1|^2}.$$

Equation (19 . 2) therefore gives, in this case, the value of the possible characteristic constant λ_c. The additional function $\phi_c(s)$ is found to have the same value as for the pair (15 . 1), (15 . 3), namely $\rho(s)+A\sigma(s)$ or $\sigma(s)$, according as (16 . 4) or (16 . 5) holds. The discussion can now be completed as before.

20. We have, in the preceding pages, discussed the pair of equations (2 . 1), (2 . 5), and the pseudo-orthogonal system of functions arising therefrom in all cases except that in which $\beta = 0$ and $0 < a < a^2$. In conclusion, we should explicitly remark upon a fact which may otherwise escape notice amidst the detailed discussion : this is that, for the purpose of expansion of an "arbitrary" function,* the pseudo-orthogonal system contains one function too many. Thus, with the assumption that $\beta = 0$, it was shown, in § 5, that the functions $\phi_n(s)$ $(n \geqslant 2)$ form a closed system; again, in the cases discussed in § 11, we were still left with a closed system after the function $\psi(s)$ had been excluded from the solutions; and, lastly, in §§ 15–19 only one linear combination of the functions $\rho(s)$, $\sigma(s)$ had to be included in the system in place of the two solutions $\rho(s)\pm i\sigma(s)$.† If, then, in any of these cases, the excluded function can be expanded in terms of the functions forming the closed system, we shall have an expansion of zero in a series in which *all* the functions $\phi(s)$ of the pseudo-orthogonal system are used. Consequently the expansion of a given function $H(s)$ in terms of *all* the functions $\phi(s)$, when at all possible, will not be unique, but can, for example, be effected subject to an arbitrary linear condition imposed on the coefficients

* In the sense, of course, of mathematical physics. Actually we may suppose the function to be formed "quellenmässig" from the nucleus or iterated nucleus of the problem.

† Any case, such as those treated in § 9 and in § 13, in which (2 . 5) has a repeated root, naturally forms an exception to the above remarks. For then two or more of the functions $\phi_n'(s)$ have coalesced, and the complete system is either just adequate, or, in the case of a triple root of (2 . 5), inadequate for the representation of an "arbitrary" function.

of the expansion. This is in conformity with what Proudman finds in the special case which he discusses.

We cannot now enter into a discussion of the conditions under which the above requirement that the excluded function shall be expressible as a series of multiples of the remaining functions is satisfied, and so will merely remark that this expression of the excluded function is certainly possible if $k(s,t)$ is continuous and positive-definite, $\beta \geqslant 0$, and the function $f^{(-1)}(s)$ of (5 . 7) exists. For, for example, when $\beta = 0$, we easily show that the existence of $f^{(-1)}(s)$ involves the existence of a function $p(s)$, such that

$$f(s) = \int_0^1 H(s,\, t)\, p(t)\, dt,$$

where $H(s,t)$ is defined by (5 . 6). The required expression for $f(s)$ now follows immediately from the uniformity of the convergence of the series in (6 . 6); similarly for $\psi(s)$, when $\beta > 0$, we use the series deducible from (6 . 6) by the theory of § 11.*

* In the special case of equations (2 . 1), (2 . 5), discussed by Proudman, and to which reference was made in § 1, $k(s,t)$ is positive-definite, and $f^{(-1)}(s)$ exists, but β has a particular negative value. Thus we are again faced by the case of difficulty (in whatever concerns expansion-theorems) to which attention was drawn in foot-note * of p. 93.

A CERTAIN SERIES OF BESSEL FUNCTIONS

By T. J. I'A. BROMWICH.

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1. *Introductory Remarks.*

Mr. T. A. Lumsden has recently published a paper* calling attention to an oversight in former discussions (from the point of view of pure analysis) of a series of Bessel functions arising in the treatment of a vibrating membrane by the method of normal functions.

As pointed out by Mr. Lumsden, the previous analytical discussion appeared to show that the series of normal functions would give an erroneous result at the centre of the membrane; a conclusion which is manifestly contrary to the ordinary experiences which arise in beating a drum.† And, as might have been anticipated, the error has crept in through defects in our analytical methods (see, for example, the end of § 4 below).

After reading Mr. Lumsden's paper, I was led to apply the methods of my earlier paper‡ to the two special examples worked out numerically by Mr. Lumsden; and in this way I obtained first of all the formulae of § 5 below. On considering these, I concluded that it might be of interest to give more details of the argument sketched generally in § 8 of my earlier paper, and I wrote out the substance of § 4 below. On doing this, I observed that the discussion in the neighbourhood of $r = 0$ could hardly

* "A certain type of Fourier-Bessel Series", *Proc. London Math. Soc.* (2), 22 (1924), 381.

† For the centre of the drum-head vibrates in a manner which is perfectly continuous with the surrounding parts of the membrane.

‡ "Normal Co-ordinates in Dynamical Systems", *Proc. London Math. Soc.* (2), 15 (1916), 401.

be regarded as completely settled by my earlier paper, on account of the fact that the analytical condition laid down at $r = 0$ is of a different type from those contemplated on page 435 of my earlier paper. And I have accordingly been led to discuss the whole problem again.

The results of my discussion appear to justify the remarks added at the end of my earlier paper (page 447), where it is pointed out that the complex integrals built up by the method of Green's function are usually more convenient than those which are constructed by first calculating the series of normal functions, and then devising a complex integral for summation of the series.

I have written out my solution here with the same notation as I used in my earlier work, but I may repeat once more the opinion expressed in later papers* that Heaviside's symbolical notation is the more effective for ordinary applications to problems which can have any real physical interest.

2. *Solution by means of a Green's function.*

The fundamental differential equation of the symmetrical vibrations of a membrane is the same as that of Ex. 11, page 424, of my paper already quoted, namely,

$$\frac{\partial^2 u}{\partial t^2} - c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{du}{dr}\right) = 0;$$

from this we derive the auxiliary equation

$$\lambda^2 v - c^2\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) = \lambda u_0 + u_1, \tag{2.1}$$

where u_0 and u_1 are functions of r, representing the initial values of u and $\partial u/\partial t$ respectively. When v is found, the value of u (the displacement at time t) is given by the integral [equation (39) of my former paper]

$$u = \frac{1}{2\pi\iota}\int_{\alpha-\iota\infty}^{\alpha+\iota\infty} v e^{\lambda t}\, d\lambda \quad (\alpha > 0); \tag{2.11}$$

the path of integration being a straight line (parallel to the imaginary axis) in the λ-plane, passing on the positive side of the origin $\lambda = 0$.

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* See, for instance, "Symbolical Methods in the Conduction of Heat", *Proc. Camb. Phil. Soc.*, 20 (1921), 411.

Following the notation of Mr. Lumsden's paper, let us write $c = 1$, $a = 1$, by special choice of units;* and let us take further

$$u_0 = f(r), \quad u_1 = 0.$$

Thus equation (2 . 1) takes the form

$$\lambda^2 v - \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) = \lambda f(r), \tag{2 . 12}$$

or

$$\lambda^2 v - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \lambda f(r).$$

To obtain the general theoretical† solution, we may follow the method of § 8 in my former paper, and construct a *Green's function* $G(r, s)$ defined by the following conditions :—

$$\text{(2 . 2)} \quad \begin{cases} (\alpha) & \lambda^2 G - \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial G}{\partial r}\right) = 0 \quad \text{from } r = 0 \text{ to } r = 1, \\ (\beta) & G(1, s) = 0 \text{ at } r = 1, \\ (\gamma) & G(0, s) \text{ is continuous‡ at } r = 0, \\ (\delta) & G(r, s) \text{ is continuous at } r = s, \text{ but} \\ & \qquad r\left(\dfrac{\partial G}{\partial r}\right)_{s-0} - r\left(\dfrac{\partial G}{\partial r}\right)_{s+0} = 1. \end{cases}$$

Using the conditions (α), (β), (γ), we find that

$$\text{(2 . 21)} \quad \begin{cases} G(r, s) = A I_0(\lambda r) \text{ if } r < s, \\ G(r, s) = B\{I_0(\lambda r) K_0(\lambda) - K_0(\lambda r) I_0(\lambda)\} \text{ if } r > s. \end{cases}$$

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* The unit of length is then the radius of the circular boundary: and the unit of time is the period taken by the vibrations to travel from the centre to the circumference of the membrane.

† In most examples of any practical interest, simpler methods are available; see, for instance, § 5 below.

‡ This form of condition differs from the general types used in § 8 of my former paper, and it is on account of this point of difference that a certain amount of the analysis has to be recast.

The notation adopted is explained in Chapter II of Gray and Mathews and MacRobert, *Bessel Functions** (1922); we write in fact

$$I_0(x) = 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \frac{x^6}{2^2.4^2.6^2} + \cdots,$$

$$K_0(x) = I_0(x)\{\log(2/x) - C\} + L_0(x),$$

where
$$L_0(x) = \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2}(1+\tfrac{1}{2}) + \frac{x^6}{2^2.4^2.6^2}(1+\tfrac{1}{2}+\tfrac{1}{3}) + \cdots$$

and $C = 0{\cdot}57721\ldots$ is Euler's constant.

The general solution of (2.2), (α) is of the form

$$AI_0(\lambda r) + BK_0(\lambda r),$$

but $K_0(\lambda r)$ tends to behave like $\log(2/\lambda r)$ when r is small.

It is convenient to note that†

(2.22) $$I_0'(x)\,K_0(x) - I_0(x)\,K_0'(x) = 1/x.$$

Also that when x is larger (with its real part positive), the following asymptotic formulae hold:‡

(2.23) $$I_0(x) \sim \frac{e^x}{\sqrt{(2\pi x)}}, \quad K_0(x) \sim \sqrt{\left(\frac{\pi}{2x}\right)} e^{-x}.$$

We now introduce the conditions (δ), which lead to the two equations

$$AI_0(\lambda s) = B\{I_0(\lambda s)\,K_0(\lambda) - K_0(\lambda s)\,I_0(\lambda)\},$$

and
$$\lambda s\,[AI_0'(\lambda s) - B\{I_0'(\lambda s)\,K_0(\lambda) - K_0'(\lambda s)\,I_0(\lambda)\}] = 1.$$

From these equations, with the aid of (2.22), we see that

(2.24) $$\begin{cases} A = K_0(\lambda s) - I_0(\lambda s)\,K_0(\lambda)/I_0(\lambda), \\ B = \qquad\quad -I_0(\lambda s)/I_0(\lambda). \end{cases}$$

Thus, if we substitute from (2.24) in (2.21), we obtain the following formulae for the Green's function of the present problem:—

(2.3) $$G(r, s) = \frac{I_0(\lambda r)}{I_0(\lambda)}\{K_0(\lambda s)\,I_0(\lambda) - I_0(\lambda s)\,K_0(\lambda)\} \quad \text{if} \quad r < s$$

or
$$\frac{I_0(\lambda s)}{I_0(\lambda)}\{K_0(\lambda r)\,I_0(\lambda) - I_0(\lambda r)\,K_0(\lambda)\} \quad \text{if} \quad r > s.$$

Thus, as might be anticipated, *the Green's function is symmetrical in the two letters r, s.*

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* For brevity, the above text-book will be quoted as *Bessel Functions* in what follows.

† *Bessel Functions*, Ch. 2, § 4, Equation (52).

‡ *Bessel Functions*, Ch. 5, §§ 3, 4, 58–59.

It is of importance, in what follows, to know the character of $G(r, s)$, *regarded as a function of* λ. In the first place, on substituting for K_0 in terms of I_0 and L_0, we see that

$$\frac{1}{I_0(\lambda)}\{K_0(\lambda s)\, I_0(\lambda) - I_0(\lambda s)\, K_0(\lambda)\}$$

$$= I_0(\lambda s)\left[\left\{\log\left(\frac{2}{\lambda s}\right) - C\right\} - \left\{\log\left(\frac{2}{\lambda}\right) - C\right\}\right] + L_0(\lambda s) - I_0(\lambda s)\, L_0(\lambda)/I_0(\lambda)$$

$$= I_0(\lambda s)\{\log(1/s) - L_0(\lambda)/I_0(\lambda)\} + L_0(\lambda s).$$

Thus we have

$$G(r, s) = I_0(\lambda r)\, I_0(\lambda s)\{\log(1/s) - L_0(\lambda)/I_0(\lambda)\} + L_0(\lambda s)\, I_0(\lambda r) \quad \text{if} \quad r < s, \tag{2.31}$$

with a similar formula when $r > s$ (found by interchanging r and s).

Since $L_0(x)$ is an integral function of x, it follows from (2 . 31) that $G(r, s)$ has no singularities in the finite part of the λ-plane, other than those given by the roots of $I_0(\lambda) = 0$, and that these roots are simple poles of the function $G(r, s)$. In particular, $G(r, s)$ is regular at $\lambda = 0$, and takes the value $\log(1/s)$ {or $\log(1/r)$} at the point $\lambda = 0$.

Further, since $I_0(x)$ and $L_0(x)$ are even functions of x, the same property holds for $G(r, s)$, regarded as a function of λ; and so the character of $G(r, s)$, for large values of λ, can be obtained by restricting the real part of λ to be positive. Then, since r, s are both positive, the formulae (2 . 23) give

$$G(r, s) \sim \frac{1}{2\lambda\sqrt{(rs)}}\{e^{-\lambda|s-r|} - e^{-\lambda(2-r-s)}\}, \tag{2 . 32}$$

where the real part of λ *is positive, and* r, s *are unequal and neither of them is zero.*

The formula (2 . 32) requires modification when, say, r is zero; then (2 . 3) becomes

$$G(0, s) = K_0(\lambda s) - I_0(\lambda s)\, K_0(\lambda)/I_0(\lambda), \tag{2 . 33}$$

and the asymptotic formula is

$$G(0, s) \sim \sqrt{\left(\frac{\pi}{2\lambda s}\right)}\{e^{-\lambda s} - e^{-\lambda(1-s)}\} \tag{2 . 34}$$

provided that s is not zero nor unity.

Next, if we write $r = 0$ in (2 . 31), we have

$$G(0, s) = I_0(\lambda s)\{\log(1/s) - L_0(\lambda)/I_0(\lambda)\} + L_0(\lambda s),$$

so that $G(0, s)$ behaves like $\log(1/s)$ for small values of s; and $G(0, 1) = 0$.

It will be convenient to obtain similar formulae for $s\partial G/\partial s$; the results are easily found to be

$$(2.35) \qquad \begin{cases} s\dfrac{\partial G}{\partial s} \to +\frac{1}{2} & \text{as} \quad s \to r-0 \\ \text{or} \qquad -\frac{1}{2} & \text{as} \quad s \to r+0 \end{cases}$$

and for other values of s, r (not zero), $s\partial G/\partial s \to 0$ exponentially. Also, when r is zero and s is not, $s\partial G/\partial s \to 0$ in a similar way.

But, when $r = 0$, we have

$$(2.36) \qquad s\frac{\partial}{\partial s}\{G(0, s)\} \to -1 \quad \text{as} \quad s \to +0.$$

Finally, if we differentiate (2.3) with respect to s, and make use of (2.22), we find that

$$s\frac{\partial G}{\partial s} \to -\frac{I_0(\lambda r)}{I_0(\lambda)} \quad \text{as} \quad s \to 1-0,$$

and this tends to zero exponentially if $r < 1$; while if $r = 1$, $G(1, s)$ is always zero, and so $s\partial G/\partial s = 0$.

3. *The solution expressed by means of a series of normal functions.*

If we multiply equation (2.12) by $G(r, s)$, and then use (2.2) (α), we see that

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(v\frac{\partial G}{\partial r} - G\frac{\partial v}{\partial r}\right)\right\} = \lambda f(r)\, G(r, s).$$

Thus, if we multiply by r and integrate from $r = 0$ to $r = 1$, we have the result

$$\left[r\left(v\frac{\partial G}{\partial r} - G\frac{\partial v}{\partial r}\right)\right]_{r=0}^{r=s-0} + \left[r\left(v\frac{\partial G}{\partial r} - G\frac{\partial v}{\partial r}\right)\right]_{r=s+0}^{r=1} = \lambda\int_0^1 rf(r)\, G(r, s)\, dr.$$

At the limit $r = 0$, v, $\partial v/\partial r$, G, and $\partial G/\partial r$ are all continuous, while v and G both vanish at $r = 1$. Also v, $\partial v/\partial r$, and G are continuous at $r = s$; so, using the property (2.2) (δ), we see that

$$v(s) = \lambda\int_0^1 rf(r)\, G(r, s)\, dr.$$

Thus, interchanging r, s, we find that *the solution of* (2 . 12) *is given by the definite integral*

$$v = \lambda \int_0^1 s f(s)\, G(r, s)\, ds. \tag{3 . 1}$$

Accordingly the solution of the original problem is given by the formulae

$$u = \frac{1}{2\pi\iota} \int \lambda d\lambda e^{\lambda t} \int_0^1 s f(s)\, G(r, s)\, ds \tag{3 . 2}$$

$$= \int_0^1 s f(s)\, ds \int \frac{\lambda d\lambda}{2\pi\iota}\, e^{\lambda t}\, G(r, s),$$

on reversing the order of integration.

To express the solution (3 . 2) in the form of a series of normal functions, we reduce the complex integral to a series. Now, in view of the fact that $G(r, s)$ is regular in the whole λ-plane, except at the poles [which are the roots of $I_0(\lambda) = 0$], we see that

$$\sqrt{\frac{\lambda d\lambda}{2\pi\iota}}\, e^{\lambda t}\, G(r, s) = \text{the sum of the residues of } \lambda e^{\lambda t}\, G(r, s), \text{ taken for the roots of } I_0(\lambda) = 0. \tag{3 . 21}$$

These roots are of the form $\pm\iota\omega$, where ω is any positive root of

$$J_0(\omega) = 0. \tag{3 . 3}$$

If we take the upper sign, the residue is seen to be*

$$\iota\omega e^{\iota\omega t} \frac{J_0(\omega r)}{\iota J_1(\omega)} \{0 - J_0(\omega s)\, K_0(\iota\omega)\} \quad \text{if} \quad r < s. \tag{3 . 31}$$

Also, on writing $x = \iota\omega$, in (2 . 22) we see that

$$\iota\omega \{\iota J_1(\omega)\, K_0(\iota\omega) - 0\} = 1,$$

so that

$$K_0(\iota\omega) = -1/\{\omega J_1(\omega)\}.$$

* The first formula of (2 . 3) is used; it may be noted that

$$I_0'(\lambda) = \frac{\lambda}{2} + \frac{\lambda^3}{2^2 . 4} + \frac{\lambda^5}{2^2 . 4^2 . 6} + \ldots = \frac{\iota\omega}{2}\left(1 - \frac{\omega^2}{2 . 4} + \frac{\omega^4}{2 . 4^2 . 6} - \ldots\right) = \iota J_1(\omega).$$

Hence (3 . 31) becomes

$$\frac{J_0(\omega r)\, J_0(\omega s)}{\{J_1(\omega)\}^2}\, e^{\iota\omega t}, \tag{3 . 32}$$

and, from the symmetry of this result, it is evident that r may be greater or less than s.

Consequently the sum of the two residues, for the two roots $\lambda = \pm \iota\omega$, will be

$$\frac{2J_0(\omega r)\, J_0(\omega s)}{\{J_1(\omega)\}^2} \cos \omega t. \tag{3 . 33}$$

And so, finally, if we substitute from (3 . 21) and (3 . 33) in (3 . 2), we obtain the formula

$$u = 2\Sigma \cos \omega t\, \frac{J_0(\omega r)}{\{J_1(\omega)\}^2} \int_0^1 s J_0(\omega s) f(s)\, ds. \tag{3 . 4}$$

And, except for notation, the formula (3 . 4) is the same as (1 . 1) of Mr. Lumsden's paper.

4. *Discussion of the series of normal functions.*

We have now to consider whether the equation is correct which is suggested by writing $t = 0$ in the solution (3 . 4), namely,

$$f(r) = 2\Sigma\, \frac{J_0(\omega r)}{\{J_1(\omega)\}^2} \int_0^1 s J_0(\omega s) f(s)\, ds. \tag{4 . 1}$$

The method is sketched on pages 439–443 of my paper; we assume that $f(r)$ is subject to Dirichlet's conditions, as laid down in his theory of Fourier series, and then (using the second theorem of the mean) we have

$$\int_0^1 sf(s)\, G(r, s)\, ds = f(0) \int_0^{\sigma_1} sG(r, s)\, ds + f(r-0) \int_{\sigma_1}^{r} sG(r, s)\, ds + f(r+0) \int_r^{\sigma_2} sG(r, s)\, ds + f(1) \int_{\sigma_2}^1 sG(r, s)\, ds, \tag{4 . 2}$$

where $$0 < \sigma_1 < r < \sigma_2 < 1.$$

Now, if we integrate the differential equation for $G(r, s)$, we obtain

$$\lambda^2 \int rG(r, s)\, dr = \left[r \frac{\partial G}{\partial r} \right],$$

or, interchanging r and s, we have

$$\lambda^2 \int sG(r, s)\, ds = \left[s \frac{\partial G}{\partial s} \right]. \tag{4.21}$$

It follows from (4.21) and (2.35) that, when $|\lambda|$ is large, the first and last of the four integrals on the right-hand side of (4.2) will tend to zero exponentially, while the second and the third each tend to $1/(2\lambda^2)$. Consequently

$$\int_0^1 sf(s)\, G(r, s)\, ds = \tfrac{1}{2}\lambda^2 \{f(r-0)+f(r+0)\} + \text{terms which tend to zero exponentially.} \tag{4.22}$$

And, using (4.22) in (3.2), we see that, when $t = 0$, u will take the form

$$\tfrac{1}{2}\{f(r-0)+f(r+0)\} \quad \text{or} \quad f(r), \tag{4.23}$$

the latter form holding when $f(r)$ is continuous.

Consider next the case when $r = 0$; we have then

$$\int_0^1 sf(s)\, G(0, s)\, ds = f(0) \int_0^\sigma sG(0, s)\, ds + f(1) \int_\sigma^1 sG(0, s)\, ds, \tag{4.3}$$

where

$$0 < \sigma < 1.$$

Thus

$$\lambda \int_0^1 sf(s)\, G(0, s)\, ds = \frac{1}{\lambda} f(0) \left[s \frac{\partial}{\partial s} G(0, s) \right]_0^\sigma + \frac{1}{\lambda} f(1) \left[s \frac{\partial}{\partial s} G(0, s) \right]_\sigma^1$$

and so, using (2.36), we have

$$\lambda \int_0^1 sf(s)\, G(0, s)\, ds = \frac{1}{\lambda} f(0) + \text{terms which tend to zero exponentially.} \tag{4.31}$$

Thus for $t = 0$ and $r = 0$, u takes the form $f(0)$ as might have been predicted on physical grounds.

We are now in a position to see why the statement made in *Bessel Functions*, 8, § 2, is wrong; the fact is that the asymptotic formula (15) does *not* hold* for $r = 0$ nor for $x = 0$ (the x used there corresponds to our s). No doubt a formula similar to (15) can be constructed to hold for these excepted values; the necessary results are deducible from those given in § 2 above, but it seems unnecessary to write them out here at full length.

5. *Two special functions.*

The two special functions discussed numerically (as well as theoretically) by Mr. Lumsden are

$$\text{(i)}\quad f(r) = 1, \qquad \text{(ii)}\quad f(r) = 1-r^2.$$

I would remark in this connexion that it is usually simpler to solve the equation (2 . 12) *directly*, rather than by applying the formulae of § 3. Thus we may proceed as follows :—

$$\text{(i)}\quad f(r) = 1.$$

Then we have to solve

$$\lambda^2 v - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \lambda, \tag{5 . 1}$$

subject to the continuity of v at $r = 0$, and with the boundary condition $v = 0$ at $r = 1$.

The general solution of (5 . 1) is easily seen to be given by

$$\lambda v = 1 + AI_0(\lambda r) + BK_0(\lambda r), \tag{5 . 11}$$

and B must be zero to make v continuous at $r = 0$. Thus the final solution is

$$v = \frac{1}{\lambda}\left\{1 - \frac{I_0(\lambda r)}{I_0(\lambda)}\right\}. \tag{5 . 12}$$

Proceeding as in § 3, we note first that $\lambda = 0$ is not a pole of v (regarded as a function of λ), and that the residue of $ve^{\lambda t}$ at the pole $\lambda = \iota\omega$, is

$$-\frac{1}{\iota\omega}\frac{J_0(\omega r)}{\iota J_1(\omega)}e^{\iota\omega t} = +\frac{J_0(\omega r)}{\omega J_1(\omega)}e^{\iota\omega t}.$$

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* This is evident since the formula contains $\surd(xr)$ *in the denominator.*

Thus the series of normal functions giving the solution is

$$u = \Sigma \frac{2J_0(\omega r)}{\omega J_1(\omega)} \cos \omega t, \tag{5.13}$$

agreeing with the formula given by Mr. Lumsden on page 385 of his paper. It is proved there that the series given by writing $r = 0$, $t = 0$, namely

$$2\Sigma 1/\{\omega J_1(\omega)\},$$

converges to a value (calculated by Euler's method) which differs from 1, that is, $f(0)$, by less than 6×10^{-6}.

The reader will find no difficulty in repeating the arguments used in § 4, when $r = 0$ is substituted in (5.12), so that we are left with the function of λ,

$$v_0 = \frac{1}{\lambda}\left\{1 - \frac{1}{I_0(\lambda)}\right\}. \tag{5.14}$$

(ii) $f(r) = 1 - r^2$.

This is effectively the same as the example worked out (for another purpose) on page 424 of my earlier paper;* but it is perhaps worth while to repeat a few of the results here.

The equation for v is now

$$\lambda^2 v - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \lambda(1 - r^2), \tag{5.2}$$

which gives, by considerations similar to those used in (i),

$$v = \frac{1}{\lambda}(1 - r^2) - \frac{4}{\lambda^3}\left\{1 - \frac{I_0(\lambda r)}{I_0(\lambda)}\right\}. \tag{5.21}$$

Here again, observing that when λ is small

$$\frac{I_0(\lambda r)}{I_0(\lambda)} = 1 + \frac{\lambda^2}{4}(r^2 - 1) + O(\lambda^4),$$

we see that $\lambda = 0$ is not a pole of v. The residue of $ve^{\lambda t}$ at the pole $\lambda = \iota\omega$ is then seen to be

$$\frac{4}{(\iota\omega)^3}\frac{J_0(\omega r)}{\iota J_1(\omega)} e^{\iota\omega t}.$$

* Taken from Lord Rayleigh's *Theory of Sound*, Art. 204.

Thus the series of normal functions is now found to be

$$u = \Sigma \frac{8}{\omega^3} \frac{J_0(\omega r)}{J_1(\omega)} \cos \omega t, \tag{5.22}$$

agreeing with the formulae given by Mr. Lumsden and by myself.

Mr. Lumsden shows that the sum of the series (given by writing $r = 0$, $t = 0$),

$$\Sigma 8/\{\omega^3 J_1(\omega)\} \tag{5.23}$$

differs from 1, that is, $f(0)$ by less than 7×10^{-6}.

Again there is no difficulty in applying the methods of § 4 to the function given by writing $r = 0$ in (5.21), namely,

$$v_0 = \frac{1}{\lambda} - \frac{4}{\lambda^3}\left\{1 - \frac{1}{I_0(\lambda)}\right\}, \tag{5.24}$$

if the reader wishes to complete the argument in every detail.

ANALYTIC COMPLETENESS OF SETS OF POSTULATES

By C. H. LANGFORD.

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THE following discussion is concerned with a type of *completeness* which certain sets of postulates can be shown to possess. The type of completeness in question may be called *completeness of general law*, in accordance with Professor Huntington's distinction* between *general laws* and *existence* postulates.

Consider the following very simple set of *general laws* which hold for *Serial Order*. We have

I. A class K.

II. A dyadic relation R.

III. The following logical functions of K and R :

(1) For every element a, in K, $R(aa)$ fails.

(2) For every pair of distinct elements a, b, in K, $R(ab)$ fails $.\supset.$ $R(ba)$ holds.

(3) For every triad of distinct elements a, b, c, in K, $R(ab)$ holds $.$ $R(bc)$ holds $.\supset.$ $R(ac)$ holds.

(4) For every pair of distinct elements a, b, in K, $R(ab)$ holds $.\supset.$ $R(ba)$ fails.

These four postulates are all general laws in this sense :—(1) They do not demand the existence of elements in K, and (2) they make

* E. V. Huntington and J. R. Kline, "Sets of Independent Postulates for Betweenness", *Trans. Amer. Math. Soc.*, 18 (1917), 302.

universal statements about *every* element, *every* pair of elements, and *every* triad of elements in the class K. By a general law ϕ we shall mean any postulate of the form "If α be *any* subset of the elements of K of cardinality n, then $\psi(\alpha, R)$", where ψ is any logical function of the elements of α and of no other K-elements.* A general law does not demand the existence of elements, since it is always satisfied when K does not have at least n elements, and it makes exactly the same statement about every set of n elements in K.

We may now raise this question : Is it possible to add to the above set of postulates any other general law on the dyadic relation R which shall be consistent with that set, but which shall not be implied by it? In other words, is any general law on the base (K, R) such that either it is implied by the set 1–4, or else its contradictory is implied by the set? If so, these general laws are exhaustive of the possibilities of statement, in terms of general law, on a dyadic R, and the set 1–4 is a complete set of general laws. In the form in which these postulates stand, the answer to this question is not at all evident.

An equivalent set of postulates may be formulated as follows :

For convenience we introduce the definitions,

$$R(ab) \text{ holds} \,.=.\, ab. \qquad \text{(Df.)}$$

$$R(ab) \quad \text{fails} \,.=.\, \overline{ab}. \qquad \text{(Df.)}$$

The concept on which the formulation of the set is based is what will be called the *form* of an n-ad of elements. To explain this concept : Postulate 2 of the above set says : "For any two distinct K-elements a, b, $\overline{ab} \,.\supset.\, ba$", while Postulate 4 says : "For any two distinct K-elements a, b, $ab \,.\supset.\, \overline{ba}$". The force of Postulates 2 and 4 is expressed

* A value of ϕ is obtained by substituting a particular value for α. ψ is a function on (α, R) in the same sense as the postulate set itself is a function on (K, R). The postulate "If a, b are members of K, then $R(ab)$ or $R(ba)$" is a general law, since the elements referred to in the conclusion are those given in the hypothesis, whereas the postulate "If a, b are members of K, then for some x in K, $R(ax)$ and $R(xb)$ or $R(bx)$ and $R(xa)$" is not a general law. If "successor in α" be defined in an obvious way, then "If an element in α has a successor in α, then it has an immediate successor in α" is a general law, since reference is confined to the elements of α. Here "successor in α" is a variable sub-relation of R with values corresponding to different values of α.

In the above set of postulates the maximum number of elements involved is three. In the present investigation we shall be concerned with general laws involving any finite number of elements, but shall not consider the non-finite case.

by saying: For any two distinct K-elements a, b, $ab \,.\equiv.\, \overline{ba}$. And the force of this latter postulate is given by saying: Any two distinct K-elements *have the form* $AB \,.\, \overline{BA}$.

This postulate assigns to every pair of distinct elements in K a unique *form*, since it involves all of the combinations into which two elements, on dyads of distinct elements, can enter. Now consider any postulate, $\phi(x, y)$, in the form of a general law, which refers to dyads of distinct elements. And, further, consider any two elements a, b to which this postulate refers. Quite aside from any information given by the postulate $\phi(x, y)$, we know that a and b are such that either (1) $ab \,.\, ba$, or (2) $\overline{ab} \,.\, \overline{ba}$, or (3) $ab \,.\, \overline{ba}$, or (4) $\overline{ab} \,.\, ba$. Any two of these alternatives are mutually exclusive, when a and b are particular elements, so that all that $\phi(x, y)$ can do is to determine, or partially determine, the selection of one or the other of the alternatives. In doing this it must be *true of* at least one of the alternatives. But a further consideration enters here. A postulate about *any* two distinct K-elements cannot distinguish between (3) and (4). It cannot select a as against b where a and b are particular elements, so that (3) and (4) are indistinguishable.* The *forms* $(AB \,.\, BA)$, $(\overline{AB} \,.\, \overline{BA})$, $(AB \,.\, \overline{BA})$ are, however, distinct and mutually exclusive, and any postulate $\phi(x, y)$ must be true of† at least one of these forms if it is to be consistent with the existence of at least two elements in the class K. If it is true of only one of the forms, then it selects that form uniquely; and a postulate $\phi(x, y)$ which selects a unique form has the following important property: Any other general law $\phi'(x, y)$ is either such that it is implied by $\phi(x, y)$ or else such that its contradictory is implied by $\phi(x, y)$.

For, suppose, first that there are at least two elements in the class K. Denote any one of the above forms by X, and any other by Y, and the third by Z. Now postulate ϕ, by hypothesis, is true of one and only one of the forms X, Y, Z. Call this form X. Then ϕ determines X uniquely and is false of Y and Z. Suppose ϕ' to be true also of X. (X may or may not be the only form which satisfies ϕ'.)

Then ϕ is true implies that every pair of distinct elements have the

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* Considerations of this sort are due to Dr. H. M. Sheffer, who has used them extensively in unpublished work. He refers to the "postulational indistinguishability" of different permutations of the same set of elements. Clearly, any general statement which is satisfied by $ab \,.\, \overline{ba}$ is also satisfied by $\overline{ab} \,.\, ba$, and conversely. The particular permutation of the symbols used in a general statement is not relevant. The use of "form" here is also due to Sheffer. But, whereas he uses the concept in connexion with an n-ad of elements or individuals, we are here extending its use to an n-ad of variables which denote individuals.

† A general law ϕ is true of a and b which are such that $ab \,.\, \overline{ba}$ if $ab \,.\, \overline{ba}$ satisfies ϕ. It is true of the form $AB \,.\, \overline{BA}$ if any two elements which have this form satisfy ϕ.

form X. But, by hypothesis, any pair of elements a', b' having the form X satisfy ϕ'. Therefore ϕ implies ϕ' is *always* satisfied, *i.e.* that ϕ' is true.

If a pair of elements having the form X does not satisfy ϕ', then such a pair satisfies the contradictory of ϕ', and, as above, ϕ implies the contradictory of ϕ'.*

If there are not at least two elements in K, then any general law $\phi'(x, y)$ is satisfied, so that the argument is independent of the cardinality of K.

For example, suppose ϕ' to be true of X and, in addition, say, true of Y but false of Z. Now $\phi(a, b)$ is equivalent to "a and b have the form X". For if $\phi(a, b)$ is true, then a and b have the form X, and if a and b have the form X, then $\phi(a, b)$ is true. Similarly, $\phi'(a, b)$ is true is equivalent to "a and b have the form X or the form Y". For if $\phi'(a, b)$ is true, then a and b have either the form X or the form Y, and if a and b have either the form X or the form Y, then $\phi'(a\ b)$ is true. It follows that $\phi(a, b)$ implies $\phi'(a, b)$. For the proposition "a and b have the form X" implies the proposition "a and b have the form X or a and b have the form Y". But $\phi(a, b)$ and $\phi'(a, b)$ are equivalent, respectively, to those two statements.

Similarly if ϕ' is false of the forms X and Y and true of Z, then $\phi(a, b)$ implies that $\phi'(a, b)$ is false.

If ϕ' is true of X and false of Y and Z, then ϕ and ϕ' are equivalent. In no case can ϕ' be independent of ϕ.

Thus the postulate "Any two distinct K-elements *have the form* $AB\,.\,\overline{BA}$" is complete within its domain of reference. It selects directly a unique *form*. The postulate is equivalent to the statement "For any two distinct K-elements a, b, $ab\,.\,\overline{ba}$, or $\overline{ab}\,.\,ba$". But the concepts involved in the two formulations are different. In the latter form of statement specific mention is made of the elements a, b which enter into the combinations, and this necessitates the disjunctive expression. Whereas in the former statement we abstract from the identity of the particular elements and express merely their form.

Consider any n *particular* elements a, b, c, d, ..., n of a class K, and consider a dyadic (or m-adic) relation R, which either holds or fails for every ordered dyadic (or m-adic) combination of these elements. If R holds for the dyad ab we may write $ab+$, and if R fails for ab we may write $ab-$. Form all possible ordered dyads of the elements

* It is to be noted that this argument proves more than is strictly required, since the contradictory of the universal statement ϕ' is a particular negative, whereas a universal negative is established.

$a, b, c, d, \ldots, n$. We get

$$\begin{array}{r} ab,\ ac,\ ad,\ \ldots,\ an \\ ba,\ ca,\ da,\ \ldots,\ na \\ bc,\ bd,\ \ldots,\ bn \\ cb,\ db,\ \ldots,\ nb \\ cd,\ \ldots,\ cn \\ dc,\ \ldots,\ nc \\ \ldots,\ dn \\ \ldots,\ nd \\ \ldots\ldots \\ \ldots\ldots \\ aa,\ bb,\ cc,\ dd,\ \ldots,\ nn. \end{array}$$

Quite aside from any postulational information, we know that each of the above ordered dyads must have a "+" or a "−" validity value,* and, of course, not both. We know, therefore, that

$$\begin{array}{r} ab\pm\ .\ ac\pm\ .\ ad\pm\ \ldots\ an\pm \\ ba\pm\ .\ ca\pm\ .\ da\pm\ \ldots\ na\pm \\ bc\pm\ .\ bd\pm\ \ldots\ bn\pm \\ cb\pm\ .\ db\pm\ \ldots\ nb\pm \\ cd\pm\ \ldots\ cn\pm \\ dc\pm\ \ldots\ nc\pm \\ \ldots\ dn\pm \\ \ldots\ nd\pm \\ \ldots\ldots \\ \ldots\ldots \\ aa\pm\ .\ bb\pm\ .\ cc\pm\ .\ dd\pm\ \ldots\ nn\pm\ . \end{array}$$

Thus there will be 2^n selections of positive and negative validity values possible for n elements, and any selection gives rise to a compound asser-

* An ordered dyad ab is said to have a *positive* validity value if $R(ab)$ holds and a *negative* validity value if $R(ab)$ fails.

tion about the holdure and failure of the relation R for the several ordered dyads which is incompatible with a statement determined by any other selection of validity values.

Denote particular selections of sets of validity values for the elements $a, b, c, d, \ldots, n$ by S_1, S_2, etc. Consider a particular selection S_1. The ordered dyads with these values attached make a definite compound statement. Take any other set of n elements $a', b', c', d', \ldots, n'$, and establish in any way a one-to-one correspondence between these elements and $a, b, c, d, \ldots, n$.

(These two sets may be entirely distinct, or they may have common elements, or the two classes may be identical, and the correspondence may then hold between different permutations of the same set of elements.)

For each element in S_1 substitute the element to which it corresponds in the set $a', b', c', \ldots, n'$. This substitution will give rise to a new statement, and this new statement will be said to have the same form as the first statement. Consider the class of all possible statements which may be generated in this manner from a definite statement. Any two of these statements will be said to have the same *form*. The form of a statement is therefore what might be called a "validity function" of the elements involved in the statement.*

Now consider any general law ϕ which is a logical function on the base (K, R), any value of which involves just n elements, and consider the particular statement S_1, involving the elements $a, b, c, d, \ldots, n$. This statement involves all possible combinations of these elements in dyads of distinct and identical elements, and assigns a single validity value to each. By substituting the elements $a, b, c, \ldots, n$ in ϕ we get $\phi(a, b, c, \ldots, n)$, which is a definite statement about these elements, and $\phi(a, b, c, \ldots, n)$ will be true or false according as it agrees or disagrees with the unique assignment of validity values given by S_1.

But ϕ is more abstract than S_1, for it is a general statement about *any* n elements; so that if ϕ is true of the elements $a, b, c, \ldots, n$ in the statement S_1, it will be true of any statement derived from S_1 by the method prescribed above. That is, ϕ will be true of *any* set of elements which have the *form* which the elements of S_1 have in the statement S_1. Otherwise ϕ is not a general law, and does not assert a relation to hold among *any* n elements.

For example, take the following statement about the three elements a, b, c : $ab \,.\, bc \,.\, ac \,.\, \overline{ba} \,.\, \overline{cb} \,.\, \overline{ca} \,.\, \overline{aa} \,.\, \overline{bb} \,.\, \overline{cc}$, and consider the general law

* For this method of defining a function, see W. E. Johnson, *Logic*, Part II, 48–52.

of transitivity $XY \,.\, YZ \,.\supset.\, XZ$. This law is true of the statement on a, b, c, and it will be true of any other statement derived from this one by substituting other elements. Whereas, take the statement

$$a'b' \,.\, b'c' \,.\, \overline{a'c'} \,.\, \overline{b'a'} \,.\, \overline{c'b'} \,.\, c'a' \,.\, \overline{a'a'} \,.\, \overline{b'b'} \,.\, \overline{c'c'}.$$

The general law is false for a', b', c', and it will be false for any derived statement.

The truth or falsity of a general statement rests, therefore, only on the *form* of the elements to which it refers, and not at all on the particular elements or on the particular permutation of the elements involved. A general law may, however, be true of two statements having different forms, provided they have enough in common to satisfy the law. For example, $XZ \,.\, YZ \,.\supset.\, XZ$ is true of the first of the above statements, and is also true of $de \,.\, ef \,.\, df \,.\, ed \,.\, fe \,.\, fd \,.\, \overline{dd} \,.\, \overline{ee} \,.\, \overline{ff}$. It will therefore be true of any set of elements having either of these forms.

A general law, involving n variables, denoting individuals, which is expressed in terms of the concept of *form*, is equivalent to a general law of the usual kind, which involves reference to any n *given* elements and which is expressed by $n!$ disjunctions, corresponding to the $n!$ permutations of the particular elements, for a determination of the form of a set of elements cannot determine a particular permutation. Any substitution of particular elements in a general form will, as different permutations of the elements are substituted, always give rise to incompatible statements if the statements resulting from different permutations are not identical and, therefore, redundant. Take the form $AB \,.\, BA$. Then, if for the elements a and b we have $ab \,.\, ba$, we also have $ba \,.\, ab$ by permuting a and b. But these two statements are identical. That any two such permutations must be either redundant or incompatible may be seen as follows:

Any permutation of elements substituted in a form involves all possible dyadic (or m-adic) combinations of these elements, and to each dyad (or m-ad) a single validity value is assigned. Now any other permutation of the elements involves these same dyads (or m-ads). If each ordered dyad in the one permutation has the same validity value as that dyad in the other, then the two forms of statement are identical and redundant. If some dyad differs in validity value in the two cases, the statements are incompatible. It follows that a form which involves all possible dyadic (or m-adic) combinations of n variables involves all that can consistently be said about *any* n elements.

The following general laws for Serial Order are expressed in terms of

the concept of *form*. We demand

I′. A class K.

II′. A dyadic relation R.

III′. The following logical functions of K and R:

1′. Any single K-element *has the form*

$$\overline{AA}.$$

2′. Any two distinct K-elements *have the form*

$$AB \,.\, \overline{BA}.$$

3′. Any three distinct K-elements *have the form*

$$\begin{matrix} AB \,.\, BC \,.\, AC \\ \overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}. \end{matrix}$$

Postulates 2′ and 3′ refer to the same class of R-dyads, whereas 1′ refers to an entirely distinct class, since 2′ and 3′ refer to dyads formed of distinct elements, while 1′ refers to dyads formed of identical elements.

The following theorems establish the equivalence of sets 1–4 and 1′–3′.

THEOREM I (Postulate 1).—*For every element a in K, $\overline{aa}$.*

Proof.—This follows immediately from 1′, since, if we substitute any element a of the class K in the form $\overline{AA}$, we get $\overline{aa}$.

THEOREM II (Postulate 2).—*For every pair of distinct elements a, b in K, $\overline{ab} \,.\supset.\, ba$.*

Proof.—Suppose that for some a, b $(a \neq b)$ in K, $\overline{ab}$. Then if $\overline{ba}$, a, and b have the form $\overline{AB} \,.\, \overline{BA}$, contrary to Postulate 2′.

THEOREM III (Postulate 4).—*For every pair of distinct elements a, b in K, $ab \,.\supset.\, \overline{ba}$.*

Suppose that for some a, b in K $(a \neq b)$, ab. Then if ba, a and b have the form $AB \,.\, BA$, contrary to Postulate 2′.

THEOREM IV (Postulate 3).—*For every triad of distinct elements a, b, c in K, $ab \,.\, bc \,.\supset.\, ac$.*

Proof.—Suppose that, for some triad of elements, a, b, c, $ab \,.\, bc \,.\, \overline{ac}$.

Then, by Theorems II and III, we have $\overline{ba} \,.\, \overline{cb} \,.\, ca$; so that we have $\dfrac{ab \,.\, bc \,.\, \overline{ac}}{\overline{ba} \,.\, \overline{cb} \,.\, ca}$ which is contrary to the form demanded by 3'.*

Theorems I–IV show that the set 1–4 follows from the set 1'–3' The following theorems establish the converse relation.

THEOREM V (Postulate 1').—*Any single K-element has the form* $\overline{AA}$.

Proof.—By Postulate 1, for every element a in K, $\overline{aa}$. Hence any K-element has the form $\overline{AA}$.

THEOREM VI (Postulate 2').—*Any two distinct K-elements have the form* $AB \,.\, BA$.

Proof.—Let a, b be any two distinct K-elements. If ab, then $\overline{ba}$, by 4. If $\overline{ab}$, then ba, by 2. This gives the required form.

THEOREM VII (Postulate 3').—*Any three distinct K-elements have the form* $\dfrac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$.

Proof.—Let a and b denote any two given elements of K. Then ab or $\overline{ab}$. If ab, then $\overline{ba}$ (by Postulate 4). If $\overline{ab}$, then ba (by Postulate 2). So that we have $\overline{ab} \,.\, ba$ or $ab \,.\, \overline{ba}$.

Suppose $\overline{ab} \,.\, ba$. Then denote any third distinct element by c. By 2 and 4, $ac \,.\, \overline{ca}$ or $\overline{ac} \,.\, ca$.

If $ac \,.\, \overline{ca}$, then $ba \,.\, ac \,.\supset.\, bc$ (by 3), and $bc \,.\supset.\, \overline{cb}$ (by 4). So that we have $\dfrac{ba \,.\, ac \,.\, bc}{\overline{ab} \,.\, \overline{ca} \,.\, \overline{cb}}$, which has the form $\dfrac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$.

If $\overline{ac} \,.\, ca$, then $ba \,.\, ca \,.\, \overline{ab} \,.\, \overline{ac}$. If bc, then $\overline{cb}$ (by 4), and we have $\dfrac{bc \,.\, ca \,.\, ba}{\overline{cb} \,.\, \overline{ac} \,.\, \overline{ab}}$, which has the form $\dfrac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$. If $\overline{bc}$, then cb (by 2), and we have $\dfrac{cb \,.\, ba \,.\, ca}{\overline{bc} \,.\, \overline{ab} \,.\, \overline{ac}}$, which has the required form.

Suppose $ab \,.\, \overline{ba}$. Denote any third distinct element by c.

If $ac \,.\, \overline{ca}$, then $ab \,.\, ac \,.\, \overline{ba} \,.\, \overline{ca}$. If bc, $bc \,.\supset.\, \overline{cb}$ (by 4), so that we have $\dfrac{ab \,.\, bc \,.\, ac}{\overline{ba} \,.\, \overline{cb} \,.\, \overline{ca}}$, which has the required form. If $\overline{bc}$, then cb, and we have $\dfrac{ac \,.\, cb \,.\, ab}{\overline{ca} \,.\, \overline{bc} \,.\, \overline{ba}}$, which has the required form.

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* Thus the general law $ab \,.\, bc \supset . ac$ is *true of* the form $\dfrac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$, *i.e.*, any set of elements having this form satisfies the general law $ab \,.\, bc \,.\supset.\, ac$.

If $\overline{ac} \,.\, ca$, then $ca \,.\, ab \,.\supset.\, cb$ (by 3), and $cb \,.\supset.\, \overline{bc}$, so that we have $\dfrac{ca \,.\, ab \,.\, cb}{\overline{ac} \,.\, \overline{ba} \,.\, \overline{bc}}$, which has the required form. Q.E.D.

Theorems I–VII establish the equivalence of the sets 1–4 and 1′–3′. The following theorems are concerned with the completeness as to general law of these sets.

THEOREM VIII.—*Any n K-elements (n finite) $a, b, c, d, \ldots, n$ are such that $\overline{aa} \,.\, \overline{bb} \,.\, \overline{cc} \,.\, \overline{dd} \,.\ldots.\, \overline{nn}$.*

This follows immediately by substitution in the form $\overline{AA}$ of Postulate 1′.

This any n dyads of identical elements have a unique form. The structure is completely determinate.

THEOREM IX.—*Any general law on the base (K, R), a value of which involves a finite number of elements, and which refers to dyads of identical elements, is either incompatible with Postulate 1′, or else it is implied by 1′.*

Proof.—Suppose that there are at least n elements in K. Any n dyads of identical elements must exhibit some combination of holdure and failure. Now Theorem VIII demands that all of the dyads fail. Any other postulate which should exclude the failure of all the dyads would be incompatible with Theorem VIII, while any other postulate which should allow for the failure of all of the dyads would be implied by Theorem VIII. For suppose that Postulate ϕ is such that it is satisfied by any set of elements having the form $\overline{AA} \,.\, \overline{BB} \,.\, \overline{CC} \,.\ldots.\, \overline{NN}$. ($\phi$ may, of course, be satisfied by sets of elements having other forms as well.) Theorem VIII implies that every set of n elements has the form $\overline{AA} \,.\, \overline{BB} \,.\, \overline{CC} \,.\ldots.\, \overline{NN}$, and any set of n elements having this form satisfies ϕ. Therefore ϕ is always satisfied. That is, ϕ is true.

If ϕ is not satisfied by a set of elements having the form $\overline{AA} \,.\, \overline{BB} \,.\, \overline{CC} \,.\ldots.\, \overline{NN}$, then the contradictory of ϕ is satisfied by such a set, and Theorem VIII implies the contradictory of ϕ.

If there are not at least n elements in K, then any such general law is satisfied. Q.E.D.

Postulate 1′ is of such a character as to determine uniquely the validity value of every dyad of identical elements. We may therefore confine the subsequent theorems to an investigation of postulates which have to do with dyads of distinct elements.

From Postulate 2′, any two distinct elements have the form $AB \,.\, \overline{BA}$. Now $AB \,.\, \overline{BA}$ involves all possible combinations of two elements in ordered dyads of distinct elements. Therefore, as shown above, any general law in two variables $\phi(x, y)$, where x and y denote elements of K, cannot be independent of Postulate 2′. Hence the theorem :

THEOREM X.—*Any general law $\phi(x, y)$ on dyads of distinct elements is such that either it is implied by Postulate 2′ or else it is incompatible with 2′.*

Examples of such general laws $\phi(x, y)$ are :

For any two distinct K-elements a, b : (1) $ab \,.\supset.\, \overline{ba}$, (2) $\overline{ab} \,.\supset.\, ba$, (3) ab or ba, (4) $\overline{ab}$ or $\overline{ba}$, (5) $ab \,.\, \overline{ba}$ or $\overline{ab} \,.\, ba$, (6) ab or $\overline{ab}$, (7) $ab \,.\supset.\, ba$, (8) $\overline{ab} \,.\supset.\, \overline{ba}$, (9) ab or $\overline{ba}$, (10) ab, and so on. General laws (1) to (6) are satisfied by any two elements having the form $AB \,.\, \overline{BA}$, while those from (7) to (10) are contrary to such a form.

THEOREM XI.—*Any general law $\phi(x, y, z)$ on dyads of distinct elements is such that either it is implied by Postulate 3′ or else it is incompatible with 3′.*

Proof.—Suppose that there are at least three elements in K. A guaranty of the truth of the theorem is given by the manner in which Postulate 3′ is constructed. In this postulate all possible combinations of three elements in ordered dyads of distinct elements are present, and to each ordered dyad a single validity value is assigned. If ϕ is satisfied by a set of elements having the form $\dfrac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$, then Postulate 3′ implies that ϕ is always satisfied. If the contradictory of ϕ is satisfied by a set having this form, then Postulate 3′ implies that the contradictory of ϕ is satisfied.

If there are not at least three elements in K, then ϕ is satisfied.

Q. E. D.

Consider all of the forms which three distinct elements formed into ordered dyads of distinct elements may have, and suppose that there are at least three elements in K. Among these forms are $\dfrac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$, and $\dfrac{AB \,.\, BC \,.\, \overline{AC}}{\overline{BA} \,.\, \overline{CB} \,.\, CA}$. Let $\phi(x, y, z)$ be a general law which is true of both of these forms and of no other form. Then Postulate 3′ implies ϕ. For when 3′ is satisfied, any three distinct K-elements have the form

$\frac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$, and the proposition : "Any three distinct K-elements have the form $\frac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$," implies the proposition, "Any three distinct K-elements have the form $\frac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$ or the form $\frac{AB \,.\, BC \,.\, \overline{AC}}{\overline{BA} \,.\, \overline{CB} \,.\, CA}$," and this latter proposition is equivalent to "ϕ is true".

If ϕ is false of both of the above forms and true of every other form, then the contradictory of ϕ is true of these forms, so that Postulate 3′ implies the contradictory of ϕ.

If ϕ is true of the form $\frac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$ and of no other form, then "ϕ is true" is equivalent to "Any three distinct K-elements have the form $\frac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$", so that ϕ is equivalent to Postulate 3′.

Theorems I–XI show that the sets 1′–3′ and 1–4 each have the following property : Any general law which has reference to dyads of distinct elements and which does not involve reference to more than three distinct elements is such that either it is implied by the set 1′–3′ (and therefore by the set 1–4), or its contradictory is implied by the set 1′–3′ (and therefore by the set 1–4). Also, any general law which has reference to dyads of identical elements, and which does not involve reference to more than a finite number of elements, is such that it is implied by either of these sets, or else its contradictory is implied by either of the sets.

THEOREM XII.—*If N be any finite subclass of K ($N \geqslant 2$), then there is one and only one element, x, in N, which has the relation R to every other element in N. (Such an element x will be referred to as the "first" element in N.)*

Proof.—Suppose N has just two elements X_1 and X_2. Then, by Postulates 2 and 4, X_1X_2 holds, or X_2X_1 holds, and not both. Thus the theorem is true when N has just two members.

Denote by N_1 any finite subclass of K-elements which, by hypothesis, has a first element, X_0, and denote any K-element not belonging to N_1 by X'. Form the class N_2 to have for members the members of N_1, and, in addition, the single element X'. It follows that N_2 has a first element. For, by Postulates 2 and 4, X_0X' holds or $X'X_0$ holds, and not both. If If X_0X', then X_0 is the first member of N_2. If $X'X_0$, then denote any third element of N_2 by Z. X_0Z holds, by hypothesis, and $X'X_0 \,.\, X_0Z \,.\supset.\, X'Z$, by Postulate 3, and ZX' is false, by 4. In this case X' is the first element in N_2.

The theorem holds for every subclass of two members. Therefore it holds for every subclass of 2+1 members, and so on. Q.E.D.

THEOREM XIII.—*Any four distinct K-elements have the form*
$AB \,.\, AC \,.\, AD \,.\, BC \,.\, BD \,.\, CD$
$\overline{BA} \,.\, \overline{CA} \,.\, \overline{DA} \,.\, \overline{CB} \,.\, \overline{DB} \,.\, \overline{DC}$.

Proof.—Denote by N any subclass of K having just four elements. Then, by Theorem XII, there exists a unique element, a, such that az holds and za fails if z is any other element in N. Form N_1 to have for members the elements of N, except the element a. Then, by Theorem XII, there exists a unique element, b, such that bz' holds and $z'b$ fails if z' is any other element in N_1. Similarly, form N_2 to have for members the elements of N_1, except the element b. There can be only two elements in N_2. Denote the first by c and the other by d. a, b, c, d are all elements of N. Therefore $\frac{ab \,.\, ac \,.\, ad}{\overline{ba} \,.\, \overline{ca} \,.\, \overline{da}}$. b, c, d are elements of N_1. Therefore $\frac{bc \,.\, bd}{\overline{cb} \,.\, \overline{db}}$. c, d are elements of N_2. Therefore $cd \,.\, \overline{dc}$. This gives $\frac{ab \,.\, ac \,.\, ad \,.\, bc \,.\, bd \,.\, cd}{\overline{ba} \,.\, \overline{ca} \,.\, \overline{da} \,.\, \overline{cb} \,.\, \overline{db} \,.\, \overline{dc}}$, which has the required form.

Q.E.D.

THEOREM XIV.—*Any n K-elements* ($n \geqslant 4$ *and finite*) *have the form*

$$\begin{array}{r}
AB \,.\, AC \,.\, AD \,.\, \ldots \,.\, AN \\
BC \,.\, BD \,.\, \ldots \,.\, BN \\
CD \,.\, \ldots \,.\, CN \\
\ldots \,.\, DN \\
\ldots\ldots \\
\overline{BA} \,.\, \overline{CA} \,.\, \overline{DA} \,.\, \ldots \,.\, \overline{NA} \\
\overline{CB} \,.\, \overline{DB} \,.\, \ldots \,.\, \overline{NB} \\
\overline{DC} \,.\, \ldots \,.\, \overline{NC} \\
\ldots \,.\, \overline{ND} \\
\ldots\ldots
\end{array}$$

Proof.—Definition: $O(abc \ldots lmn)$ is to mean that the elements denoted by any two symbols which occur in the order from left to right are such that the relation R holds for these elements if they are taken in the order of the symbols in O which denote them.

Thus $O(abc)$ means $Rab \,.\, Rac \,.\, Rbc$.

LEMMA.—*Any n K-elements ($n \geqslant 4$) are such that for some permutation $abc \ldots lmn$, $Oabc \ldots lmn$ holds.*

Proof of Lemma.—The lemma holds when n has exactly four elements, by Theorem XIII.

If the lemma holds for every M elements, then it holds for every $M+1$. For consider any $M+1$ elements. By Theorem XII, there is a unique element x such that xy holds for every other element y in $M+1$. Consider the class formed by removing this element. By hypothesis, for some permutation $abc \ldots lmn$ of the elements of this class, $Oabc \ldots lmn$ holds. But x has the relation R to each of these elements. Therefore, $Oxabc \ldots lmn$ holds.

But the lemma holds for every class of four elements.

The theorem follows by Postulate 4.

THEOREM XV.—*Any n distinct K-elements (n finite) have the form*

$$\begin{aligned}
&AB \,.\, AC \,.\, AD \ldots AN\\
&\qquad BC \,.\, BD \ldots BN\\
&\qquad\qquad CD \ldots CN\\
&\qquad\qquad\quad \ldots DN\\
&\qquad\qquad\quad \ldots\ldots\\
&\overline{BA} \,.\, \overline{CA} \,.\, \overline{DA} \ldots \overline{NA}\\
&\qquad \overline{CB} \,.\, \overline{DB} \ldots \overline{NB}\\
&\qquad\qquad \overline{DC} \ldots \overline{NC}\\
&\qquad\qquad\quad \ldots \overline{ND}\\
&\qquad\qquad\quad \ldots\ldots\\
&\overline{AA} \,.\, \overline{BB} \,.\, \overline{CC} \,.\, \overline{DD} \ldots \overline{NN}.
\end{aligned}$$

The theorem follows from Theorems VIII, XIV, and Postulates 2′ and 3′.*

We are now in a position to answer the question which was originally raised concerning the *completeness as to general law* of the set 1–4 so far

* This theorem amounts to the assertion of a whole set of laws, one for each finite value of n. It is a general law or set of laws where n varies throughout a definite range, but varies parametrically, that is, "outside the brackets".

as general laws involving reference to a finite number of elements are concerned.

Consider any general law ϕ, any value of which involves just n elements of K (n finite), and which refers to dyads of distinct elements or to dyads of identical elements or both. Suppose that ϕ is such that it is independent of the set 1–4, that is, that ϕ is such that neither ϕ nor the contradictory of ϕ is implied by the set 1–4. ϕ is therefore independent of Theorem XV.

If there are not at least n elements in K, then ϕ is satisfied.

Suppose that there are at least n elements in K. Now Theorem XV demands that any n elements shall exhibit the type of distribution of validity values given in that theorem. ϕ may or may not be true of this form (that is, ϕ may or may not be satisfied by a set of n elements which exhibit this particular type of distribution of validity values). In general, for some distributions of positive and negative validity values ϕ will be true, and for others it will be false. If ϕ should be true for all distributions of validity values it would contribute no postulational information. It cannot be false for all if it is to be consistent with the existence of at least n elements in K.

Form the class C of all those distributions of positive and negative validity values which satisfy ϕ. Then the contradictory of ϕ will be satisfied by every distribution not in C, *i.e.* by any distribution of the class not-C. Denote the particular distribution of validity values demanded by Theorem XV by D_0.

Suppose that D_0 belongs to the class C. Then any set of n elements having the form D_0 satisfies ϕ. But Theorem XV demands that every set of n elements have the form D_0. Therefore ϕ is always satisfied, and the set 1–4 implies ϕ.

Similarly, if D_0 belongs to the class not-C, Theorem XV implies the contradictory of ϕ.

It follows that the set 1–4 is complete as to general law as far as general laws whose values involve only a finite number of elements are concerned.

Note on the Set 1′–3′.

In any case in which Postulates 1′–3′ are used in connexion with other conditions which assure that K does not have exactly two elements Postulate 2′ becomes redundant, as is shown in the following theorem.

THEOREM XVI.—*K does not have exactly two elements and Postulate 3′ holds implies that 2′ holds.*

Proof.—If K does not have more than one element, $2'$ is clearly satisfied. Suppose that K has at least three elements. $3'$ demands that any three elements have the form $\frac{AB \,.\, BC \,.\, AC}{\overline{BA} \,.\, \overline{CB} \,.\, \overline{CA}}$. It follows that any two elements involved in such a form have the form $AB \,.\, \overline{BA}$. Q.E.D.

The Completeness of Sets of General Laws for "Betweenness".

Huntington and Kline* have offered eleven sets of independent postulates for "betweenness" which are all known to be equivalent. In the course of the discussion of these sets, the authors refer to the possibility that other independent general laws involving reference to more than four distinct elements might be added to any one of the sets.† It is the purpose of the following investigation to close this possibility as far as logical functions whose values involve reference to a finite number of elements are concerned, and to show that any one of these sets is complete as to general law within this limitation.

For the present purpose we shall employ the first‡ of the eleven sets, which may be formulated as follows. We demand

I. A class K.

II. A triadic relation R.

III. The following logical functions of K and R:

Postulate A.—For any three K-elements a, x, b, $axb \,.\supset.\, bxa$.

Postulate B.—For any three K-elements a, b, c,

$$a \neq b \,.\, b \neq c \,.\, a \neq c \,.\supset.\, bac \text{ or } cab \text{ or } abc \text{ or } cba \text{ or } acb \text{ or } bca.$$

Postulate C.—For any three K-elements a, x, y,

$$a \neq x \,.\, x \neq y \,.\, a \neq y : \supset : axy \,.\, ayx \,.= .\, 0.\S$$

Postulate D.—For any three K-elements a, b, c,

$$abc \,.\supset.\, a \neq b \,.\, b \neq c \,.\, a \neq c.$$

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* E. V. Huntington and J. R. Kline, "Sets of Independent Postulates for Betweenness", *Trans. Amer. Math. Soc.*, 18 (1917), 301–325.

† *Ibid.*, 323.

‡ *Ibid.*, 322.

§ That is, axy and ayx cannot both be true.

Postulate 1.—For any four distinct K-elements a, b, x, y,

$$xab \,.\, aby \,.\supset.\, xay.$$

Postulate 2.—For any four distinct K-elements a, b, x, y,

$$xab \,.\, ayb \,.\supset.\, xay.$$

These six postulates are all general laws. As different sets of existence postulates are added to the list different types of system will result, all of which will have the properties assigned by Postulates A–2. It is proposed to show that no further independent general laws, whose values involve only a finite number of elements, can be true of any such system.

THEOREM I.—*Any single K-element has the form* $\overline{AAA}$.

Proof.—By Postulate D, any K element a is such that $\overline{aaa}$. But $\overline{aaa}$ is of the form $\overline{AAA}$. Q.E.D.

THEOREM II.—*Any two distinct K-elements have the form*

$$\overline{AAB} \,.\, \overline{ABA} \,.\, \overline{BAA} \,.\, \overline{BBA} \,.\, \overline{BAB} \,.\, \overline{ABB}.$$

Proof.—By Postulate D, if a and b are any two K-elements, then $\overline{aab} \,.\, \overline{aba} \,.\, \overline{baa} \,.\, \overline{bba} \,.\, \overline{bab} \,.\, \overline{abb}$. This involves all possible combinations of two distinct elements into triads of two distinct elements, and any pair of distinct elements has this form.*

THEOREM III.—*Any three distinct K-elements a, b, c are such that abx or axb or xab.*

The theorem follows from Postulates A and B.

THEOREM IV.—*For any four distinct K-elements a, b, x, y,*

$$xab \,.\, yab \,.\supset.\, xya \text{ or } yxa.$$

Proof.—By Theorem III, xyb or xby or bxy.

If xyb, then $xyb \,.\, yab \,.\supset.\, xya$ (by Postulate 2).

If xby, then $yab \,.\supset.\, bay$ (by A), and $xby \,.\, bay \,.\supset.\, xba$ (by 2). But $xab \,.\, xba \,.=.\, 0$ (by C).

* Permutation of a pair of elements having this form gives rise to identical statements, *i.e.* to redundant statements. But if we had $aab.\overline{aba}.\overline{baa}.\overline{bba}.\overline{bab}.\overline{abb}$, then a permutation of a and b would give rise to a different but *incompatible* statement. In either case the assignment of a form to a and b says all that can be said in terms of general law.

If bxy, then $bxy \,.\supset.\, yxb$ (by A), and $yxb \,.\, xab \,.\supset.\, yxa$ (by 2).
Q.E.D.

THEOREM V.—*For any finite non-empty class N in K there is at least one element s in N such that $\overline{xsy}$ for every pair of elements x, y in N.*

Proof.—The theorem holds when N does not have more than three elements (by A, B, C, D).

Suppose the theorem true for a class of M ($M \geqslant 3$) elements, and let s denote a term which is such that $\overline{xsy}$ for every x, y in M. Denote by z any element not in M. Form the class M_1 of the members of M and the single element z.

If there is no term a in M such that zsa holds or asz holds, then s satisfies the demands of the theorem.

Suppose there is at least one term a in M such that zsa or asz. zsa implies that s and a are distinct (by D), $zsa \,.\equiv.\, asz$ (by A). Now sya or say, where y is any element in M other than s and a (by Theorem III and hypothesis).

If sya, $zsa \,.\, sya \,.\supset.\, zsy$ (by Postulate 2).

If say, $zsa \,.\, say \,.\supset.\, zsy$ (by Postulate 1).

In either case zsy holds, where y is any element in M other than s and a, and we have zsa. So that zsy holds where y is any element in M other than s. Then $\overline{szy}$ and $\overline{yzs}$ (by A and C).

Suppose a' and b' in M such that $a'zb'$ holds. Neither a' nor b' can be identical with s. Since zsy, zab' holds, and $a'zb' \,.\, zsb' \,.\supset.\, a'zs$ (by 2). But since zsy, zsa', and $zsa' \,.\supset.\, a'sz$ (by A), contrary to $a'zs$ (by C). Therefore the element z satisfies the demands of the theorem.

It follows that M_1 satisfies the theorem if M does. But the theorem holds for any three elements. Q.E.D.

Definition.—We define a relation O as follows: $Oabc \ldots n$ (n finite) is to mean that the terms denoted by any three symbols which occur in the order from left and right are such that the relation R holds for these terms if they are taken in the order of occurrence of the symbols in O which denote them.

Thus $Oabc$ means $Rabc$ holds, and $Oabcd$ means $abc \,.\, abd \,.\, acd \,.\, bcd$ holds, and so on. If $Oabc \ldots n$ holds, then $a, b, c, \ldots, n$ must, of course, all be distinct (by D).

THEOREM VI.—*If $Oabc \ldots lmn$ holds, and if z is any element not among the elements $a, b, c, \ldots, l, m, n$, and such that for every a' and b' among these elements $a'\overline{zb'}$, then either $Ozabc \ldots lmn$ holds or $Oabc \ldots lmnz$ holds.*

Proof.—Either zab or abz, where a and b are the first and second elements of $Oabc \ldots lmn$ (by Theorem III).

Case 1.—Suppose zab. Denote the class of elements $a, b, c, \ldots, l, m, n$ by N. Now aby for every y other than a and b in N. $zab \,.\, aby \,.\supset.\, zay$ (by Postulate 1). And since zab, we have zay, where y is any element in N other than a.

$ayx \,.\supset.\, zyx$, where y and x are any two elements of N other than a. For zyx or yxz (by Theorem III and hypothesis), and if yxz, $ayx \,.\, yxz \,.\supset.\, ayz$ (by Postulate 1), and $ayz \,.\supset.\, zya$, contrary to zay (by C).

Let w, y, x be any three elements in N such that $Oabc \ldots lmn$ asserts wyx. Then since a is the first element in O, if w is other than a we have $awy \,.\, awx \,.\, ayx$. This implies $zwy \,.\, zwx \,.\, zyx$. If w is identical with a, we have $zay \,.\, zax \,.\, zyx$.

It follows that $Ozabc \ldots lmn$. For $Oabc \ldots lmn$, and $zwy \,.\, zwx \,.\, zyx$ is true for any wyx *in* O.

Case 2.—Suppose abz. Then mnz *or* zmn (by Theorem III and hypothesis). We have amn, and $amn \,.\, zmn \,.\supset.\, azm$ or zam (by Theorem IV). $\overline{azm}$ by hypothesis. Therefore zam. $zam \,.\, amn \,.\supset.\, zan$ (by Postulate 1). $zan \,.\supset.\, naz$. $naz \,.\, abz \,.\supset.\, nab$ (by Postulate 2), contrary to abn (by C). Therefore mnz.

$mnz \,.\supset.\, znm$ and $Oabc \ldots lmn \,.\supset.\, Onml \ldots cba$ (by A). Therefore (by Case 1) $Oznml \ldots cba$. But $Oznml \ldots cba \,.\supset.\, Oabc \ldots lmnz$ (by A).

Q.E.D.

THEOREM VII.—*Any* n *elements* ($n \geqslant 3$ *and finite*) *are such that for some permutation* $abcd \ldots n$, $Oabcd \ldots n$ *holds.*

Proof.—The theorem holds for any class of three elements (by Postulate B).

If the theorem is true for *any* N elements, where N is a definite number, then it is true for any $N+1$ elements. For consider any $N+1$ elements. By Theorem V there is an element s such that no two terms a, b are such that asb. Consider the class formed by removing s from $N+1$. This class has N elements and the theorem holds for it, by hypothesis. It follows from Theorem VI that the theorem holds for $N+1$.

But the theorem holds for *any* three elements. Q.E.D.

THEOREM VIIIA.—*Any four distinct K-elements on triads of distinct*

elements have the form

$$ABC \,.\, ABD \,.\, ACD \,.\, BCD$$

$$CBA \,.\, DBA \,.\, DCA \,.\, DCB$$

$$\overline{CAB} \,.\, \overline{DAB} \,.\, \overline{DAC} \,.\, \overline{DBC}$$

$$\overline{BCA} \,.\, \overline{BDA} \,.\, \overline{CDA} \,.\, \overline{CDB}$$

$$\overline{ACB} \,.\, \overline{ADB} \,.\, \overline{ADC} \,.\, \overline{BDC}$$

$$\overline{BAC} \,.\, \overline{BAD} \,.\, \overline{CAD} \,.\, \overline{CBD}.$$

Proof.—By Theorem VII, any four distinct K-elements are such that for some permutation *abcd*, *Oabcd* holds. $Oabcd \,.\supset.\, Odcba$ (by A). So that we have $\begin{matrix} abc \,.\, abd \,.\, acd \,.\, bcd \\ cba \,.\, dba \,.\, dca \,.\, dcb \end{matrix}$

By Postulate C, if *xyz* and *zyx*, then $\overline{xzy}$ and $\overline{zxy}$. If *yzx*, then *xzy*, and if *yxz*, *then zxy* (by A). Therefore $\overline{yzx}$ and $\overline{yxz}$. So that if *xyz* and *zyx*, then all other permutations fail. This gives the required form.

Q.E.D.

THEOREM VIIIB.—*Any three distinct K-elements on triads of distinct elements have the form*

$$ABC \,.\, CBA \,.\, \overline{CAB} \,.\, \overline{BCA} \,.\, \overline{ACB} \,.\, \overline{BAC}.$$

Proof.—By Postulate B, some permutation, *abc*, holds. $abc \,.\supset.\, cba$ (by A). $abc \,.\supset.\, \overline{acb}$ and $cba \,.\supset.\, \overline{cab}$ (by C). If *bca*, then *acb*, and if *bac*, then *cab* (by A). Therefore $\overline{bca}$ and $\overline{bac}$. Q.E.D.

THEOREM IX.—*Any n K-elements (n finite) have the following form* :—

(1) $O(ABC \ldots LMN) \,.\, O(NML \ldots CBA)$.

(2) *Every ordered triad of distinct variables which can be formed of the variables A, B, C, ..., L, M, N and which does not appear in* (1) *fails.*

(3) *Every ordered triad which involves the same variable more than once fails.**

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* This method of formulating statements about a set of elements so as to secure a determinate description is due to H. M. Sheffer. It is here used in connection with variables which denote elements.

Proof.—(*a*) If n does not have at least three elements, then (1) and (2) are satisfied. By Theorem VII, any n distinct K-elements ($n \geqslant 3$) are such that for some permutation $abc \ldots lmn$, $Oabc \ldots lmn$ holds. $Oabc \ldots lmn \,.\supset.\, Onml \ldots cba$ (by A). This satisfies condition (1).

(*b*) Consider all of the triads of distinct elements which can be formed of the elements $a, b, c, \ldots, l, m, n$. For each triad two permutations xyz and zyx hold, from (*a*). $xyz \,.\supset.\, \overline{xzy}$ and $zyx \,.\supset.\, \overline{zxy}$ (by C). If yzx, $yzx \,.\supset.\, xzy$, and if yxz, $yxz \,.\supset.\, zxy$ (by A). Therefore $\overline{yzx}$ and $\overline{yxz}$. That is, all ordered triads other than xyz and zyx *fail.*

It follows that all ordered triads of distinct elements other than those which hold in (*a*) fail. This satisfies condition (2).

(*c*) By Postulate D, any triad not involving more than two distinct elements fails. This satisfies condition (3). Q.E.D.

The form demanded by Theorem IX involves all possible combinations of n variables into ordered triads of three distinct variables, of two distinct variables, and of a single variable. Let $a, b, c, \ldots, n$ be a set of elements which exhibit the type of distribution of validity values demanded in the theorem. Denote by S_1 the set of all possible ordered triads of these elements with their respective validity values. Consider any other set of elements $a', b', c', \ldots, n'$, and let S_2 denote the set of all possible ordered triads of these elements with their several validity values. If there exists a one-to-one correspondence between the elements of S_1 and the elements of S_2 such that when each element of S_1 is replaced by the element to which it corresponds in S_2 the statement S_1 is transformed into the statement S_2, then S_1 and S_2 have the same form. If such a correspondence does not exist, then S_1 and S_2 have different forms.

Let ϕ be any general law a value of which involves just n elements (n finite), and which has reference to ordered triads involving three distinct elements, or to ordered triads involving two distinct elements, or to ordered triads involving a single element, or to any two or to all three of these. If the particular statement S_1 satisfies ϕ, that is if the elements $a, b, c, \ldots, n$ in ordered triads with their respective validity values satisfy ϕ, then any other set of elements having the same form as S_1 satisfies ϕ.

Suppose that there are at least n elements in the class K, and denote the form demanded in Theorem IX by F_0. If ϕ is satisfied by a certain set of elements having this form, then ϕ is satisfied by every such set. But Theorem IX demands that every set of elements have the form F_0. Therefore ϕ is always satisfied.

If ϕ is not satisfied by a set of elements of the form F_0, then the contradictory of ϕ is satisfied by any such set. Hence the contradictory of ϕ is satisfied, and Theorem IX implies the contradictory of ϕ.

If there are not at least n elements in K, then ϕ is true.

It follows that Postulates A–2 form a complete set with respect to general laws which have for values finite sets of the elements of K.

An Equivalent Set of Postulates for Betweenness.

We may formulate an equivalent set of postulates for Betweenness by the use of Theorems I, II, VIIIA, and VIIIB:

1′. Any single K-element has the form

$$\overline{AAA}.$$

2′. Any two distinct K-elements on ordered triads of two distinct elements have the form

$$\overline{AAB} \,.\, \overline{ABA} \,.\, \overline{BAA} \,.\, \overline{BBA} \,.\, \overline{BAB} \,.\, \overline{ABB}.$$

3′. Any three distinct K-elements on triads of distinct elements have the form

$$ABC \,.\, CBA \,.\, \overline{CAB} \,.\, \overline{BCA} \,.\, \overline{ACB} \,.\, \overline{BAC}.$$

4′. Any four distinct K-elements on triads of distinct elements have the form

$$ABC \,.\, ABD \,.\, ACD \,.\, BCD$$

$$CBA \,.\, DBA \,.\, DCA \,.\, DCB$$

$$\overline{CAB} \,.\, \overline{DAB} \,.\, \overline{DAC} \,.\, \overline{DBC}$$

$$\overline{BCA} \,.\, \overline{BDA} \,.\, \overline{CDA} \,.\, \overline{CDB}$$

$$\overline{ACB} \,.\, \overline{ADB} \,.\, \overline{ADC} \,.\, \overline{BDC}$$

$$\overline{BAC} \,.\, \overline{BAD} \,.\, \overline{CAD} \,.\, \overline{CBD}.$$

It has already been shown that the set 1′–4′ follows from the set A–2. The following theorems establish the converse relation.

THEOREM X (Postulate A).—*For any three K-elements a, x, b, $axb \,.\supset.\, bxa$.*

Proof.—By Postulates 1′ and 2′, if axb, then a, x, b are all distinct. But in this case $axb \,.\supset.\, bxa$ (by Postulate 3′).

THEOREM XI (Postulate B).—*For any three K-elements* a, b, c, $a \neq b \,.\, b \neq c \,.\, a \neq c \,.\supset.$ *bac or cab or abc or cba or acb or bca.*

This follows immediately from Postulate 3′.

THEOREM XII (Postulate C).—*For any three K-elements* a, x, y, $a \neq x \,.\, x \neq y \,.\, a \neq y \,.\supset.\, axy \,.\, ayx \,.=.\, 0.$

Proof.—Suppose axy and ayx. Then yxa and xya (by Theorem X). But such a triad of elements has a form contrary to that demanded by Postulate 3′.

THEOREM XIII (Postulate D).—*For any three K-elements* a, b, c, $abc \,.\supset.\, a \neq b \,.\, b \neq c \,.\, a \neq c.$

Proof.—By 1′ and 2′ every ordered triad of non-distinct elements fails.

THEOREM XIV (Postulate 1).—*For any four distinct K-elements* a, b, x, y, $xab \,.\, aby \,.\supset.\, xay.$

Proof.—Suppose $xab \,.\, aby \,.\, \overline{xay}$. Now any four distinct elements have the form demanded by Postulate 4′. An examination of the form in 4′ shows that $xab \,.\, aby \,.\, \overline{xay}$ is always false.

THEOREM XV (Postulate 2).—*For any four distinct K-elements* a, b, x, y, $xab \,.\, ayb \,.\supset.\, xay.$

An examination of the form which any four distinct elements must have by 4′ shows that $xab \,.\, ayb \,.\, \overline{xay}$ is always false.

Sets A–2 and 1′–4′ are therefore equivalent sets.

Note on Postulate 3′.

If the set 1′–4′ is given in connexion with other conditions which assure that K does not have exactly three elements, then 3′ becomes redundant, as shown by the following theorem.

THEOREM XVI.—*K does not have exactly three elements, and Postulate 4′ holds implies that 3′ holds.*

Proof.—If K does not have more than two elements, then 3′ is satisfied. If K has at least four elements, then it follows from 4′ that any three distinct elements involved in the form demanded for any four distinct elements by 4′ must have the form

$$ABC \,.\, CBA \,.\, \overline{CAB} \,.\, \overline{BCA} \,.\, \overline{ACB} \,.\, \overline{BAC}.$$

The Completeness of Sets of Postulates for Cyclic Order.

Exactly the same method can be applied to postulates for Cyclic Order as has been used in connexion with the Serial Order and Betweenness sets. Three sets of completely independent postulates for cyclic order have been given by Huntingdon.* For the present purpose the first† of these sets will be used. We demand

I. A class K.

II. A triadic relation $R(abc)$.

III. The following logical functions of K and R :—

E.‡ If a, b, c are distinct, then $abc \,.\supset.\, bca$.

B. If a, b, c are distinct, then at least one of the orders abc, bca, cab, cba, acb, bac is true.

C. If a, b, c are distinct, then abc and acb cannot both be true.

D. If abc is true, then a, b, and c are distinct.

2. If a, b, x, y are distinct, and xab and ayb, then xay.

It is proposed to show that any finite general law on the base (K, R) is either incompatible with the set E–2 or else it is implied by this set. To do this it is sufficient to show that any n elements in K have a determinate form.

THEOREM I.—*Any single K-element has the form* $\overline{AAA}$.

The theorem follows from Postulate D.

THEOREM II.—*Any two distinct K-elements have the form*

$$\overline{AAB} \,.\, \overline{ABA} \,.\, \overline{BAA} \,.\, \overline{BBA} \,.\, \overline{BAB} \,.\, \overline{ABB}.$$

The theorem follows from Postulate D.

THEOREM III.—*Any three distinct K-elements have the form*

$$ABC \,.\, BCA \,.\, CAB \,.\, \overline{CBA} \,.\, \overline{BAC} \,.\, \overline{ACB}.$$

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* E. V. Huntington, "Sets of Completely Independent Postulates for Cyclic Order", *Proc. Nat. Acad. Sci.*, 10 (1924), 74–78.

† *Ibid.*, 74–75.

‡ This set is numbered in such a way as to exhibit its relations to sets for betweenness. See page 136.

Proof.—By B, R holds for some permutation $abc \,.\, abc \,.\supset.\, bca \,.\supset.\, cab$ (by E). Hence $\overline{acb} \,.\, \overline{bac} \,.\, \overline{cba}$ (by C). Q.E.D.

THEOREM IVA.—If a and b are distinct, abb fails for every a, b.

The theorem follows from Postulate D.

THEOREM IVB.—*If a, b, c are distinct,* $\overline{abc} \,.\supset.\, acb$.

Proof.—$bca \,.\supset.\, cab \,.\supset.\, abc$ (by E). Hence $\overline{abc} \,.\supset.\, \overline{cab} \,.\supset.\, \overline{bca}$. Therefore acb (by Theorem III). Q.E.D.

THEOREM IVC.—*If a, b, c, d are distinct,* $abc \,.\, acd \,.\supset.\, abd$.

Proof.—$acd \,.\supset.\, cda \,.\supset.\, dac$ (by E). $dac \,.\, abc \,.\supset.\, dab$ (by 2). $dab \,.\supset.\, abd$ (by E). Q.E.D.

THEOREM IVD.—*If a, b, c are distinct,* $abc \,.\supset.\, \overline{acb}$.

The theorem follows from Postulate C.

From Theorems IVA–IVD it will be seen that if any element a be fixed parametrically in the first place of the relation R, then the variables in the second and third places have all of the properties demanded in the postulates for serial order (page 115), provided the element a is never substituted in the second or third place of the relation.

We define a dyadic relation "Ra" as follows: $Ra(bc) \,.=.\, Rabc$. Df. It follows from Theorems IVA–IVD that any set of elements over which the relation Ra holds and which does not involve the element a is such that all of the laws for serial order hold for the relation Ra with respect to this set. In particular, Theorem XV (page 128) holds for any such set.

Definition.—$a(bcd \ldots n)$ is to mean that the elements denoted by any two symbols taken in the order from left to right are such that the relation Ra holds for these elements if they are taken in the order of the symbols which denote them.

Thus $a(bcd)$ means $Ra(bc) \,.\, Ra(bd) \,.\, Ra(cd)$.

THEOREM V.—*If N be any finite subclass of K ($N \geqslant 3$) and if a be any element in N, then for some permutation $bcd \ldots n$ of the elements in N other than a, $a(bcd \ldots n)$ holds.*

Since Ra, with respect to the elements in N other than a, has all of the properties of serial order, the theorem follows from Theorem XV (page 128).

Definition.—$O(abcd \dots n)$ is to mean that the terms denoted by any three symbols which occur in the order from left to right are such that the relation R holds for these terms if they are taken in the order of occurrence of the symbols in O which denote them.

THEOREM VI.—*$a(bcd \dots n)$ implies $O(abcd \dots n)$.*

Proof.—$a(bcd \dots n)$ implies that $a, b, c, d, \dots, n$ are all distinct (by D). Since $a(bcd \dots n)$ and since $Ra(xy) . \equiv . Raxy$, the theorem holds so far as triads in O which contain the element a are concerned. If x_1, x_2, x_3 in $a(bcd \dots n)$ occur in the order from left to right, then ax_1x_2 and ax_2x_3. $ax_1x_2 . \supset . x_1x_2a$ and $ax_2x_3 . \supset . x_2x_3a$ (by E). $x_1x_2a . x_2x_3a . \supset . x_1x_2x_3$ (by 2). Q.E.D.

THEOREM VII.—*If N be any finite subclass of K ($N \geqslant 3$), then for some permutation $abcd \dots n$ of the elements of N, $O(abcd \dots n)$ holds.*

The theorem follows from Theorems V and VI.

Definition.—$R'(abc) . = . R(abc) . R(bca) . R(cab)$. Df.

Definition.—$C(abcd \dots n)$ is to mean that the terms denoted by any three symbols which occur in the order from left to right are such that the relation R' holds for these terms if they are taken in the order of occurrence of the symbols in C which denote them.

THEOREM VIII.—*If N be any finite subclass of K ($N \geqslant 3$), then for some permutation $abcd \dots n$ of the elements in N, $C(abcd \dots n)$ holds.*

Proof.—For some permutation $abcd \dots n$, $O(abcd \dots n)$ holds, by Theorem VII. $O(abcd \dots n)$ implies $C(abcd \dots n)$, by E. Q.E.D.

THEOREM IX.—*Any n K-elements (n finite) have the following form:*

(1) *$C(ABCD \dots N)$ holds.*

(2) *Any ordered triad of distinct variables, formed of the variables in* (1), *which is not asserted in* (1), *fails.*

(3) *Any ordered triad which involves the same variable more than once fails.*

Proof.—(1) follows from Theorem VIII. (2) follows from (1) by Theorem III. (3) follows from Theorems I and II. Q.E.D.

Theorem IX involves all possible combinations of n variables into ordered triads of distinct and identical variables. Therefore a determinate form is assigned to any n K-elements. It follows that no further independent finite general laws can be added to the set E–2.

An equivalent set of postulates for cyclic order may be formulated as follows :

I. Any single K-element has the form $\overline{AAA}$.

II. Any two distinct K-elements on ordered triads of two distinct elements have the form $\overline{AAB} \,.\, \overline{ABA} \,.\, \overline{BAA} \,.\, \overline{BBA} \,.\, \overline{BAB} \,.\, \overline{ABB}$.

III. Any three distinct K-elements on ordered triads of three distinct elements have the form $ABC \,.\, BCA \,.\, CAB \,.\, \overline{CBA} \,.\, \overline{BAC} \,.\, \overline{ACB}$.

IV. Any four distinct K-elements on ordered triads of three distinct elements have the form

$$ABC \,.\, ABD \,.\, ACD \,.\, BCD$$

$$BCA \,.\, BDA \,.\, CDA \,.\, CDB$$

$$CAB \,.\, DAB \,.\, DAC \,.\, DBC$$

$$\overline{CBA} \,.\, \overline{DBA} \,.\, \overline{DCA} \,.\, \overline{DCB}$$

$$\overline{BAC} \,.\, \overline{BAD} \,.\, \overline{CAD} \,.\, \overline{CBD}$$

$$\overline{ACB} \,.\, \overline{ADB} \,.\, \overline{ADC} \,.\, \overline{BDC}.$$

The set I–IV follows from E–2. For Postulates I, II, and III are identical with Theorems I, II, and III respectively, while Postulate IV follows immediately from Theorem IX, since $C(ABCD)$ means

$$ABC \,.\, ABD \,.\, ACD \,.\, BCD$$

$$BCA \,.\, BDA \,.\, CDA \,.\, CDB$$

$$CAB \,.\, DAB \,.\, DAC \,.\, DBC.$$

The following theorems show that the set E–2 follows from I–IV.

THEOREM X(E).—*If a, b, c are distinct, then* $abc \,.\supset.\, bca$.

Since any three distinct elements have the form demanded by Postulate III, $abc \,.\, \overline{bca}$ cannot occur.

THEOREM XI(B).—*If a, b, c are distinct, then at least one of the orders abc, bca, cab, cba, acb, bac is true.* (By Postulate III.)

THEOREM XII(C).—*If a, b, c are distinct, then abc and acb cannot both be true.*

$abc \,.\, acb$ would be contrary to the form demanded by Postulate III.

THEOREM XIII(D).—*If abc is true, then a, b, and c are distinct.*

This follows from Postulates I and II.

THEOREM XIV(2).—*If a, b, x, y are distinct, and xab and ayb, then xay.*

An examination of the form demanded by Postulate IV shows that $xab \,.\, ayb \,.\, \overline{xay}$ cannot occur.

THEOREM XV.—*K does not have exactly three elements, and Postulate IV holds implies that Postulate III holds.*

Proof.—If K does not have more than two elements, then III holds. If K has at least four elements, then Postulate IV implies that any three elements of K have the form $ABC \,.\, BCA \,.\, CAB \,.\, \overline{CBA} \,.\, \overline{BAC} \,.\, \overline{ACB}$.

ON THE REPRESENTATION OF NUMBERS IN THE FORM $ax^2+by^2+cz^2+dt^2$

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1. *Introduction.*

1.1. The object of this paper is to treat by means of analytical methods some special cases of the problem of the representation of positive integral numbers in the form $ax^2+by^2+cz^2+dt^2$ (x, y, z, t integers >0, $=0$, or <0), where a, b, c, and d are given positive integers. Let $r(n)$ be the number of representations of n in this form.* Then we may ask in the first place for an asymptotic formula for $r(n)$ for large values of n. Problems of this kind are generally attacked by means of a well known method of Hardy and Littlewood. However, this method does not enable us to prove an asymptotic formula for $r(n)$, since the ultimate result it leads to is

$$r(n) = \frac{\pi^2}{\sqrt{(abcd)}}\, nS(n)+O(n^\kappa), \tag{1.11}$$

where $S(n)$ is the singular series and κ may be any number >1, which is not sufficient for the conclusion that

$$r(n) \sim \frac{\pi^2}{\sqrt{(abcd)}}\, nS(n). \tag{1.12}$$

For the proof of (1.12) it would be necessary to prove (1.11) for some $\kappa < 1$.† I have not been able to do this, even in the simplest case $a=b=c=d=1$, though in this special case it is known, from other

* Two representations $n = ax_1^2+by_1^2+cz_1^2+dt_1^2$ and $n = ax_2^2+by_2^2+cz_2^2+dt_2^2$ will be considered as the same if and only if $x_1=x_2$, $y_1=y_2$, $z_1=z_2$, $t_1=t_2$.

† Of course the proof of (1.11) for some $\kappa<1$ would not be sufficient to prove (1.12). It would be also necessary to prove $S(n)>\delta>0$.

considerations, that the error term in (1.11) must be 0. Therefore we arrive at the very remarkable fact that it must be very difficult to prove an asymptotic formula for the number of representations of n as a sum of four squares by means of the method of Hardy and Littlewood. Yet it seems to be highly probable that (1.11) is also true for some $\kappa < 1$.

1.2. It has been shown by Hardy* that both sides of the relation (1.12) are equal in the special case $a = b = c = d = 1$ (sum of four squares). His proof is based on a theorem from the theory of modular functions, following a method which is in principle due to Mordell.† In this way he finds a new proof for the well known formulae of Jacobi for $r(n)$. The importance of this method is that it can also be applied to study the representation of numbers as a sum of 5, 6, 7, and 8 squares. In the present paper some special cases of the quadratic form

$$ax^2+by^2+cz^2+dt^2$$

will be treated by the same method.

For the sake of simplicity we confine ourselves to the form

$$x^2+y^2+cz^2+ct^2.$$

We shall prove what we shall call "exact formulae" for the number of representations (as opposed to "asymptotic formulae") for $c = 3, 5, 6, 7$, but the method used is also applicable for $c = 1, 2, 4, 8, 16$ as well as in a great many other special cases of the form $ax^2+by^2+cz^2+dt^2$.

1.3. A great many special results have been obtained by Liouville.‡ As far as the form $x^2+y^2+cz^2+ct^2$ is concerned, he proved exact formulae for $c = 2, 3, 4, 8, 16$ and partly for $c = 5$ (for even values of n only). The exact formulae for $c = 5, 6, 7$ which are proved in this paper are new.

1.4. In § 2 the singular series will be obtained by merely formal calculations. In the same section it will be proved that the singular series is absolutely convergent and its sum will be calculated.

In § 3 the exact formulae will be proved.

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* "On the representation of a number as the sum of any number of squares, and in particular of five", *Trans. Amer. Math. Soc.*, 21 (1920), 255–284.

† "On the representation of numbers as a sum of $2r$ squares", *Quart. J. of Math.*, 48 (1917), 93–104. "On the representation of a number as a sum of an odd number of squares", *Trans. Camb. Phil. Soc.*, 22 (1919), 361–372.

‡ For the history of the subject I refer to L. E. Dickson, *History of the Theory of Numbers*, 3, Ch. 10.

2. *The singular series.*

2.1. *Preliminary lemmas.*

2.11.* The *Gaussian sums* are defined by the equation

$$S_{h,k} = \sum_{j=1}^{k} \exp \frac{2\pi i h j^2}{k}, \tag{2.111}$$

where h and k are two positive integers. In calculating these sums we may suppose $0 < h < k$† and $(h, k) = 1$.‡ Then, if also $(h, k') = 1$, $(k, k') = 1$, we have

$$S_{h,kk'} = S_{hk',k} S_{hk,k'}. \tag{2.112}$$

Further, let $k = 2^\alpha m$ (m odd). Then

$$\begin{cases} S^2_{h,k} = (-1)^{(k-1)/2} k & \text{if } \alpha = 0; \\ S^2_{h,k} = 0 & \text{if } \alpha = 1; \\ S^2_{h,k} = 2(-1)^{(m-1)/2} k i^{hm} & \text{if } \alpha > 1. \end{cases} \tag{2.113}$$

If $c_1 = c/(k, c)$, $k_1 = k/(k, c)$, then

$$S_{ch,k} = (k, c)\, S_{c_1 h, k_1}. \tag{2.114}$$

From (2.113) and (2.114) we easily derive the following results ($k = 2^\alpha m$, $k_1 = 2^{\alpha_1} m_1$, m and m_1 odd):

$$\begin{cases} S^2_{h,k} S^2_{ch,k} = (-1)^{(k-1)/2+(k_1-1)/2}(k, c)\, k^2, \text{ if } k \text{ is odd}; \\ \qquad = 0, \text{ if at least one of the numbers } k \text{ and } k_1 \text{ is congruent to 2 (mod 4)}; \\ \qquad = 2(-1)^{(m-1)/2+(k_1-1)/2}(k, c)\, k^2 \exp(\tfrac{1}{2} hm\pi i), \text{ if } k \text{ is congruent to 0 (mod 4) and } k_1 \text{ is odd}; \\ \qquad = 4(-1)^{(m-1)/2+(m_1-1)/2}(k, c)\, k^2 \exp(\tfrac{1}{2} hm\pi i + \tfrac{1}{2} c_1 h m_1 \pi i), \\ \qquad \text{if both } k \text{ and } k_1 \text{ are congruent to 0 (mod 4).} \end{cases} \tag{2.115}$$

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* For proofs of the assertions of this section see Bachmann, *Zahlentheorie*, 2 (1894), 145–187.

† The case $h = 0$, $k = 1$ is trivial.

‡ As usual, we write (h, k) for the greatest common divisor of h and k.

2.12. Ramanujan's sums* are defined by the equation

$$c_k(n) = \sum_h \exp \frac{2\pi i h n}{k}, \tag{2.121}$$

where h runs through the $\phi(k)$ positive integers, less than and prime to k.

It follows immediately that

$$c_k(-n) = c_k(n). \tag{2.122}$$

If $(k, k') = 1$, then

$$c_{kk'}(n) = c_k(n)\, c_{k'}(n). \tag{2.123}$$

Further

$$c_k(n) = \Sigma \delta \mu \left(\frac{k}{\delta}\right), \tag{2.124}$$

where δ runs through all common divisors of k and n, and where μ is the well known arithmetical function of Möbius.

If $k = p^\lambda$, where p is prime, and if p^ξ is the highest power of p, that divides n, then it follows easily from (2.124), that

$$\begin{cases} c_k(n) = 0 & \text{if} \quad \xi < \lambda - 1; \\ c_k(n) = -p^{\lambda-1} & \text{if} \quad \xi = \lambda - 1; \\ c_k(n) = p^\lambda - p^{\lambda-1} = \phi(p^\lambda) & \text{if} \quad \xi > \lambda - 1. \end{cases} \tag{2.125}$$

2.2. *Formal construction of the singular series.*

The formal construction of the singular series for $x^2+y^2+cz^2+ct^2$ is nearly identical with that for $x^2+y^2+z^2+t^2$, as it is carried out by Hardy in his paper.†

I write (with $q = \exp \pi i \tau$)

$$f(\tau) = 1 + \sum_{n=1}^{\infty} r(n) q^n = (1+2q+2q^4+\ldots)^2 (1+2q^c+2q^{4c}+\ldots)^2$$

$$= \vartheta_3^2(0\,|\,\tau)\, \vartheta_3^2(0\,|\,c\tau).$$

In order to determine the behaviour of $f(q)$, when q tends radially to the point $\exp 2\pi i h/k$ of the unit circle, I write

$$q = Q \exp 2\pi i h/k,$$

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* "On certain trigonometrical sums and their applications in the theory of numbers", *Trans. Camb. Phil. Soc.*, 22 (1918), 259–276.

† See footnote * on 144.

so that $0 \leqslant Q < 1$ and $Q \to 1$. Then

$$\vartheta_3(0\,|\,\tau) \sim \sqrt{\pi}\,\frac{S_{h,k}}{k}\left(\log\frac{1}{Q}\right)^{-\frac{1}{2}}.*$$

and

$$\vartheta_3(0\,|\,c\tau) \sim \sqrt{\frac{\pi}{c}}\,\frac{S_{ch,k}}{k}\left(\log\frac{1}{Q}\right)^{-\frac{1}{2}}.*$$

Hence

$$f(\tau) \sim \frac{\pi^2}{c}\left(\frac{S_{h,k}}{k}\right)^2\left(\frac{S_{ch,k}}{k}\right)^2\left(\log\frac{1}{Q}\right)^{-2}.\dagger$$

Now the function

$$\sum_{n=1}^{\infty} nQ^n - \left(\log\frac{1}{Q}\right)^{-2}$$

is regular at $Q=1$. Therefore

$$f(\tau) \sim \frac{\pi^2}{c}\left(\frac{S_{h,k}}{k}\right)^2\left(\frac{S_{ch,k}}{k}\right)^2 \sum_{n=1}^{\infty} nq^n \exp\left(-\frac{2\pi ihn}{k}\right) = F_{h,k} = \sum_{n=1}^{\infty} c_{h,k,n}q^n.$$

The fundamental idea of the Hardy-Littlewood method is to approximate to $r(n)$ by the sum

$$\rho(n) = \sum_{h,k} c_{h,k,n} = \frac{\pi^2}{c}\,n \sum_{k=1}^{\infty} A_k = \frac{\pi^2}{c}\,nS(n), \tag{2.21}$$

where

$$A_1 = 1, \quad A_k = k^{-4} \sum_{h} S_{h,k}^2\, S_{ch,k}^2 \exp(-2h\pi in/k), \tag{2.22}$$

the summation being extended over the $\phi(k)$ values of h less than and prime to k.

The series $S(n)$ is the *singular series.*

2.3. *Convergence of the singular series.*

2.31. We shall now prove that the singular series is absolutely convergent. Therefore we substitute in the formula (2.22) for A_k the value of $S_{h,k}^2 S_{ch,k}^2$ as it is given by the formulae (2.115). Then it follows that

$|A_k| < K \cdot k^{-2}|c_k(n)|$, if k is odd;

$A_k = 0$, if one at least of the numbers k and k_1 is congruent to 2 (mod 4);

$|A_k| < K \cdot k^{-2}|c_k(mk/4-n)|$, if $k \equiv 0$ (mod 4) and k_1 is odd;

$|A_k| < K \cdot k^{-2}|c_k(mk/4+m_1c_1k/4-n)|$, if both k and k_1 are congruent to 0 (mod 4);

where K is a constant, not depending on k and n.

* Compare the formula (3.111).

† This formula must be read as $f(\tau) = o(\log 1/Q)^{-2}$ if $S_{h,k}S_{ch,k} = 0$.

Making use of (2.124) we find that, for all values of k,

$$(2.311) \qquad |A_k| < K.k^{-2} \sum_{\delta|k,\ \delta|4n} \delta \leqslant K.k^{-2} \sum_{\delta|4n} \delta,$$

which proves the absolute convergence of the singular series.

2.32. From (2.311) we can prove something about the behaviour of the singular series for large values of n. For we have

$$(2.321) \quad |S(n)| \leqslant \sum_{k=1}^{\infty} |A_k| < K \sum_{k=1}^{\infty} k^{-2} \sum_{\delta|k,\ \delta|4n} \delta = K \sum_{\delta|4n} \delta \sum_{r=1}^{\infty} \frac{1}{r^2\delta^2}$$
$$< K \sum_{\delta|4n} 1 = O(n^\epsilon),$$

making use of the well known fact that the number of divisors of n is $O(n^\epsilon)$.

2.4. *Summation of the singular series.*

The summation of the singular series is easily effected by means of the general rules given by Hardy in his paper.*

2.41. We first prove: *if* $(k, k') = 1$, *then*

$$(2.411) \qquad A_{kk'} = A_k A_{k'}.$$

For we have

$$(2.412) \quad A_k A_{k'} = (kk')^{-4} \sum_h \sum_{h'} S^2_{h,k} S^2_{ch,k} S^2_{h',k'} S^2_{ch',k'} \exp(-2\pi i n H/kk'),$$

where $H = hk' + h'k$, and where h runs through all positive integers, less than and prime to k and h' through all positive integers, less than and prime to k'. Then H runs through all positive integers, less than and prime to kk', since $(k, k') = 1$.

Now let $D = (c, k)$, $D' = (c, k')$, $c = Dc_1$, $k = Dk_1$, $c = D'c_1'$, $k' = D'k_1'$, $c = DD'c_2$. Then $DD' = (c, kk')$, and it follows from (2.114) and (2.112), that

$$S_{ch,k} = DS_{c_1 h, k_1}, \quad S_{ch',k'} = D' S_{c_1' h', k_1'},$$

$$S_{cH,kk'} = DD' S_{c_2 H, k_1 k_1'} = DD' S_{c_2 Hk_1', k_1} S_{c_2 Hk_1, k_1'}.$$

But
$$S_{c_2 Hk_1', k_1} = \sum_{j=0}^{k_1-1} \exp 2\pi i c_2 Hk_1' j^2/k_1 = \sum_{j=0}^{k_1-1} \exp 2\pi i c_2 hk' k_1' j^2/k_1$$
$$= \sum_{j=0}^{k_1-1} \exp 2\pi i c_1 h(k_1' j)^2/k_1 = S_{c_1 h, k_1}.$$

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* See footnote * on 144

In the same way we prove that

$$S_{c_2 Hk_1, k_1'} = S_{c_1 h', k_1'}.$$

Hence

$$S_{cH, kk'} = S_{ch', k'} S_{ch, k}, \quad S_{H, kk'} = S_{h', k'} S_{h, k}. \tag{2.413}$$

Substituting (2.413) in (2.412), we get

$$A_k A_{k'} = (kk')^{-4} \sum_H S^2_{H, kk'} S^2_{cH, kk'} \exp(-2\pi i n H/kk') = A_{kk'}.$$

Since the singular series is absolutely convergent, this result enables us to write it as an infinite product as follows:

$$S(n) = \prod_p \chi_p, \tag{2.414}$$

where p runs through all prime numbers and where

$$\chi_p = 1+A_p+A_{p^2}+A_{p^3}+\ldots. \tag{2.415}$$

2.42. We write

$$n = 2^\alpha m, \quad c = 2^\gamma d, \quad D = (c, k), \quad c = Dc_1, \quad k = Dk_1, \tag{2.421}$$

where m and d are odd.

For the calculation of χ_2 we derive the following results from (2.115):—

$$\begin{aligned} S^2_{h, k} S^2_{ch, k} &= 0, && \text{if } k = 2; \\ &= 2^{3\lambda+1} \exp \tfrac{1}{2} h\pi i, && \text{if } k = 2^\lambda, \quad 2 \leqslant \lambda \leqslant \gamma; \\ &= 0, && \text{if } k = 2^\lambda, \quad \lambda = \gamma+1; \\ &= (-)^{(d-1)/2} 2^{2\lambda+\gamma+2}, && \text{if } k = 2^\lambda, \quad \lambda > \gamma+1. \end{aligned}$$

Hence, if $2 \leqslant \lambda \leqslant \gamma$, then

$$\begin{aligned} A_{2^\lambda} &= 2^{-\lambda+1} \sum_h \exp\left(-\frac{2\pi i h n}{2^\lambda} + \tfrac{1}{2} h\pi i\right) \\ &= 2^{-\lambda+1} \exp\left(-\frac{2\pi i n}{2^\lambda} + \frac{\pi i}{2}\right) \sum_{\mu=0}^{2^{\lambda-1}-1} \exp\left(-\frac{2\pi i \mu n}{2^{\lambda-1}} + \pi i \mu\right). \end{aligned}$$

The last sum is 0, except for $\lambda = \alpha+2$, in which case it is $(-1)^{(m-1)/2}$. But the value $\lambda = \alpha+2$ occurs only in the interval $2 \leqslant \lambda \leqslant \gamma$ if $0 \leqslant \alpha \leqslant \gamma-2$.

For $\lambda > \gamma+1$, we have

$$A_{2^\lambda} = (-1)^{(d+1)/2} 2^{-2\lambda+\gamma+2} c_{2^\lambda}(n),$$

and this is [as follows from (2.125)]

$$0, \quad (-1)^{(d-1)/2}2^{-\lambda+\gamma+1}, \quad \text{or} \quad (-1)^{(d+1)/2}2^{-\lambda+\gamma+1},$$

according as $\lambda > \alpha+1$, $\lambda = \alpha+1$, or $\lambda < \alpha+1$.

Collecting the results, we arrive at the following formulae :—

$$(2.422) \quad \begin{cases} \chi_2 = 1+(-1)^{(m-1)/2}, & \text{if } 0 \leqslant \alpha \leqslant \gamma-2; \\ = 1, & \text{if } \gamma-2 < \alpha < \gamma+1; \\ = 1+(-1)^{(d-1)/2}\dfrac{2^{\alpha-\gamma}-3}{2^{\alpha-\gamma}}, & \text{if } \gamma+1 < \alpha. \end{cases}$$

2.43. In order to calculate χ_p for odd values of p, we have from (2.115)

$$S^2_{h,k}S^2_{ch,k} = (-1)^{(D-1)/2}Dk^2 \quad (k \text{ odd}).$$

Hence $$A_{p^\lambda} = (-1)^{(D-1)/2}D \cdot p^{-2\lambda}c_{p^\lambda}(n),$$

and if we substitute the value of the Ramanujan sum from (2.125)

$$(2.431) \qquad \chi_p = 1-(-1)^{[(c,p)-1]/2}\frac{(c,p)}{p} \quad \text{if} \quad p \nmid n$$

and

$$(2.432) \quad \chi_p = 1+\sum_{\lambda=1}^{\mu_n}(-1)^{[(c,p^\lambda)-1]/2}(c,p^\lambda)p^{-\lambda-1}(p-1) - (-1)^{[(c,p^{\mu_n+1})-1]/2}(c,p^{\mu_n+1})p^{-\mu_n-2},$$

if $p \mid n$. Here we have written μ_n for the exponent, determined by $p^{\mu_n} \mid n$, $p^{\mu_n+1} \nmid n$.

If μ_c is determined by $p^{\mu_c} \mid c$, $p^{\mu_c+1} \nmid c$, then we get the following special results from the general result (2.432) :—

If $p \mid n$, $\mu_c \geqslant \mu_n+1$, then

$$(2.433) \qquad \chi_p = \left(1-\frac{1}{p}\right)(\mu_n+1), \quad 1+\frac{1}{p}, \quad \text{or} \quad 0,$$

according as $p \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$ and μ_n even, $p \equiv 3 \pmod 4$ and μ_n odd.

If $p \mid n$, $\mu_c \leqslant \mu_n$, then

$$(2.434) \quad \chi_p = \left(1-\frac{1}{p}\right)\left\{\mu_c+\left(1+\frac{1}{p}\right)\left(1+\frac{1}{p}+\ldots+\frac{1}{p^{\mu_n-\mu_c}}\right)\right\},$$

$$\left(1+\frac{1}{p}\right)\left(1-\frac{1}{p^{\mu_n-\mu_c+1}}\right), \quad \text{or} \quad \frac{1}{p^{\mu_n-\mu_c+1}}\left(1+\frac{1}{p}\right)$$

according as

$p \equiv 1 \pmod 4$, $p \equiv 3 \pmod 4$, and μ_c even, $p \equiv 3 \pmod 4$ and μ_c odd.

2.44. The expression which can be obtained from 2.42 and 2.43 for the sum of the singular series is very complicated. Therefore we shall consider only some special values of c.

2.441. Let c be prime and $\equiv 1 \pmod 4$. We write $n = 2^\alpha c^\beta m$, where m is prime to $2c$. Then we get from (2.431)

$$\chi_p = 1 - \frac{1}{p^2} \quad \text{if} \quad p \nmid n,$$

and from (2.434)

$$\chi_p = \left(1-\frac{1}{p^2}\right)\left(1+\frac{1}{p}+\ldots+\frac{1}{p^{\mu_n}}\right) \quad \text{if} \quad p \mid n,\ p \neq c,$$

$$\chi_c = 2 - c^{-\beta} - c^{-\beta-1}.$$

Further, we have $\chi_2 = 1$ if $\alpha = 0$ and $\chi_2 = 3.2^{-\alpha}$ if $\alpha > 0$. Noticing that

$$\prod_p \left(1-\frac{1}{p^2}\right) = \frac{6}{\pi^2},$$

$$m \prod_{p \mid m} \left(1+\frac{1}{p}+\frac{1}{p^2}+\ldots+\frac{1}{p^{\mu_n}}\right) = \sigma(m)$$

(the sum of the divisors of m), we now get by multiplication

$$\frac{\pi^2}{c} nS(n) = 24 \frac{2c^{\beta+1}-c-1}{c^2-1} \sigma(m) \quad \text{if} \quad \alpha > 0, \tag{2.4411}$$

$$= 8 \frac{2c^{\beta+1}-c-1}{c^2-1} \sigma(m) \quad \text{if} \quad \alpha = 0.$$

2.442. In the same way we get the following formulae:—

If c is prime and $\equiv 3 \pmod 4$, then

$$\frac{\pi^2}{c} nS(n) = \frac{8}{c-1}(2^{\alpha+1}-3)\sigma(m) \quad \text{or} \quad \frac{8}{c-1}\sigma(m), \tag{2.4421}$$

according as $\alpha > 0$ or $\alpha = 0$.

If $c = 2d$, d prime and $\equiv 1 \pmod 4$, $n = 2^\alpha d^\beta m$ (m prime to $2d$), then

$$\frac{\pi^2}{c} nS(n) = A \frac{\sigma(m)}{d^2-1}(2d^{\beta+1}-d-1), \tag{2.4422}$$

where $A = 4$, 8, or 24 according as $\alpha = 0$, $\alpha = 1$, $\alpha \geqslant 2$.

If $c = 2d$, d prime and $\equiv 3 \pmod 4$, $n = 2^\alpha d^\beta m$ (m prime to $2d$), then

$$\frac{\pi^2}{c}\, nS(n) = B\,\frac{\sigma(m)}{d-1}, \tag{2.4423}$$

where $B = 4$, 8, or $8(2^\alpha - 3)$, according as $\alpha = 0$, $\alpha = 1$, $\alpha \geqslant 2$.

3. *Proof of the exact formulae.*

3.1. *Preliminary lemmas.*

3.11. The functions $\vartheta_2(0|\tau)$, $\vartheta_3(0|\tau)$, and $\vartheta_4(0|\tau)$* are analytic functions of the complex variable τ, regular in the half-plane $I(\tau) > 0$. They have no zeros in this half-plane and they have the real axis as a singular line. In order to determine the behaviour of the functions when τ tends to a rational point λ/μ on the real axis along a curve, which does not touch the real axis, I write $\tau = \lambda/\mu - 1/T$. Then $\exp \pi i T \to 0$. We may suppose that $(\lambda, \mu) = 1$.

Then if one of the numbers λ or μ is even, we have

$$\begin{aligned}
\vartheta_3(0|\tau) &= \sum_{n=-\infty}^{\infty} \exp n^2 \pi i \left(\frac{\lambda}{\mu} - \frac{1}{T}\right) \qquad (3.111)\\
&= \sum_{j=0}^{\mu-1} \sum_{l=-\infty}^{\infty} \exp (l\mu + j)^2 \pi i \left(\frac{\lambda}{\mu} - \frac{1}{T}\right)\\
&= \sum_{j=0}^{\mu-1} \exp\left(\frac{\lambda j^2 \pi i}{\mu} - \frac{j^2 \pi i}{T}\right) \cdot \vartheta_3\left(\frac{\mu j}{T}\,\middle|\, -\frac{\mu^2}{T}\right)\\
&= \sqrt{\left(\frac{-iT}{\mu^2}\right)} \cdot \sum_{j=0}^{\mu-1} \exp \frac{\lambda j^2 \pi i}{\mu} \cdot \vartheta_3\left(\frac{j}{\mu}\,\middle|\, \frac{T}{\mu^2}\right).\dagger
\end{aligned}$$

If both λ and μ are odd, then

$$\vartheta_3(0|\tau) = \sqrt{\left(\frac{-iT}{\mu^2}\right)} \cdot \sum_{j=0}^{\mu-1} \exp \frac{\lambda j^2 \pi i}{\mu} \cdot \vartheta_2\left(\frac{j}{\mu}\,\middle|\, \frac{T}{\mu^2}\right). \tag{3.112}$$

For the other ϑ-functions we have similar formulae :

$$\vartheta_2(0|\tau) = \sqrt{\left(\frac{-iT}{\mu^2}\right)} \cdot \sum_{j=0}^{\mu-1} \exp \frac{\lambda (j+\frac{1}{2})^2 \pi i}{\mu} \cdot \vartheta_\eta\left(\frac{j+\frac{1}{2}}{\mu}\,\middle|\, \frac{T}{\mu^2}\right), \tag{3.113}$$

where $\eta = 3$ if μ is odd and $\eta = 2$ if μ is even.

$$\vartheta_4(0|\tau) = \sqrt{\left(\frac{-iT}{\mu^2}\right)} \cdot \sum_{j=0}^{\mu-1} (-1)^j \exp \frac{\lambda j^2 \pi i}{\mu} \cdot \vartheta_\zeta\left(\frac{j}{\mu}\,\middle|\, \frac{T}{\mu^2}\right), \tag{3.114}$$

---

* As far as the theory of ϑ-functions is concerned, the notation of Tannery and Molk, *Eléments de la théorie des fonctions elliptiques*, will be used.

† All square roots, occurring in the transformation formulae for the ϑ-functions, have their real part positive.

where $\xi = 3$ if λ is odd and $\xi = 2$ if λ is even.

Since the right-hand sides of the equations are power series in $\exp \pi i T$, which tends to 0, they actually show the behaviour of the ϑ-functions, when $\tau \to \lambda/\mu$.

3.12. In the following pages there will frequently occur series of the type $\sum_h \sum_k (h-k\tau)^{-2}$, where h and k run through certain systems of integral numbers and where $I(\tau) > 0$. In every special case it can easily be shown that the *double series* diverges, but that the two *repeated series* formed from it are absolutely convergent.

Suppose, for instance, that h and k run through all integral numbers and have opposite parity. Then it is easy to see that the repeated series

$$\sum_k \left(\sum_h (h-k\tau)^{-2} \right) \quad \text{and} \quad \sum_h \left(\sum_k (h-k\tau)^{-2} \right)$$

are absolutely convergent. For let $q = \exp \pi i \tau$ and $q' = \exp(-\pi i/\tau)$. Then

$$\sum_h (h-k\tau)^{-2} = \pi^2 \frac{(-q)^k}{\{1+(-q)^k\}^2} \quad \text{and} \quad \sum_k (h-k\tau)^{-2} = \frac{\pi^2}{\tau^2} \frac{(-q')^h}{\{1+(-q')^h\}^2},$$

and since $|q| < 1$, $|q'| < 1$, the series

$$\frac{\pi^2}{4} + 2\pi^2 \sum_{k=1}^{\infty} \frac{(-q)^k}{\{1+(-q)^k\}^2} \quad \text{and} \quad \frac{1}{\tau^2} \left(\frac{\pi^2}{4} + 2\pi^2 \sum_{h=1}^{\infty} \frac{(-q')^h}{\{1+(-q')^h\}^2} \right)$$

are absolutely convergent.

Since the double series is not absolutely convergent it is not permissible to change the order of summation and, in fact, the sums of the two repeated series are not equal. For let

$$\alpha(\tau) = \sum_k \left(\sum_h (h-k\tau)^{-2} \right). \tag{3.121}$$

Then

$$\sum_h \left(\sum_k (h-k\tau)^{-2} \right) = \tau^{-2} \alpha(-\tau^{-1}).$$

We have also

$$\alpha(\tau) = \frac{\pi^2}{4} + 2\pi^2 \sum_{k=1}^{\infty} \frac{(-q)^k}{\{1+(-q)^k\}^2} = -\tfrac{1}{4} \left(\frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_4''(0)}{\vartheta_4(0)} \right) = -\pi i \frac{d}{d\tau} \log \vartheta_2 \vartheta_4. \tag{3.122}$$

(See T.M., 30 and 39.*)

* This refers to the tables of formulae in the second volume of Tannery and Molk.

If we apply the formulae T.M., 43 (14) and (16), we get

$$a(\tau) = \pi i/\tau + \tau^{-2} a(-\tau^{-1}), \tag{3.123}$$

or also

$$\sum_k \Big(\sum_h (h - k\tau)^{-2} \Big) = \frac{\pi i}{\tau} + \sum_h \Big(\sum_k (h - k\tau)^{-2} \Big). \tag{3.124}$$

It is also possible to prove relations of the type (3.124), without using the theory of ϑ-functions.*

3.13. It is well known that all linear substitutions

$$\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$$

with integral coefficients α, β, γ, δ such that $\alpha\delta - \beta\gamma = 1$ form a *group*, the so-called *modular group*. Important subgroups of it are the cyclic groups. If V is the generating substitution of a cyclic group, then this group consists of the substitutions

$$\ldots, \; V^{-2}, \; V^{-1}, \; V^0 = 1, \; V, \; V^2, \; V^3, \; \ldots.$$

The simplest fundamental polygon of a cyclic group, generated by a *hyperbolic* ($\alpha + \delta > 2$) or *parabolic* ($\alpha + \delta = 2$) substitution is given by the following rules:†

G consists of the whole τ-plane with the exception of the interior of the following two circles:

1°. The circle whose centre is $-(\delta/\gamma)$ and whose radius is $1/\gamma$ (circumference *in*cluded). 2°. The circle whose centre is α/γ and whose radius is $1/\gamma$ (circumference *ex*cluded).

In the case of a parabolic substitution the two circles have the point $(-\delta + 1)/\gamma = (\alpha - 1)/\gamma$ in common.

3.14. LEMMA.—*A modular function, whose group has a finite index in the group of all modular substitutions, and which is bounded in its fundamental polygon, is a constant.*

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* A. Hurwitz, "Grundlagen einer independenten Theorie der elliptischen Modulfunctionen und Theorie der Multiplicatorgleichungen erster Stufe", *Math. Annalen*, 18 (1881), 549.

† Klein-Fricke, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, 1 (1890), 194–196.

If the function exists in the whole plane, we find, by applying all substitutions of its group, that it is bounded all over the plane and therefore it must be a constant.

If the real axis is a singular line for the function, the lemma can be proved as follows:

Let $\phi(\tau)$ be the given function and Γ its group. Let $J(\tau)$ be the function which is defined by the equations (notation of Weierstrass)

$$J(\tau) = g_2^3/\Delta, \quad \Delta = g_2^3 - 27g_3^2, \quad \tau = \omega_3/\omega_1.$$

The fundamental region of $J(\tau)$ consists of two triangles, bounded by circles. The fundamental region of $J(\tau)$ consists of 2μ triangles, if μ is the index of Γ in the group of all modular substitutions. In its own fundamental polygon the function $J(\tau)$ takes every value (except 0, 1, and ∞) exactly once and therefore exactly μ times in the fundamental polygon of Γ, from which it follows that the function $J(\tau)$ transforms the fundamental region of Γ into a Riemann surface, consisting of μ sheets. The function ϕ, considered as a function of J, is one-valued and bounded on this surface if $\phi(\tau)$ is bounded in its fundamental polygon. But a function, which is one-valued and bounded on a Riemann surface, must be a constant.

3.2. *The function $F(\tau)$.*

We write $$\rho(n) = \frac{\pi^2}{c} nS(n), \quad F(\tau) = 1 + \sum_{n=1}^{\infty} \rho(n) q^n.$$

LEMMA.—*If c is prime and congruent to* 3 (mod 4), *then*

$$\left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F(\tau) = -\frac{1}{c} \sum_k \sum_h (h-k\tau)^{-2} + \sum_k \sum_h (h-kc\tau)^{-2}, \tag{3.21}$$

where h and k run through all integers of opposite parity.

We have

$$F(\tau) = 1 + \frac{\pi^2}{c} \sum_{n=1}^{\infty} nq^n \sum_{k=1}^{\infty} A_k. \tag{3.22}$$

From (2.321) we get $$\sum_{k=1}^{\infty} |A_k| = O(n^{\epsilon}).$$

Therefore the *repeated series*

$$\sum_{n=1}^{\infty} n|q|^n \sum_{k=1}^{\infty} |A_k|$$

converges, and this includes the absolute convergence of the *double series*

$$\sum_{n,\,k} nq^n A_k,$$

so that we may change the order of summation in (3 . 22) :

$$F(\tau) = 1+\frac{\pi^2}{c}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} nq^n A_k$$

$$= 1+\frac{\pi^2}{c}\sum_{k=1}^{\infty} k^{-4}\sum_{h} S^2_{h,\,k} S^2_{ch,\,k} \sum_{n=1}^{\infty} nq^n \exp\left(-\frac{2n\pi i h}{k}\right).$$

Using the identity

$$\sum_{n=1}^{\infty} ne^{-ny} = \frac{e^{-y}}{(1-e^{-y})^2} = \sum_{m=-\infty}^{\infty} (y+2m\pi i)^{-2},$$

we get

$$F(\tau) = 1-c^{-1}\sum_{k=1}^{\infty} k^{-2}\sum_{h} S^2_{h,\,k} S^2_{ch,\,k} \sum_{m=-\infty}^{\infty} \Big(2(h+mk)-k\tau\Big)^{-2}.$$

Now let $c \equiv 3 \pmod 4$. Write $(k, c) = \delta$. Then it follows from (2 . 115) that

$$S^2_{h,\,k} S^2_{ch,\,k} = 4k^2\delta(-1)^{(\delta-1)/2}, \quad \text{if} \quad k \equiv 0 \pmod 4;$$

$$= 0, \qquad \text{if} \quad k \equiv 2 \pmod 4;$$

$$= k^2\delta(-1)^{(\delta-1)/2}, \qquad \text{if} \quad k \text{ is odd.}$$

Hence

$$F(\tau) = 1-c^{-1}\sum_{\delta\,|\,c} (-1)^{(\delta-1)/2}\,\delta \sum_{(k,\,c)=\delta;\ k>0} \eta_k \sum_{h} \sum_{m=-\infty}^{\infty} \Big(2(h+mk)-k\tau\Big)^{-2},$$

where $\eta_k = 4$, 0, or 1, according as $k \equiv 0 \pmod 4$, $\equiv 2 \pmod 4$, or odd. Now we have

$$\sum_{h} \sum_{m=-\infty}^{\infty} \Big(2(h+mk)-k\tau\Big)^{-2} = \sum_{h} (2h-k\tau)^{-2},$$

where on the right-hand side h runs through all values, positive and negative, which are prime to k. Therefore, if c is *prime*,

$$F(\tau) = 1-c^{-1} \sum_{(k,\,c)=1;\ k>0} \eta_k \sum_{h} (2h-k\tau)^{-2} + \sum_{(k,\,c)=c;\ k>0} \eta_k \sum_{h} (2h-k\tau)^{-2}$$

$$= 1-c^{-1} F_1(\tau)+F_2(\tau).$$

Further

$$(3.23)\qquad F_1(\tau) = \sum_{(k,c)=1}\sum_{h}\eta_k(2h-k\tau)^{-2}$$

$$= \sum_{(k,c)=1,\ k\text{ odd}}\sum_{h}(h-k\tau)^{-2} + \sum_{(k,c)=1,\ k\text{ even}}\sum_{h}(h-k\tau)^{-2}$$

$$= \sum_{k}\sum_{h}(h-k\tau)^{-2},$$

where now in the last sum k runs through all positive integers that are not divisible by c, and h through all integers, positive and negative, which are prime to k and have opposite parity to k.

Similarly

$$(3.24)\qquad F_2(\tau) = \sum_{k}\sum_{h}(h-kc\tau)^{-2},$$

where k runs through all positive integers and h through all integers, positive and negative, which are prime to c, prime to k, and have opposite parity to k.

We now consider the sum

$$\Sigma_1 = \sum_{K}\sum_{H}(H-K\tau)^{-2},$$

where H and K are subject to the same restrictions as h and k on the right-hand side of (3.23) with the exception of $(h, k) = 1$. Then we have

$$\Sigma_1 = \sum_{K}\sum_{2d+1\,|\,K}\sum_{(H,K)=2d+1}(H-K\tau)^{-2}$$

$$= \sum_{K}\sum_{2d+1\,|\,K}(2d+1)^{-2}\sum_{(H_1,K_1)=1}(H_1-K_1\tau)^{-2},$$

where $H=(2d+1)H_1$ and $K=(2d+1)K_1$. I write

$$\sum_{(H_1,K_1)=1}(H_1-K_1\tau)^{-2} = u_{K_1},$$

so that

$$\Sigma_1 = \sum_{K}\sum_{2d+1\,|\,K}(2d+1)^{-2}u_{K_1}.$$

Collecting those terms of this sum which have the same K_1, we find

$$(3.25)\qquad \Sigma_1 = \sum_{k}u_k\Sigma(2m+1)^{-2},$$

where $2m+1$ runs through all positive odd integers that are prime to c, and k through all positive integers that are prime to c.

This last step can be justified by the following considerations. We have

$$\Sigma_1 = \lim_{N\to\infty} \sum_{K\leqslant N} \sum_{2d+1\mid K} (2d+1)^{-2} u_{K_1} = \lim_{N\to\infty} \sum_{k\leqslant N} u_k \sum_{(2m+1)k\leqslant N} (2m+1)^{-2}$$

$$= \lim_{N\to\infty} \sum_{k\leqslant N} u_k \sum_{m} (2m+1)^{-2} - \lim_{N\to\infty} \sum_{k\leqslant N} u_k \sum_{(2m+1)k>N} \frac{1}{(2m+1)^2}$$

$$= \sum_k u_k \sum (2m+1)^{-2} - \lim_{N\to\infty} \sum_{k\leqslant N} u_k \sum_{(2m+1)k>N} (2m+1)^{-2}.$$

Therefore it remains to prove that this last limit is 0. Now we have

$$\sum_{(2m+1)k>N} (2m+1)^{-2} < Ak/N,$$

where A is independent of k and N. Hence

$$\left| \sum_{k\leqslant N} u_k \sum_{(2m+1)k>N} (2m+1)^{-2} \right| < A \cdot \frac{1}{N} \sum_{k=1}^{N} k\,|u_k|.$$

Since the series $\Sigma|u_k|$ is convergent, the right-hand side tends to 0, if $N\to\infty$ and (3.25) is proved.

Further we have $\Sigma(2m+1)^{-2} = \frac{1}{8}\pi^2(1-c^{-2})$

and $$\sum_k u_k = \sum_k \sum_h (h-k\tau)^{-2} = F_1(\tau).$$

Hence $$\tfrac{1}{8}\pi^2(1-c^{-2})\,F_1(\tau) = \sum_K \sum_H (H-K\tau)^{-2}.$$

Similarly, we can prove

$$\tfrac{1}{8}\pi^2(1-c^{-2})\,F_2(\tau) = \sum_K \sum_H (H-Kc\tau)^{-2},$$

where K and H are subject to the same restrictions as k and h in (3.24) *with the exception of* $(h, k) = 1$.

Therefore, writing again h and k instead of H and K,

$$\tfrac{1}{8}\pi^2(1-c^{-2})\,F(\tau) = \tfrac{1}{8}\pi^2(1-c^{-2}) - c^{-1}\sum_k \sum_h (h-k\tau)^{-2} + \sum_k \sum_h (h-kc\tau)^{-2},$$

where now the ranges of summation are as follows:—in the first sum $k\geqslant 1$, $(k, c) = 1$, and h has opposite parity to k; in the second sum $k\geqslant 1$, $(h, c) = 1$, and h has opposite parity to k.

Finally, we have for the first sum

$$-c^{-1}\sum_k \sum_h (h-k\tau)^{-2} = -c^{-1}\sum_k \sum_h (h-k\tau)^{-2} + c^{-1}\sum_k \sum_h (h-kc\tau)^{-2},$$

and for the second sum

$$\sum_k \sum_h (h-kc\tau)^{-2} = \sum_k \sum_h (h-kc\tau)^{-2} - c^{-2} \sum_k \sum_h (h-k\tau)^{-2},$$

where now on the right-hand sides $k \geqslant 1$, and h has opposite parity to k.

Therefore

$$\tfrac{1}{8}\pi^2(1-c^{-1})F(\tau) = \tfrac{1}{8}\pi^2(1-c^{-1}) - c^{-1} \sum_k \sum_h (h-k\tau)^{-2} + \sum_k \sum_h (h-kc\tau)^{-2},$$

and from this equation we obtain the lemma after multiplication by 2, and observing that

$$\sum_h (h-k\tau)^{-2} = \sum_h (h+k\tau)^{-2} \quad \text{and} \quad \sum_{h=-\infty}^{\infty} h^{-2} = \tfrac{1}{4}\pi^2.$$

3.3. *Determination of a group of modular substitutions, for which $F(\tau)$ is an invariant.*

3.31. By means of the lemma of the preceding section, we shall try to obtain a group of modular substitutions, for which $F(\tau)$ is an invariant. We have from (3.21)

$$\left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F\left(\frac{a\tau+\beta}{\gamma\tau+\delta}\right) = -\frac{(\gamma\tau+\delta)^2}{c} \sum_k \sum_h \{\tau(h\gamma-ka)+h\delta-k\beta\}^{-2} + (\gamma\tau+\delta)^2 \sum_k \sum_h \{\tau(h\gamma-cka)+h\delta-ck\beta\}^{-2}.$$

We are now going to rearrange the terms of the double series on the right-hand side *formally* by writing

$$(3.311) \qquad \begin{cases} h_1 = \delta h - \beta k, & k_1 = -\gamma h + \alpha k, \\ h_2 = \delta h - c\beta k, & k_2 = -(\gamma/c)h + \alpha k; \end{cases}$$

and

$$(3.312) \qquad \left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = -\frac{(\gamma\tau+\delta)^2}{c} \sum_{k_1} \sum_{h_1} (h_1-k_1\tau)^{-2} + (\gamma\tau+\delta)^2 \sum_{k_2} \sum_{h_2} (h_2-ck_2\tau)^{-2}.$$

We shall try to determine the coefficients α, β, γ, δ in such a way that h_1 and k_1, as well as h_2 and k_2 run through all integers of opposite parity, like h and k. It follows from (3.311) that this will be the case if α, β, γ, δ satisfy the following conditions:—

(3.313) $\alpha\delta-\beta\gamma = 1$, $\gamma \equiv 0 \pmod{c}$, γ and δ opposite parity, α and β opposite parity.

If these conditions are satisfied the right-hand side of (3 . 312) is equal to $\frac{1}{4}\pi^2(1-c^{-1})(\gamma\tau+\delta)^2F(\tau)$, so that we may expect $F(\tau)$ to be invariant for the substitutions.

It has not been proved that the rearrangement of the terms of the double series is permissible. However, it follows from (3 . 124) that the order of summation in the two double series on the right-hand side of (3 . 31) may be changed simultaneously, though not in each series separately. This leads us to the conjecture that the above considerations are also true, which will be proved rigorously in 3 . 35. The object of this section is only to determine the conditions (3 . 313) *formally*.

Now let a_r' and β_r' $\{r = 1, 2, \ldots, (c-1)/2\}$ be determined by the Diophantine equation

$$2ra_r'-c\beta_r' = 1, \tag{3 . 314}$$

and the subsidiary conditions

$$-c < a_r' < c, \quad a_r' \text{ even}. \tag{3 . 315}$$

Let $\quad a_r = a_r', \quad$ if $\quad a_r'+2r > 0; \quad a_r = -a_r', \quad$ if $\quad a_r'+2r < 0;$

$$\beta_r = \beta_r'; \tag{3 . 316}$$

$$\delta_r = 2r, \quad \text{if} \quad a_r'+2r > 0; \quad \delta_r = -2r, \quad \text{if} \quad a_r'+2r < 0.$$

Then it is easily verified that the $(c-1)/2$ substitutions

$$\tau' = \frac{a_r\tau+\beta_r}{c\tau+\delta_r} \quad \left(r = 1, 2, \ldots, \frac{c-1}{2}\right) \tag{3 . 317}$$

satisfy the conditions (3 . 313). These conditions are also satisfied by the substitutions

$$\tau' = \tau+2, \quad \tau' = \frac{\tau}{2c\tau+1}. \tag{3 . 318}$$

3 . 32. LEMMA.—*The substitutions* (3 . 317) *are hyperbolic.*

It must be proved that $|a_2+2r| > 2$. Now in the first place $a_r+2r \neq 0$. For if $a_r+2r = 0$, then it would follow from (3 . 314) that $2ra_r \equiv 1 \pmod{c}$, and therefore $(2r)^2 \equiv -1 \pmod{c}$, which is impossible, because -1 is not a quadratic residue of prime numbers which are $\equiv 3 \pmod 4$.

In the second place a_r+2r cannot be -1 or $+1$, since a_r and $2r$ are both even.

In the third place $a_r+2r = 2$ would imply $(2r-1)^2 \equiv 0 \pmod{c}$, which is impossible, because $2r-1$ is one of the numbers $1, 3, 5, \ldots, c-2$.

Finally it would follow from $a_r+2r = -2$ that $(2r+1)^2 \equiv 0 \pmod{c}$, and from this we deduce successively $2r+1 \equiv 0 \pmod{c}$, $2r+1 = c$, $a = -c-1$, which is again impossible because $a_r > -c$.

3.33. LEMMA.—*The system of numbers*

$$\alpha_r \text{ and } -\delta_r \ \{r = 1, 2, \ldots, (c-1)/2\}$$

is identical with the following system:—

$$-(c-1),\ -(c-3),\ \ldots,\ -2,\ +2,\ +4,\ \ldots,\ c-1$$

(*all even numbers between* $-c$ *and* $+c$, *except* 0).

The numbers α_r and $-\delta_r$ being even, $\neq 0$, $> -c$, and $< c$, it is sufficient to show that they are all different. The numbers α_r and $-\delta_r$ fall into four classes:—

(A) The numbers α'_r, such that $\alpha'_r+2r > 2$;

(B) The numbers $-2r$, such that $\alpha'_r+2r > 2$;

(C) The numbers $-\alpha'_r$, such that $\alpha'_r+2r < -2$;

(D) The numbers $2r$, such that $\alpha'_r+2r < -2$.

1°. Obviously no two numbers of one class are the same.

2°. A number from (A) cannot be equal to a number from (B). For if $\alpha'_r = -2s$, it would follow from $2r\alpha'_r \equiv 1$, $2s\alpha'_s \equiv 1 \pmod{c}$, that $2r+\alpha'_s \equiv 0 \pmod{c}$. But this is impossible. For in the first place $-2c < 2r+\alpha'_s < 2c$. Secondly $2r+\alpha'_s$ cannot be c or $-c$, because $2r$ and α'_s are both even. In the third place $2r+\alpha'_s$ cannot be 0, for then we would have $2r+\alpha'_s = 0$, $2s+\alpha'_r = 0$, contrary to $\alpha'_r+2r > 2$, $\alpha'_s+2s > 2$.

3°. A number from (A) cannot be equal to a number from (C). For if $\alpha'_r = -\alpha'_s$, it would follow from $2r\alpha'_r \equiv 1$, $2s\alpha'_s \equiv 1 \pmod{c}$, that $r+s \equiv 0 \pmod{c}$, but this is impossible because $0 < r+s \leqslant c-1$.

4°. A number from (A) cannot be equal to a number from (D). For if $\alpha'_r = 2s$, it would follow from $2r\alpha'_r \equiv 1$, $2s\alpha'_s \equiv 1 \pmod{c}$, that $2r-\alpha'_s \equiv 0 \pmod{c}$. Also $0 < 2r-\alpha'_s < 2c$ (since $\alpha'_s < 0$ in consequence of $\alpha'_s+2s < -2$). Therefore it would follow that $2r-\alpha'_s = c$, which is impossible, since α'_s is even and c is odd.

5°. A number from (B) cannot be equal to a number from (C). For if $2r = \alpha'_s$, then $\alpha'_r \equiv 2s \pmod{c}$. Now $-2c < \alpha'_r-2s < 2c$, and therefore (since α'_r-2s is even) $\alpha'_r-2s = 0$. From this it would follow that $2r+\alpha'_r = 2s+\alpha'_s$, contrary to $\alpha'_r+2r > 2$, $\alpha'_s+2s < -2$.

6°. A number from (B) cannot be equal to a number from (D), because the first is negative, the latter positive.

7°. A number from (C) cannot be equal to a number from (D). For if $-a'_r = 2s$, it would follow from $2ra'_r \equiv 2sa'_s \pmod{c}$, that $2r+a'_s \equiv 0 \pmod{c}$. Now $-2c < 2r+a'_s < 2c$, and since $2r+a'_s$ is even, it cannot be equal to $-c$ or $+c$. Therefore $2r+a'_s = 0$ and $a'_r+2s+a'_s+2r = 0$, contrary to $a'_r+2r < -2$, $a'_r+2s < -2$.

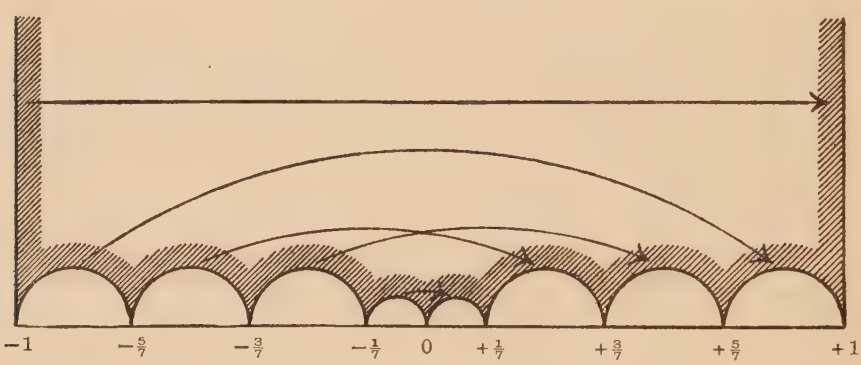
Therefore the numbers a_r and $-\delta_r$ are all different and the lemma is proved.

3.34. The substitutions (3.317) and (3.318) generate a group which we shall call γ_c.

LEMMA.—*That part of the strip* $-1 < R(\tau) < +1$, *which is situated "above" the circles with their centres in*

$$-\frac{c-1}{c}, \quad -\frac{c-3}{c}, \quad \ldots, \quad -\frac{2}{c}, \quad +\frac{2}{c}, \quad +\frac{4}{c}, \quad \ldots, \quad \frac{c-3}{c}, \quad \frac{c-1}{c}$$

and radius $1/c$ *and the circle with its centre in the origin and radius* $1/2c$, *can be taken as a fundamental polygon for* γ_c. (See the figure for $c = 7$.)*



The lemma follows from well known principles of the theory of modular substitutions as follows:—

The substitutions (3.317) are hyperbolic (Lemma 3.32) and the second substitution (3.318) is parabolic. Therefore the fundamental polygons of these substitutions are given by the rule of 3.13. According to this rule we have to describe the circles with centres in a_r/c and $-\delta_r/c$ and with radius $1/c$ for the substitutions (3.317), and the circle with the origin as centre and $1/2c$ as its radius for the second of the substitutions

* The arrows indicate how the sides of the polygon are conjugated to one another by means of the generating substitutions of the group.

(3.318). It follows from Lemma 3.33 that these circles are exactly the circles described in the statement of the present lemma. Further, the strip $-1 \leqslant R(\tau) < 1$ is a fundamental region for the substitution $\tau' = \tau+2$. Therefore the region described in the statement of the lemma is the common part of the fundamental polygons of the generating substitutions of γ_c, that is to say the fundamental polygon of γ_c.

3.35. LEMMA.—*For any substitution of* γ_c

$$F\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^2 F(\tau) \quad \textit{and} \quad f\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^2 f(\tau).$$

It is sufficient to consider the generating substitutions of γ_c only. For the proof of the first part of the lemma we deduce from (3.21), (3.121), and (3.122), that

(3.351)

$$\left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F(\tau) = -\frac{1}{c}\,a(\tau)+a(c\tau) = \frac{\pi i}{c}\frac{d}{d\tau}\log\frac{\vartheta_2(0\,|\,\tau)\,\vartheta_4(0\,|\,\tau)}{\vartheta_2(0\,|\,c\tau)\,\vartheta_4(0\,|\,c\tau)}.$$

Introducing the functions $h(\tau)$ and $\chi(\tau)$ defined by the equations

$$h(\tau) = q^{\frac{1}{12}}\prod_{n=1}^{\infty}(1-q^{2n}), \quad \chi(\tau) = 2^{\frac{1}{2}}q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1+q^{2n-1})^{-1},$$

we get [T.M., 38 (10)]

$$\left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F(\tau) = \frac{2\pi i}{c}\frac{d}{d\tau}\log\frac{h(\tau)\chi(\tau)}{h(c\tau)\chi(c\tau)}.$$

From this formula the lemma follows by a straightforward application of the formulae T.M., 46 (2) and 46 (6).

The second part of the lemma is merely an application of the formulae T.M., 42 (3), to the function

$$f(\tau) = \vartheta_3^2(0\,|\,\tau)\,\vartheta_3^2(0\,|\,c\tau).$$

3.4. *Behaviour of the functions* $F(\tau)$ *and* $f(\tau)$ *in the angles of the fundamental polygon of* γ_c.

3.41. If τ tends to infinity in a direction parallel to the imaginary axis, then $q \to 0$ and therefore both $F(\tau)$ and $f(\tau)$ tend to 1.

3.42. The points λ/c $\{\lambda = \pm 1, \pm 3, \ldots, \pm(c-2)\}$. We write, as in 3.11, $\tau = \lambda/c - 1/T$, and expand the functions in power series in

$\exp \pi i T$. From (3.112) we have

$$\vartheta_3\Big(0\Big|\frac{\lambda}{c}-\frac{1}{T}\Big)=\frac{1}{c}\sqrt{(-iT)}\sum_{j=0}^{c-1}\exp\frac{\lambda\pi i j^2}{c}\,.\,\vartheta_2\Big(\frac{j}{c}\Big|\frac{T}{c^2}\Big).$$

The first term in the expansion on the right-hand side is

$$\frac{2}{c}\sqrt{(-iT)}\exp\frac{\pi i T}{4c^2}\,.\sum_{j=0}^{c-1}\exp\frac{\lambda\pi i j^2}{c}\,.\cos\frac{\pi j}{c}$$

Further, since λ is odd,

$$\vartheta_3\Big(0\Big|\lambda-\frac{c}{T}\Big)=\vartheta_4\Big(0\Big|-\frac{c}{T}\Big)=\sqrt{\Big(\frac{-iT}{c}\Big)}\,.\,\vartheta_2\Big(0\Big|\frac{T}{c}\Big)$$

$$=2\sqrt{\Big(\frac{-iT}{c}\Big)}\exp\frac{\pi i T}{4c}+\ldots.$$

Hence $$f\Big(\frac{\lambda}{c}-\frac{1}{T}\Big)=-\frac{16T^2}{c^3}\,\sigma_{\lambda,c}^2\exp\frac{\pi i T}{2c^2}(c+1)+\ldots,$$

where $$\sigma_{\lambda,c}=\sum_{j=0}^{c-1}\exp\frac{\lambda\pi i j^2}{c}\,.\cos\frac{\pi j}{c}=\sum_{j=0}^{c-1}\exp\frac{\lambda j^2+j}{c}\,\pi i.$$

This sum can be easily calculated. Therefore we determine p by the congruence

$$2\lambda p+1\equiv 0 \pmod{c} \tag{3.421}$$

and write $j+p$ instead of p. Then we get

$$\sigma_{\lambda,c}=\sum_{j=0}^{c-1}\exp\frac{\lambda(j+p)^2+j+p}{c}\,\pi i=\exp\frac{p+\lambda p^2}{c}\,\pi i\,.\sum_{j=0}^{c-1}\exp\frac{\lambda\pi i j^2}{c}\,.\,(-1)^j$$

$$=\exp\frac{p+\lambda p^2}{c}\,\pi i\,.\,S_{\frac{1}{2}(\lambda+c),\,c}.$$

Hence, in consequence of (2.113),

$$\sigma_{\lambda,c}^2=-c\exp\frac{2p+2\lambda p^2}{c}\,\pi i=-c\exp\Big(-\frac{2\lambda p^2\pi i}{c}\Big),$$

so that

$$f\Big(\frac{\lambda}{c}-\frac{1}{T}\Big)=\frac{16T^2}{c^2}\exp\Big(\frac{\pi i T(c+1)}{2c^2}-\frac{2\lambda p^2\pi i}{c}\Big)+\ldots. \tag{3.422}$$

In order to determine the analogous formula for $F(\tau)$, we write

$$G(\tau)=\frac{h(\tau)\,\chi(\tau)}{h(c\tau)\,\chi(c\tau)}.$$

Then we find from T.M., 53 (7) and (10),

$$G(\tau) = K \cdot \prod_r \frac{\vartheta_1(r\tau \mid c\tau)}{\vartheta_3(r\tau \mid c\tau)},$$

where K is a constant and r runs through the numbers

$$1,\ 2,\ 3,\ \ldots,\ (c-1)/2.$$

Now writing $\tau = \lambda/c - 1/T$ and applying T.M., 43, (1), (3), (5), and (8),

$$G\left(\frac{\lambda}{c}-\frac{1}{T}\right) = K\prod_r \frac{\vartheta_1\left(\frac{r\lambda}{c}-\frac{r}{T}\,\middle|\,\lambda-\frac{c}{T}\right)}{\vartheta_3\left(\frac{r\lambda}{c}-\frac{r}{T}\,\middle|\,\lambda-\frac{c}{T}\right)} = K\cdot\prod_r \frac{\vartheta_1\left(\frac{r\lambda}{c}-\frac{r}{T}\,\middle|\,-\frac{c}{T}\right)}{\vartheta_4\left(\frac{r\lambda}{c}-\frac{r}{T}\,\middle|\,-\frac{c}{T}\right)}$$

$$= K\prod_r \frac{\vartheta_1\left(\frac{r\lambda T}{c^2}-\frac{r}{c}\,\middle|\,\frac{T}{c}\right)}{\vartheta_2\left(\frac{r\lambda T}{c^2}-\frac{r}{c}\,\middle|\,\frac{T}{c}\right)},$$

and therefore, using the infinite products for the ϑ-functions,

$$(3.423)\quad \left(1-\frac{1}{c}\right)\frac{\pi^2}{4}F\left(\frac{\lambda}{c}-\frac{1}{T}\right) = \frac{2\pi i}{c}T^2\frac{d}{dT}\log G\left(\frac{\lambda}{c}-\frac{1}{T}\right)$$

$$= \frac{8\pi^2\lambda T^2}{c^3}\sum_r \frac{r\exp 2\pi i\left(\frac{r\lambda T}{c^2}-\frac{r}{c}\right)}{1-\exp 4\pi i\left(\frac{r\lambda T}{c^2}-\frac{r}{c}\right)}$$

$$+\frac{8\pi^2T^2}{c^2}\sum_s\sum_{n=1}^{\infty}\frac{\left(n-\frac{s\lambda}{c}\right)\exp 2\pi i\left(\frac{nT}{c}-\frac{s\lambda T}{c^2}+\frac{s}{c}\right)}{1-\exp 4\pi i\left(\frac{nT}{c}-\frac{s\lambda T}{c^2}+\frac{s}{c}\right)},$$

where s runs through the numbers

$$-\frac{c-1}{2},\quad -\frac{c-3}{2},\quad \ldots,\quad -2,\quad -1,\quad +1,\quad +2,\quad \ldots,\quad \frac{c-3}{2},\quad \frac{c-1}{2}.$$

Expanding in powers of $\exp \pi i T$, we get the desired result

$$(3.424)\quad \left(1-\frac{1}{c}\right)\frac{\pi^2}{4}F\left(\frac{\lambda}{c}-\frac{1}{T}\right) = \frac{8\pi^2T^2}{c^3}\exp 2\pi i\left(\frac{T}{c^2}+\frac{y}{c}\right)+\ldots,$$

where the integer y must be determined from

$$cx-\lambda y = 1,\quad -\tfrac{1}{2}c < y < \tfrac{1}{2}c.$$

3.43. The points $0, \pm 1$.

The analogous expansions for these points are very simple. We find

(3.431)

$$f\left(-\frac{1}{T}\right) = -\frac{T^2}{c} + \ldots, \quad f\left(\pm 1 - \frac{1}{T}\right) = -\frac{16T^2}{c} \exp \frac{\pi i T(c+1)}{2c} + \ldots,$$

and

(3.432) $$\left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F\left(-\frac{1}{T}\right) = -\left(1-\frac{1}{c}\right)\frac{\pi^2}{4}\frac{T^2}{c} + \ldots,$$

(3.433) $$\left(1-\frac{1}{c}\right)\frac{\pi^2}{4} F\left(\pm 1 - \frac{1}{T}\right) = -\frac{8\pi^2 T^2}{c^2} \exp \frac{2\pi i T}{c} + \ldots.$$

3.5. *The exact formula for* $c = 3$.

3.51. It has already been stated in the introduction that Hardy has proved that $\rho(n) = r(n)$ [or what comes to the same thing: $F(\tau) = f(\tau)$], if $c = 1$. It is therefore natural to ask whether there are other cases where this is true. Now it is easy to see that $F(\tau) \neq f(\tau)$ for all prime values of $c \equiv 3 \pmod 4$, except for $c = 3$. For if $F(\tau)$ were identical with $f(\tau)$, then the behaviour of these functions at the point $\tau = 1$ would be the same. Therefore it would follow from (3.431) and (3.433) that $(c+1)/2c = 2/c$ or $c = 3$.

3.52. THEOREM.—*If* $c = 3$, *then* $F(\tau)$ *is identical with* $f(\tau)$.

We consider the function

$$\Psi(\tau) = \frac{F(\tau)}{f(\tau)}.$$

It follows from Lemma 3.35 that, for any substitution of γ_3,

$$\Psi\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = \Psi(\tau),$$

that is to say $\Psi(\tau)$ is a modular function belonging to the group γ_3. This function is obviously bounded and one-valued in the interior and in those points of the boundary of the fundamental polygon which are situated in the half-plane $I(\tau) > 0$ [the function $f(\tau)$ being a product of ϑ-functions having no zeros in this half-plane]. Therefore, in order to show that it is one-valued and bounded throughout the fundamental polygon, we need

only consider the points $\tau = 0$, $\pm\frac{1}{3}$, ± 1, which the fundamental polygon has in common with the real axis. It follows from the formulae (3.422) and (3.424) for $c = 3$, $\lambda = \pm 1$, and the formulae (3.431), (3.432), and (4.433), that $\Phi(\tau) \to 1$, if τ tends to one of these points. It follows that $\Psi(\tau)$ is bounded throughout the fundamental polygon and therefore, according to Lemma 3.14, it must be a constant, which is obviously one.

3.53. Equating coefficients in the identity $f(\tau) = F(\tau)$, we get

$$r(n) = \frac{\pi^2}{3} nS(n).$$

The sum of the singular series is given by (2.4421). Hence

THEOREM.—*If* $n = 2^\alpha 3^\beta m$ $[(m, 6) = 1]$, *then the number of representations of* n *in the form* $x^2+y^2+3z^2+3t^2$ *is* $4(2^{\alpha+1}-3)\sigma(m)$ *if* $\alpha > 0$ *and* $4\sigma(m)$ *if* $\alpha = 0$.

This result has already been obtained by Liouville by means of elementary methods.

3.6. *The exact formula for* $c = 7$.

3.61. It has been shown in 3.51 that $F(\tau)$ and $f(\tau)$ are not identical if $c = 7$. We may ask whether a simple function $X(\tau)$ can be found, such that $f(\tau) = F(\tau)+X(\tau)$. We shall try to express $X(\tau)$ in the form

$$M\vartheta_2^\alpha(0|\tau)\,\vartheta_3^\beta(0|\tau)\,\vartheta_4^\gamma(0|\tau)\,\vartheta_2^\lambda(0|7\tau)\,\vartheta_3^\mu(0|7\tau)\,\vartheta_4^\nu(0|7\tau),$$

where M is a constant and where α, β, γ, λ, μ, ν are real numbers but not necessary integers. Since the theta functions have no zeros in the half-plane $I(\tau) > 0$, the expression is a one-valued function of τ. We shall first prove that:

If there exists a relation of the form

$$(3.611)\quad f(\tau) = F(\tau)+M\vartheta_2^\alpha(0|\tau)\,\vartheta_3^\beta(0|\tau)\,\vartheta_4^\gamma(0|\tau)\,\vartheta_2^\lambda(0|7\tau)\,\vartheta_3^\mu(0|7\tau)\,\vartheta_4^\nu(0|7\tau)$$
$$= F(\tau)+X(\tau),$$

then $\qquad M = \frac{4}{3}, \quad \alpha = \gamma = \lambda = \nu = \frac{1}{2}, \quad \beta = \mu = 1.$

In order to prove this, we consider the effect of a substitution of the group γ_7 to the three functions $f(\tau)$, $F(\tau)$, and $X(\tau)$. It has been proved in 3.35 that both $f(\tau)$ and $F(\tau)$ will be multiplied by the square of the

denominator of the substitution. Hence

$$\alpha+\beta+\gamma+\lambda+\mu+\nu = 4. \tag{3.612}$$

Next we consider the substitution

$$\tau' = \frac{6\tau+5}{7\tau+6},$$

which belongs to γ_7. This substitution changes a ϑ_2 into a ϑ_4, a ϑ_3 into a ϑ_3, and a ϑ_4 into a ϑ_2. Hence

$$\alpha = \gamma, \quad \lambda = \nu. \tag{3.613}$$

In order to find further relations between the unknown quantities, we consider the expansions of the three functions in power series in q.

We have

$$f(\tau) = \vartheta_3^2(0|\tau)\vartheta_3^2(0|7\tau) = 1+4q+4q^2+0\cdot q^3+\ldots$$

and $\quad \frac{1}{4}(1-\frac{1}{7})\pi^2 F(\tau) = -\frac{1}{7}a(\tau)+a(7\tau) = 1+\frac{4}{3}q+\frac{4}{3}q^2+\frac{16}{3}q^3+\ldots.$

Further, we find by straightforward calculation,

$$X(\tau) = M\cdot 2^{\alpha+\lambda}\exp\frac{\pi i\tau(\alpha+7\lambda)}{4}$$

$$\times[1+2(\beta-\gamma)q+\{2\beta(\beta-1)+2\gamma(\gamma-1)+\alpha-4\beta\gamma\}q^2+\ldots].$$

Since the coefficients of q in the expansions of $F(\tau)$ and $f(\tau)$ are not the same, we must have

$$\alpha+7\lambda = 4. \tag{3.614}$$

Equating coefficients of q, q^2, and q^3 on both sides of

$$f(\tau) = F(\tau)+X(\tau),$$

we find

$$M\cdot 2^{\alpha+\lambda} = \tfrac{8}{3}, \quad \beta-\gamma = \tfrac{1}{2}, \quad 2\beta(\beta-1)+2\gamma(\gamma-1)+\alpha-4\beta\gamma = -2. \tag{3.615}$$

The quantities M, α, β, γ, λ, μ, ν can be determined from the equations (3.612)–(3.615) and we find the values stated in the lemma.

Therefore, *if there exists a relation of the form* (3.611), *then*

$$X(\tau) = \tfrac{4}{3}\sqrt{\{\vartheta_2(0|\tau)\vartheta_3^2(0|\tau)\vartheta_4(0|\tau)\vartheta_2(0|7\tau)\vartheta_3^2(0|7\tau)\vartheta_4(0|7\tau)\}}. \tag{3.616}$$

This can also be written in the form

$$(3.617)\qquad X(\tau) = \tfrac{8}{3}q \prod_{n=1}^{\infty} (1-q^{2n})^2 (1+q^{2n-1})(1-q^{14n})^2 (1+q^{14n-7}),$$

and also

$$(3.618)\qquad X(\tau) = \tfrac{8}{3}\sqrt[3]{2} \cdot \frac{h^2(\tau)\, h^2(7\tau)}{\chi(\tau)\, \chi^2(7\tau)}.$$

Some simple calculations, making use of the transformation formulae for the functions h and χ [T.M. 46, (2), 1° and 5°, (5) and (6)], show that, for a substitution of γ_7,

$$(3.619)\qquad X\left(\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right) = (\gamma\tau+\delta)^2\, X(\tau).$$

3.62. THEOREM.—*If* $c = 7$, *then*

$$f(\tau) = F(\tau)+X(\tau).$$

We consider the function

$$\Phi(\tau) = \frac{F(\tau)+X(\tau)}{f(\tau)}.$$

It follows from Lemma 3.35 and (3.619), that $\Phi(\tau)$ is a modular function belonging to the group γ_7. Therefore, in order to prove the theorem, it is sufficient to show (Lemma 3.14) that the function $\Phi(\tau)$ is bounded throughout the fundamental polygon of γ_7. As in 3.52 the only difficulty is to prove that $\Phi(\tau)$ tends to one, if τ tends to one of the points which the fundamental polygon has in common with the real axis. These points are $\tau = 0,\ \pm 1,\ \pm\frac{1}{7},\ \pm\frac{3}{7},\ \pm\frac{5}{7}$.

3.621. The behaviour of $f(\tau)$ and $F(\tau)$ for $\tau \to 0$ is given by (3.431) and (3.432)

$$(3.6211)\qquad f\left(-\frac{1}{T}\right) = -\tfrac{1}{7}T^2+\ldots, \quad F\left(-\frac{1}{T}\right) = -\tfrac{1}{7}T^2+\ldots.$$

Further, we have [T.M. 45, (5) and (8)]

$$(3.6212)\qquad X\left(-\frac{1}{T}\right) = -\tfrac{8}{21}T^2\sqrt[3]{2} \cdot \frac{h^2(T)\, h^2(\frac{1}{7}T)}{\chi(T)\, \chi(\frac{1}{7}T)} = -\tfrac{8}{21}T^2 \exp \tfrac{1}{7}\pi i T+\ldots$$

From (3.6211) and (3.6212) it follows that $\Phi(\tau) \to 1$ if $\tau \to 0$.

3.622. For $\tau \to \pm 1$, we have from (3.431)

$$(3.6221)\quad \begin{cases} f\left(\pm 1-\frac{1}{T}\right) = -\frac{16}{7}T^2 \exp \frac{4}{7}\pi i T + \ldots, \\ F\left(\pm 1-\frac{1}{T}\right) = -\frac{16}{21}T^2 \exp \frac{2}{7}\pi i T - \frac{32}{21}T^2 \exp \frac{4}{7}\pi i T + \ldots. \end{cases}$$

For $X(\tau)$ we have

$$(3.6222)\quad X\left(\pm 1-\frac{1}{T}\right) = \frac{16}{21}T^2 \exp \frac{2}{7}\pi i T - \frac{16}{21}T^2 \exp \frac{4}{7}\pi i T + \ldots.$$

From (3.6221) and (3.6222) it follows that $\Phi(\tau) \to 1$ if $\tau \to 1$.

3.623. If $\tau \to \frac{1}{7}\lambda - 1/T$ ($\lambda = \pm 1, \pm 3, \pm 5$), we have from (3.422)

$$f\left(\frac{\lambda}{7}-\frac{1}{T}\right) = \frac{16}{49}T^2 \exp\left(\frac{4}{49}\pi i T - \frac{2\lambda\pi i p^2}{7}\right) + \ldots,$$

where $2\lambda p + 1 \equiv 0 \pmod 7$. Since $-2\lambda p^2 \equiv -16\lambda p^2 \equiv 8p \pmod{14}$, we have

$$\exp\left(-\tfrac{2}{7}\lambda\pi i p^2\right) = \exp \tfrac{8}{7}p\pi i,$$

and therefore

$$(3.6231)\quad f\left(\frac{\lambda}{7}-\frac{1}{T}\right) = \frac{16}{49}T^2 \exp\left(\frac{4\pi i T}{49} + \frac{8p\pi i}{7}\right) + \ldots.$$

Further, we get from (3.423),

$$(3.6232)\quad F\left(\frac{\lambda}{7}-\frac{1}{T}\right) = \frac{16}{147}T^2 \exp\left(\frac{2\pi i T}{49} + \frac{2\pi i y}{7}\right) + \frac{32}{147}T^2 \exp\left(\frac{4\pi i T}{49} + \frac{4\pi i y}{7}\right) + \ldots,$$

where y must be determined by the congruence $y\lambda + 1 \equiv 0 \pmod 7$.

The calculation of the analogous expansions for $X(\frac{1}{7}\lambda - 1/T)$ is rather complicated. I write

$$A_0^{(\lambda)} = \sum_{s=0}^{6} \exp \tfrac{1}{7}\lambda(s+\tfrac{1}{2})^2 \pi i, \qquad A_1^{(\lambda)} = \sum_{s=0}^{6} \exp \tfrac{1}{7}\lambda(s+\tfrac{1}{2})^2 \pi i \,.\, \cos \tfrac{1}{7}(2s+1)\pi,$$

$$B_1^{(\lambda)} = \sum_{s=0}^{6} \exp \tfrac{1}{7}\lambda s^2 \pi i \,.\, \cos \tfrac{1}{7}s\pi, \qquad B_3^{(\lambda)} = \sum_{s=0}^{6} \exp \tfrac{1}{7}\lambda s^2 \pi i \,.\, \cos \tfrac{3}{7}\pi s,$$

$$C_0^{(\lambda)} = \sum_{s=0}^{6} (-1)^s \exp \tfrac{1}{7}\lambda s^2 \pi i, \qquad C_2^{(\lambda)} = \sum_{s=0}^{6} \exp \tfrac{1}{7}\lambda s^2 \pi i \,.\, \cos \tfrac{2}{7}\pi s.$$

We use the expression (3 . 616) for $X(\tau)$. Using the formulae (3 . 111)–(3 . 114), we get

$$\vartheta_2^{\frac{1}{2}}\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)=\left(-i\,\frac{T}{49}\right)^{\frac{1}{4}}\sqrt{A_0^{(\lambda)}}\left\{1+\frac{A_1^{(\lambda)}}{A_0^{(\lambda)}}\exp\frac{\pi iT}{49}\right.$$

$$\left.-\tfrac{1}{2}\left(\frac{A_1^{(\lambda)}}{A_0^{(\lambda)}}\right)^2\exp\frac{2\pi iT}{49}+\ldots\right\},$$

$$\vartheta_3\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)=2\left(-i\,\frac{T}{49}\right)^{\frac{1}{2}}\exp\frac{\pi iT}{96}\,.\,B_1^{(\lambda)}\left(1+\frac{B_3^{(\lambda)}}{B_1^{(\lambda)}}\exp\frac{2\pi iT}{49}+\ldots\right),$$

$$\vartheta_4^{\frac{1}{2}}\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)=\left(-i\,\frac{T}{49}\right)^{\frac{1}{4}}\sqrt{C_0^{(\lambda)}}\left\{1+\frac{C_2^{(\lambda)}}{C_0^{(\lambda)}}\exp\frac{\pi iT}{49}\right.$$

$$\left.-\tfrac{1}{2}\left(\frac{C_2^{(\lambda)}}{C_0^{(\lambda)}}\right)^2\exp\frac{2\pi iT}{49}+\ldots\right\},$$

$$\vartheta_2^{\frac{1}{2}}\left\{0\,\middle|\,7\left(\frac{\lambda}{7}-\frac{1}{T}\right)\right\}=\exp\frac{\lambda\pi i}{8}\,.\left(-i\,\frac{T}{7}\right)^{\frac{1}{4}}\left(1-\exp\frac{\pi iT}{7}+\ldots\right),$$

$$\vartheta_3\left\{0\,\middle|\,7\left(\frac{\lambda}{7}-\frac{1}{T}\right)\right\}=2\left(-i\,\frac{T}{7}\right)^{\frac{1}{2}}\exp\frac{\pi iT}{28}\left(1+\exp\frac{\pi iT}{7}+\ldots\right),$$

$$\vartheta_4^{\frac{1}{2}}\left\{0\,\middle|\,7\left(\frac{\lambda}{7}-\frac{1}{T}\right)\right\}=\left(-i\,\frac{T}{7}\right)^{\frac{1}{4}}\left(1+\exp\frac{\pi iT}{7}+\ldots\right).$$

By multiplying these equations, we get the expansion for $X(\lambda/7-1/T)$. However, in the first and third of the formulae there occur the quantities $\sqrt{A_0^{(\lambda)}}$ and $\sqrt{C_0^{(\lambda)}}$, and it is not obvious which of the values of the square roots must be taken. To settle this question we use the identity

$$\vartheta_2^{\frac{1}{2}}(0\,|\,\tau)\,\vartheta_4^{\frac{1}{2}}(0\,|\,\tau)=\sqrt[3]{2}\,.\,h(\tau)\,\chi(\tau).$$

Therefore we have

$$\vartheta_2^{\frac{1}{2}}\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)\,\vartheta_4^{\frac{1}{2}}\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)$$

$$=-\sqrt[3]{2}\,.\,\frac{\sqrt{-iT}}{\sqrt{7}}\exp\frac{7\pi i\lambda}{8}\,.\,\frac{\chi(\frac{1}{7}T)\,h(\frac{1}{7}T)}{\phi(\frac{1}{7}T)}\prod_{r=1}^{3}\frac{\vartheta_1(\frac{1}{49}rT\lambda-\frac{1}{7}r\,|\,\frac{1}{7}T)}{\vartheta_2(\frac{1}{49}rT\lambda-\frac{1}{7}r\,|\,\frac{1}{7}T)}.$$

It is easy to see that the product $\prod_{r=1}^{3}$ tends to $-i$, $-i$, $+i$, $-i$, $+i$, $+i$ respectively if $\lambda=-5$, -3, -1, $+1$, $+3$, $+5$, that is to say to the limit $-i(\lambda/7)$ [where $(\lambda/7)$ is the symbol of Legendre]. Therefore the

first term in the expansion of $\vartheta_2^{\frac{1}{2}}\vartheta_4^{\frac{1}{2}}$ is given by

$$\vartheta_2^{\frac{1}{2}}\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)\vartheta_4^{\frac{1}{2}}\left(0\,\middle|\,\frac{\lambda}{7}-\frac{1}{T}\right)=i\,\frac{\sqrt{(-iT)}}{\sqrt{7}}\exp\frac{7\pi i\lambda}{8}\cdot\left(\frac{\lambda}{7}\right)+\ldots.$$

Multiplying the six formulae obtained above, we now get

$$X\left(\frac{\lambda}{7}-\frac{1}{T}\right)=\tfrac{16}{147}T^2\cdot i\left(\frac{\lambda}{7}\right)\cdot\frac{B_1^{(\lambda)}}{\sqrt{7}}\exp\frac{2\pi iT}{49}\left[1+\left(\frac{A_1^{(\lambda)}}{A_0^{(\lambda)}}+\frac{C_2^{(\lambda)}}{C_0^{(\lambda)}}\right)\exp\frac{\pi iT}{49}\right.$$

$$\left.+\left\{\frac{B_3^{(\lambda)}}{B_1^{(\lambda)}}-\tfrac{1}{2}\left(\frac{A_1^{(\lambda)}}{A_0^{(\lambda)}}\right)^2-\tfrac{1}{2}\left(\frac{C_2^{(\lambda)}}{C_0^{(\lambda)}}\right)^2+\frac{A_1^{(\lambda)}C_2^{(\lambda)}}{A_0^{(\lambda)}C_0^{(\lambda)}}\right\}\exp\frac{2\pi iT}{49}+\ldots\right].$$

Now we have

$$A_0^{(\lambda)}=C_0^{(\lambda)}\exp\frac{7\pi i\lambda}{4},\quad A_1^{(\lambda)}=C_2^{(\lambda)}\exp\frac{3\pi i\lambda}{4},\quad \frac{B_3^{(\lambda)}}{B_1^{(\lambda)}}=\left(\frac{C_2^{(\lambda)}}{C_0^{(\lambda)}}\right)^2=\exp\frac{4p\pi i}{7},$$

where $2\lambda p+1\equiv 0 \pmod 7$.

Further (see 3.42)

$$B_1^{(\lambda)}=\exp\frac{\lambda p^2+p}{7}\pi i\cdot\left(\frac{\lambda}{7}\right)i\sqrt{7}=\exp\frac{4p\pi i}{7}\cdot\left(\frac{\lambda}{7}\right)i\sqrt{7}.$$

Hence

$$(3.6233)\quad X\left(\frac{\lambda}{7}-\frac{1}{T}\right)=-\tfrac{16}{147}T^2\exp\left(\frac{2\pi iT}{49}+\frac{4p\pi i}{7}\right)$$

$$+\tfrac{16}{147}T^2\exp\left(\frac{4\pi iT}{49}+\frac{8p\pi i}{7}\right)+\ldots.$$

It now follows again from (3.6231)–(3.6233), that $\Phi(\tau)\to 1$, if

$$\tau\to\frac{\lambda}{7}-\frac{1}{T}\quad(\lambda=\pm 1,\ \pm 3,\ \pm 5),$$

and the theorem is proved.

3.63. Equating coefficients in the relation $f(\tau)=F(\tau)+X(\tau)$, we find

$$r(n)=\frac{\pi^2}{7}nS(n)+\nu_n,$$

where ν_n is the coefficient of q^n in $X(\tau)$. Substituting the sum of the singular series as given by (2.4421), we find

$$r(n)=\tfrac{4}{3}(2^{\alpha+1}-3)\sigma(m)+\nu_n\quad\text{if}\quad\alpha>0,$$

$$=\tfrac{4}{3}\sigma(m)+\nu_n\quad\text{if}\quad\alpha=0.$$

This is the exact formula for $c=7$.

3.7. *Exact formulae for* $c=5$ *and* $c=6$.

The methods explained in the preceding sections can be applied to other values of c. For $c=2, 4, 8$ Liouville's formulae can be obtained.

For $c=5$ and $c=6$, I give the results only. The proofs in these cases are very complicated.

For $c=5$ is

$$F(\tau)=f(\tau)+\tfrac{8}{3}h^2(\tau)h^2(5\tau), \tag{3.71}$$

from which it follows that [see (2.4411)]

$$\begin{aligned} r(n) &= 2(5^{\beta+1}-3)\sigma(m) && \text{if} \quad \alpha>0,\\ &= \tfrac{2}{3}(5^{\beta+1}-3)\sigma(m)+\tfrac{8}{3}\eta_n && \text{if} \quad \alpha=0, \end{aligned}$$

where η_n is the coefficient of q^n in

$$q\prod_{m=1}^{\infty}(1-q^{2m})^2(1-q^{10m})^2.$$

The relation (3.71) can also be written in the form

$$\vartheta_3^2(0|\tau)\vartheta_3^2(0|5\tau)=\frac{2i}{3\pi}\frac{d}{d\tau}\log\frac{\vartheta_4(0|\tau)\vartheta_4(0|5\tau)}{\vartheta_2(0|\tau)\vartheta_2(0|5\tau)}+\tfrac{8}{15}\vartheta_1^2(\tfrac{1}{5}|\tau)\vartheta_1^2(\tfrac{2}{5}|\tau).$$

For $c=6$ is

$$F(\tau)=f(\tau)+2h(\tau)h(2\tau)h(3\tau)h(6\tau), \tag{3.72}$$

whence [see (2.4223)]

$$\begin{aligned} r(n) &= 2\sigma(m)+2\xi_n && \text{if} \quad \alpha=0,\\ &= 4\sigma(m) && \text{if} \quad \alpha=1,\\ &= 4(2^{\alpha}-3)\sigma(m) && \text{if} \quad \alpha\geqslant 2, \end{aligned}$$

where ξ_n is the coefficient of q^n in

$$q\prod_{m=1}^{\infty}(1-q^{2m})(1-q^{4m})(1-q^{6m})(1-q^{12m}).$$

The relation (3.72) can also be written in the form

$$\vartheta_3^2(0|\tau)\vartheta_3^2(0|6\tau)=\frac{i}{\pi}\frac{d}{d\tau}\log\frac{\vartheta_4(0|\tau)\vartheta_2(0|2\tau)}{\vartheta_4(0|3\tau)\vartheta_2(0|6\tau)}+\tfrac{1}{3}\vartheta_2(0|\tau)\vartheta_1^2(\tfrac{1}{3}|\tau)\vartheta_2(\tfrac{1}{3}|\tau).$$

THE TRITANGENT PLANES OF THE SAME SYSTEM OF THE CURVE OF INTERSECTION OF A CUBIC SURFACE AND A QUADRIC CONE.

By WILLIAM P. MILNE.

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1. *Introduction.*

The subject of the sextactic quadrics and tritangent planes of the same system of a quadri-cubic curve was dealt with from two entirely different standpoints by Roth, *Monatsh. Math. Phys.*, 22 (1911), 64-88, and by myself, *Proc. London Math. Soc.* (2), 21 (1922), 373-380. In the present paper, I propose to discuss the particular case of a quadri-cubic curve when the quadric degenerates into a cone. This curve derives its interest from the fact that it is the normal curve in space of three dimensions for the genus $p = 4$, when the two-dimensional curve of degree n possesses an ∞^1 system of adjunct ϕ_{n-3}-curves which meet the given curve in two coincident points at $p-1$ places. The normal curve in space of two dimensions for $p = 4$ is in general a quintic curve with two nodes, but in the above special case the two nodes coalesce into a simple tac-node or point, at which the quintic touches itself. The ϕ-curves are then conics that touch the quintic at its tac-node, and the ∞^1 ϕ-curves meeting the quintic in two coincident points at each of three further points are the ∞^1 straight lines drawn through the singular point, each reckoned twice over. If a point on the quadri-cubic be taken in the case when the quadric is a cone, and the curve be projected on to a plane, it is evident that the tangent plane to the cone at that point cuts the plane of the projection in the tangent line to the tac-node [L. Kraus, *Math. Annalen*, 16 (1880), 245-259].

It is found that the configuration of the tritangent planes of the same system of a quadri-cubic curve, when the quadric is a cone, differs in a

very marked way from the general case when the quadric is undegenerate. In the paper above cited I found that, of the 255 systems of sextactic quadrics, any one system contains 28 quadrics, which each degenerate into pairs of planes constituting the 28 pairs of tritangent planes of the same system, and these 28 plane-pairs correspond to the 28 bitangents of a general plane quartic curve. In the case when the quadri-cubic is the curve of intersection of a cubic surface and a quadric cone, it is found that the quartic degenerates into a conic counted twice over, with eight branch-points situated upon it. This corresponds to the hyperelliptic case of the normal curve for $p = 3$.

2. *Tritangent planes of the same system.*

It is known that the points of contact of sextactic quadrics of the same system can be regarded as those of enveloping-cones of the second degree drawn from the points of a quadri-nodal cubic surface to itself. If we regard this quadri-nodal cubic surface Γ_3' as the polo-surface of a plane γ with respect to a cubic surface C_3, *i.e.* as the envelope of the polar planes with respect to C_3 of points on γ, we see that the sextactic cones are the cones U_2 enveloped by the polar planes of points lying on the straight lines l contained in γ. If the quadri-cubic curve be the curve of intersection of the quadric cone Γ_2 and the cubic surface Γ_3', we have to find those straight lines l corresponding to which the cones U_2 touch Γ_2 at two distinct points. The curve of intersection of U_2 and Γ_2 will then be two conics, the planes of which will each be tritangent planes of the quadri-cubic curve.

Let O be the vertex of the cone Γ_2. Then the locus of points in γ, whose polar planes with respect to C_3 pass through O, is the conic s_2, which is the curve of intersection of γ with the polar quadric of O with respect to C_3. Consider, also, the locus of points in γ whose polar planes with respect to C_3 touch the conic γ_2, which is the curve of intersection of γ and Γ_2. This is plainly the locus of points whose polar planes with respect to the curve of intersection of C_3 and γ touch the conic γ_2, and is therefore a quartic curve q_4. The polar planes of the eight points of intersection of s_2 and q_4 with respect to C_3 therefore each touch the cone Γ_2 along a generator. Let the eight points of intersection of s_2 and q_4 be A_1, A_2, &c. Consider the line l_{rs} joining $A_r A_s$. Let the polar plane of A_r with respect to C_3 touch Γ_2 along the generator a_r. Then the polar planes of points on l_{rs} with respect to C_3 envelop a quadric cone touching

Γ_2 at two points lying on a_r and a_s. There are ${}_8C_2 = 28$ such lines l_{rs} joining the points A in pairs, and hence there are 28 sextactic cones of the same system which have double contact with Γ_2.

We have, therefore, the following general result :—

There are 28 *pairs of tritangent planes of the same system in the case of a quadri-cubic curve, which is the complete intersection of a cubic surface with a quadric cone. The lines of intersection of these pairs of tritangent planes cut the given cone in points lying seven by seven on eight given generators of the cone.*

Plainly, the conic s_2, reckoned twice over, and the eight points A lying upon it constitute the hyperelliptic case of the general quartic curve when Γ_2 does not degenerate into a quadric cone.

SOME REMARKS ON DIRICHLET'S SERIES

By EINAR HILLE.

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1. The main purpose of the present paper is to derive a set of sufficient conditions that the sum of a Dirichlet's series shall be an entire function.

A necessary and sufficient condition for this can be read off from the analytic representation in the principal star given by M. Riesz.* But this test is rather laborious to handle, and supplementary results along different lines seem desirable. A sufficient condition that an ordinary Dirichlet's series shall represent an entire function was obtained by Hardy in connexion with his study of the application of Borel's method of summation to Dirichlet's series.† The present paper contains similar conditions for general Dirichlet's series obtained from the Cahen-Perron integral.

2. Let the Dirichlet's series

$$D(s;\ \lambda) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s} \quad (0 < \lambda_n < \lambda_{n+1}, \quad \lambda_n \to \infty) \tag{1}$$

be convergent for $\sigma > \sigma_0$ $(s = \sigma + it)$ where $0 \leqslant \sigma_0 < \infty$. Our starting point is the formula‡

$$D(s;\ \lambda) = \frac{1}{\Gamma(s/\nu)} \int_0^{\infty} x^{(s/\nu)-1} D(x;\ l^{\nu})\, dx \tag{2}$$

where $\nu > 0$ and

$$D(u;\ l^{\nu}) = \sum_{n=1}^{\infty} a_n e^{-l_n^{\nu} u} = \sum_{n=1}^{\infty} a_n e^{-e^{\nu\lambda_n} u}. \tag{3}$$

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* *Acta Math.*, 35 (1911), 253–270 (266).

† *Proc. London Math. Soc.* (2), 8 (1910), 277–294 (290). See also Fekete, "Sur les séries de Dirichlet", *Comptes rendus*, 25 April, 1910.

‡ Perron, *Journal für Math.*, 134 (1908), 95–143 (138). The case $\nu = 1$ was considered by Cahen in his thesis. In the following the right-hand side of formula (2) will be referred to as the *Cahen-Perron integral*.

The representation is valid at least for $\sigma > \sigma_0$. The series (3) converges at least for $x > 0$, $u = x+iy$, and the function $D(u\,;\,l^\nu)$ is of finite order on the imaginary axis. In fact, Perron found that§

$$|D(x\,;\,l)| < 3C\Gamma(a+1)\left(\frac{2}{x}\right)^a e^{-\frac{1}{2}l_1 x}$$

where $a = \sigma_0+\epsilon$ and

$$C = \text{Max}\,|C_n|, \quad C_n = \sum_{k=1}^{n} a_k e^{-\lambda_k a}.$$

By exactly the same method as was used by Perron we find that

$$|D(u\,;\,l^\nu)| < C\Gamma(\tau+1)\left(1+2\frac{|u|}{x}\right)\left(\frac{2}{x}\right)^\tau e^{-\frac{1}{2}l_1^\nu x}, \tag{4}$$

where $\tau = \alpha/\nu$.

3. Let r denote a positive quantity, and write

$$\int_0^\infty x^{(s/\nu)-1} D(x\,;\,l^\nu)\,dx = \int_0^r + \int_r^\infty = F_\nu(s\,;\,r)+E_\nu(s\,;\,r). \tag{5}$$

The second integral converges uniformly as long as $|s|$ remains bounded, since

$$|E_\nu(s\,;\,r)| \leqslant |E_\nu(\sigma\,;\,r)| < 3C\Gamma(\tau+1)\,2^\tau \int_r^\infty x^{(\sigma/\nu)-\tau-1} e^{-\frac{1}{2}l_1^\nu x}\,dx. \tag{6}$$

For $\sigma \geqslant \alpha+(\nu/4)$, say, the integral is less than

$$\Gamma\left(\frac{\sigma}{\nu}-\tau\right)\left(\frac{4}{l_1^\nu}\right)^{(\sigma/\nu)-\tau} e^{-\frac{1}{2}l_1^\nu r},$$

in view of Perron's lemma. For $\sigma < \alpha+(\nu/4)$, Schwarz's inequality shows that the integral is less than

$$\left[2\left(\frac{\sigma}{\nu}-\tau\right)-1\right]^{-\frac{1}{2}} l^{(\sigma/\nu)-\tau-\frac{1}{2}}\, l^{-(\nu/2)} e^{-\frac{1}{2}l_1^\nu r},$$

---

* *Loc. cit.*, 136, formula (56), which, however, contains an additional term $Cl^a \exp(-l_1 x)$. This expression is an upper limit for the term $-C_N l_{N+1}^a \exp(-l_{N+1} u)$, obtained by a partial summation. For $N = 0$, which is the actual case, this term drops out since $C_0 = 0$ by definition.

which two inequalities together prove the statement. Hence $E_\nu(s\,;\,r)$ is an entire function of s.

Formula (4) shows that the first integral in (5) converges for $\sigma > \alpha$, and that the function $u^{-1}D(u\,;\,l^\nu)$, which is holomorphic in $|u-r|<r$, is of finite order, $\omega+1$ say, on the bounding circle. Hence $F_\nu(s\,;\,r)$ can be expanded in a factorial series in s/ν convergent for $\sigma > \nu\omega^*$, namely

$$(7) \qquad F_\nu(s\,;\,r) = r^{s/\nu} \sum_{n=0}^{\infty} (-1)^n \frac{r^n D^{(n)}(r\,;\,l^\nu)}{s/\nu\{(s/\nu)+1\}\ldots\{(s/\nu)+n\}}.$$

Thus we obtain the formula†

$$(8) \quad D(s\,;\,\lambda) = \frac{1}{\Gamma(s/\nu)} \int_r^\infty x^{(s/\nu)-1} D(x\,;\,l^\nu)\,dx + r^{(s/\nu)} \sum_{n=0}^{\infty} (-1)^n \frac{r^n D^{(n)}(r\,;\,l^\nu)}{\Gamma\{(s/\nu)+n+1\}},$$

valid for $\sigma > \nu\omega$.

4. In order to obtain an upper limit for ω we shall use a theorem due to Nörlund,‡ namely:

Let $f(z)$ be holomorphic in the interior of the unit circle on which (α, β) is an arc containing $z = 1$. Suppose a number k greater than or equal to 1 exists, such that

$$|f(z)|\,(1-|z|)^{k-1} \quad |1-z|$$

remains less than a fixed constant in the region

$$0 \leqslant |z| < 1, \quad \alpha \leqslant \arg z \leqslant \beta.$$

Then the order of $f(z)$ on the arc (α, β) is less than or equal to k.

We shall apply this theorem to the function $u^{-1}D(u\,;\,l^\nu)$. We shall try to determine a k, such that

$$|D(u\,;\,l^\nu)|\,[r-|u-r|]^{k-1}$$

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* Provided $\omega > 0$. See Nörlund, "Sur les séries de facultés", *Acta Math.*, 37 (1914), 327–387 (351).

† For ordinary binomial series a similar representation has been given by S. Wigert, *Arkiv för Mat.*, 7 (26) (1911), 1-12 (7). The reading of this paper gave me the idea for the present investigation in 1917. Formula (8) for $\nu = 1$, with the restrictive assumption that the series (1) should possess a half-plane of absolute convergence, was obtained at that time. The theorems **A**–**C** below in §§ 6–7 have obvious analogues for binomial series.

‡ *Loc. cit.*, 337, Theorem H.

remains bounded for $|u-r|<r$. Then $\omega+1 \leqslant k$, provided $k \geqslant 1$. In view of (4) we can determine a positive M_ν, independent of u, but not of ν, such that

$$|D(u;\ l^\nu)| < M_\nu |u| x^{-\tau-1} \quad (x>0).$$

If $|u-r|<r$, $|u|^2<2rx$; thus the inequality can be replaced by

$$|D(u;\ l^\nu)| < M_\nu \sqrt{(2r)}\, x^{-\tau-\frac{1}{2}} = M_\nu \sqrt{(2r)}(r+\rho\cos\theta)^{-\tau-\frac{1}{2}} \tag{9}$$

on the circle $|u-r|=\rho<r$. Hence

$$\begin{aligned}|D(u;\ l^\nu)|(r-\rho)^{k-1} &< M_\nu \sqrt{(2r)}(r+\rho\cos\theta)^{-\tau-\frac{1}{2}}(r-\rho)^{k-1} \\ &\leqslant M_\nu \sqrt{(2r)}(r-\rho)^{k-\tau-\frac{3}{2}}.\end{aligned}$$

Consequently, we can take $k=\tau+\frac{3}{2}$. Hence

$$\omega \leqslant \tau+\tfrac{1}{2} = \frac{\sigma_0+\epsilon}{\nu}+\tfrac{1}{2}$$

for every positive ϵ. We conclude that the expansion (8) holds for $\sigma > \sigma_0+(\nu/2)$. Thus, by taking ν sufficiently small, we obtain an expansion holding for every value of s in the half-plane $\sigma \geqslant \sigma_0+\delta$, δ fixed but arbitrarily small.

The factorial series (7) is absolutely convergent for $\sigma > \sigma_0+\nu$. This follows from a direct estimate of the growth of $|r^n D^{(n)}(r;\ l^\nu)|$ with n, which we shall give, assuming $r=1$, $\tau \geqslant \frac{1}{2}$; neither of these assumptions restricts the generality. We have, using (9),

$$|D^{(n)}(1;\ l^\nu)| = \left|\frac{n!}{2\pi i}\int_{|u-1|=\rho}\frac{D(u;\ l^\nu)}{(u-1)^{n+1}}\,du\right| \leqslant \frac{n!\ \sqrt{2}\,M_\nu}{2\pi\rho^n}\int_0^{2\pi}\frac{d\theta}{(1+\rho\cos\theta)^{\tau+\frac{1}{2}}}$$

$$\leqslant \frac{n!}{2\pi}\sqrt{2}\,M_\nu\rho^{-n}(1-\rho)^{-\tau+\frac{1}{2}}\int_0^{2\pi}\frac{d\theta}{1+\rho\cos\theta}$$

$$= n!\ \sqrt{2}\,M_\nu\rho^{-n}(1+\rho)^{-\frac{1}{2}}(1-\rho)^{-\tau} < n!\ \sqrt{2}\,M_\nu\rho^{-n}(1-\rho)^{-\tau}.$$

Putting $\rho=1-(1/n)$, $n\geqslant 2$, we obtain finally

$$|D^{(n)}(1;\ l^\nu)| < \sqrt{2}\,M_\nu\left(1-\frac{1}{n}\right)^{-n} n^\tau n! < 4\sqrt{2}\,M_\nu n^\tau n!. \tag{10}$$

Since

$$\frac{n^\tau n!}{|\Gamma\{(s/\nu)+n+1\}|} < \text{const}\, n^{\tau-(\sigma/\nu)},$$

the series (7) converges absolutely for $\sigma > \sigma_0+\nu+\epsilon$, $\epsilon \geqslant 0$.

5. The assumption that the series (1) possesses a half-plane of convergence can be replaced by weaker assumptions, *e.g.* by :—

(i) The series (3) converges for $x > x_0 \geqslant 0$.

(ii) $D(u\,;\,l^\nu)$ is holomorphic on the positive real axis.

(iii) There exists an $r > 0$ such that $D(u\,;\,l^\nu)$ is holomorphic in $|u-r| < r$, and $u^{-1}D(u\,;\,l^\nu)$ is of finite order k on the bounding circle.

If these conditions are fulfilled for some positive ν, the divergent Dirichlet's series for $D(s\,;\lambda)$ is summable by the Cahen-Perron integral for $\sigma > \nu(k-1)$, at least, and the representation (8) is valid in the same half-plane. In the following we shall assume the Dirichlet's series, whether convergent or divergent, to be given by the Cahen-Perron integral, and the corresponding function $D(u\,;\,l^\nu)$ shall satisfy the conditions (i-iii).

It is obvious that if an $r = r_0$ exists for which condition (iii) is satisfied, then any positive $r < r_0$ will do just as well. It would be interesting to know whether the existence of a particular $\nu = \nu_0$, for which all the conditions are satisfied, implies the existence of other such ν values. The values $\nu > \nu_0$ seem promising.

6. Formula (8) is not of much interest if it does not yield an analytic continuation of $D(s\,;\lambda)$. The first question which arises is whether the series (7) can be convergent for all values of s in special cases. This will evidently happen if $D(u\,;\,l^\nu)$ is holomorphic at the origin.* In that event we can choose r so small that $D(u\,;\,l^\nu)$ is holomorphic in $|u-r| \leqslant R$ where $R > r$. Then

$$r^n |\, D^{(n)}(r\,;\,l^\nu)| < n!\left(\frac{r}{R}\right)^n,$$

and the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma\{(s/\nu)+n+1\}}\left(\frac{r}{R}\right)^n$$

is convergent for all values of s. Thus we obtain the theorem :—

A. *If $D(u\,;l^\nu)$ is holomorphic at $u = 0$ for some positive ν, then $D(s\,;\lambda)$*

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* Cf. Pincherle, "Sur les fonctions déterminantes", *Annales Sci. de l'École Normale* (3), 22 (1905), 9-68 (50), where further references are given. See also Hurwitz: "Ueber die Anwendung eines functionentheoretischen Principes auf gewisse bestimmte Integrale", *Math. Annalen*, 53 (1900), 220-224. This paper should be consulted in connexion with Theorem **C** as well.

is an entire function which is represented by formula (8) *for all values of* s.

For $\lambda_n = \log n$, $\nu = 1$, this theorem gives Hardy's result quoted in the introduction. But the condition of Theorem **A** is far from being necessary in order that the factorial series (7) shall be convergent for all values of s. The abscissa of convergence of the series, if negative, is given by

$$\sigma' = \overline{\lim} \frac{\log |R_{n+1}|}{\log n}, \qquad R_n = \sum_{m=n}^{\infty} (-1)^m \frac{D^{(m)}(r\,;\, l^\nu)}{m!} r^m. \tag{11}$$

In order that formula (8) shall be valid for all values of s we must have $\sigma' = -\infty$, which requires

$$|R_n| = n^{-\alpha_n} \text{ where } \alpha_n \to +\infty \text{ with } n \quad (n \geqslant 2). \tag{12}$$

It is of some interest to formulate this condition differently.

B_1. *A necessary and sufficient condition in order that the formula* (8) *shall be valid for all values of* s *is that the series*

$$\sum_{n=k}^{\infty} \frac{D^{(n)}(r\,;\, l^\nu)}{(n-k)!} (u-r)^{n-k} = D^{(k)}(u\,;\, l^\nu) \text{ for } |u-r| < r \tag{13}$$

shall be convergent for $u = 0$, *for* $k = 0, 1, 2, \ldots$.

In fact, if $\sigma' = -\infty$ condition (12) shows that for $n > 2$

$$\left| \frac{D^{(n)}(r\,;\, l^\nu)}{n!} (-r)^n \right| \leqslant n^{-\alpha_n} + (n-1)^{-\alpha_{n-1}}$$

and

$$\frac{|D^{(n)}(r\,;\, l^\nu)|}{(n-k)!} r^{n-k} \leqslant \frac{n!}{(n-k)!} [n^{-\alpha_n} + (n-1)^{-\alpha_{n-1}}] r^{-k} < C_k (n-1)^{-\alpha_{n-1}+k} r^{-k},$$

if we assume that $\alpha_{n-1} \leqslant \alpha_n$. Since $\alpha_{n-1} > k+2$ from a certain point on, the series (13) is convergent for $u = 0$ and for every k. Conversely, if all the series (13) are convergent for $u = 0$ we have, for a fixed k, arbitrarily large

$$\frac{|D^{(n)}(r\,;\, l^\nu)|}{(n-k)!} r^{n-k} < 1,$$

if $n > n(k)$. Hence

$$\frac{|D^{(n)}(r\,;\, l^\nu)|}{n!} r^n < \frac{(n-k)!}{n!} r^k,$$

and

$$|R_m| < r^k \sum_{n=m}^{\infty} \frac{(n-k)!}{n!} < r^k c_k \sum_{n=m}^{\infty} \frac{1}{n^k} < \frac{2r^k c_k}{(k-1)m^{k-1}},$$

if $m > m_0$. Consequently

$$\overline{\lim} \frac{\log |R_{m+1}|}{\log m} < -k+1.$$

But k is arbitrarily large, hence $\sigma' = -\infty$.

Using a terminology due to Hardy and Littlewood,* we can give the following paraphrase of Theorem $\mathbf{B}_1$, namely :—

$\mathbf{B}_2$. *A necessary and sufficient condition in order that formula* (8) *shall be valid for all values of s is that the series*

$$\sum_{n=1}^{\infty} a_n l_n^k \quad (k = 0, 1, 2, \ldots), \tag{14}$$

shall be summable by the circle method of type l^ν with radius r.

7. The results of § 6 show that the existence of derivatives of all orders of $D(u\,; l^\nu)$ at the origin is of importance for our problem. As a matter of fact, we have the following more general theorem :—

C. *A sufficient condition in order that the sum of $D(s\,;\lambda)$ shall be an entire function is that $D^{(n)}(x\,;\, l^\nu)$ shall tend to a finite limit when $x \to 0$ through positive values for $n = 0, 1, 2, \ldots$.*

The proof follows directly from results due to Pincherle.† Suppose that

$$\lim_{x \to +0} D^{(n)}(x\,;\, l^\nu) = c_n \quad (n = 0, 1, 2, \ldots). \tag{15}$$

Then for $0 \leqslant x \leqslant r$ we have

$$D(x\,;\, l^\nu) = \sum_{n=0}^{k} c_n x^n + \frac{x^{k+1}}{k!} \int_0^1 D^{(k+1)}(\theta x\,;\, l^\nu)(1-\theta)^k d\theta. \tag{16}$$

Assuming temporarily $\sigma > 0$, and substituting in the integral for $F_\nu(s\,;\, r)$, we obtain

$$F_\nu(s\,;\, r) = r^{s/\nu} \sum_{n=0}^{k} \frac{c_n r^n}{(s/\nu)+n} + \frac{1}{k!} \int_0^r x^{(s/\nu)+k} \left\{ \int_0^1 D^{(k+1)}(\theta x\,;\, l^\nu)(1-\theta)^k d\theta \right\} dx. \tag{17}$$

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* *Rend. di Palermo*, 41 (1916), 36–53 (50). The notion of *type* in this connexion is not emphasized by these writers, who had no reason to do so in their investigation.

† *Loc. cit.*, 25–26.

The integral obviously converges for $\sigma > -(k+1)\nu$. In view of the factor $1/\Gamma(s/\nu)$ in the Cahen-Perron integral we conclude that $D(s\,;\lambda)$ is holomorphic in the half-plane $\sigma > -(k+1)\nu$. The integer k being arbitrarily large, it follows that $D(s\,;\lambda)$ is an entire function.

The same conclusion is evidently valid, if the condition (15) is fulfilled only for $n = 0, 1, 2, \ldots, k_\nu+1$, where $k_\nu \geqq k$ for $\nu \geqq \nu_0$.

If the condition (15) is fulfilled for all values of n, we can transform expression (5) by means of repeated integrations by parts operating on the integral for $F_\nu(s\,;r)$. After k such integrations we obtain the formula

$$(18)\quad D(s\,;\lambda) = \frac{1}{\Gamma(s/\nu)}\int_r^\infty x^{(s/\nu)-1} D(x\,;\,l^\nu)\,dx + r^{s/\nu}\sum_{n=0}^{k}(-1)^n \frac{r^n D^{(n)}(r\,;\,l^\nu)}{\Gamma\{(s/\nu)+n+1\}}$$
$$+(-1)^{k+1}\frac{1}{\Gamma\{(s/\nu)+k+1\}}\int_0^r x^{(s/\nu)+k} D^{(k+1)}(x\,;\,l^\nu)\,dx,$$

valid for $\sigma > -(k+1)\nu$, k being an arbitrary positive integer.* If it is permitted to modify suitably the path of integration in the remainder term in (18), this formula will yield an asymptotic representation of $D(s\,;\lambda)$ in a sector $-\alpha \leqq \arg s \leqq +\beta$, $\frac{1}{2}\pi \leqq \alpha$, $\beta < \pi$.

* Cf. Pincherle, *loc. cit.* 27.

ON SOME CASES OF TWO-DIMENSIONAL FLUID MOTIONS

By D. RIABOUCHINSKY.

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1. In the numerous researches on discontinuous motions that have been undertaken these last fifty years, it has been generally admitted, according to the initial scheme of Helmholtz and Kirchhoff, that an unlimited mass of dead water follows the obstacle. The region occupied by the moving fluid being then simply connected, the velocity-potential is necessarily always a single-valued function.

It ceases to be so if one considers finite dead spaces, limited by surfaces of constant pressure and those of the obstacles; the velocity-potential may then be a cyclic function.

The methods which allow us to obtain the solutions of such problems are equally applicable to numerous other forms of fluid motion which can be reduced to a problem of conformal representation between two areas bounded by straight lines or circular arcs.

Fig. 1 gives some typical problems* that I have studied in previous papers.†

2. The motions shown in Fig. 1 have at least one axis of symmetry ($x = 0$ or $y = 0$, say) that coincides either with equipotentials or with stream-lines. Hence for these motions the equation

$$\phi + i\psi = f(x + iy)$$

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* The last of these problems has been studied, by a different method, by Hicks, *Quart. J. of Math.*, 17 (1881), 194–202.

† *Proc. London Math. Soc.* (2), 19 (1920), 206–215. *Thèse* (Gauthier-Villars, Paris, 1922), *Comptes rendus*, 4 July, 1921; 7 November, 1921; 23 January, 1922; 8 May, 1922, *Comptes rendus du Congrès International des Mathématiciens*, Strasbourg (1920). *Association Française pour l'avancement des Sciences, Comptes rendus du Congrès de Montpellier* (1922).

is equivalent to one of the formulae

$$\phi+i|\psi|=f(x+i|y|), \quad |\phi|+i\psi=f(|x|+iy).$$

It is this remark that led me, by studying the properties of the function $|x|$, to consider the first of the discontinuous motions shown in Fig. 1, and to use the symbols $y\div|y|$, $\psi\div|\psi|$. These symbols may, however, be omitted in my formulae without altering the course of calculation, and

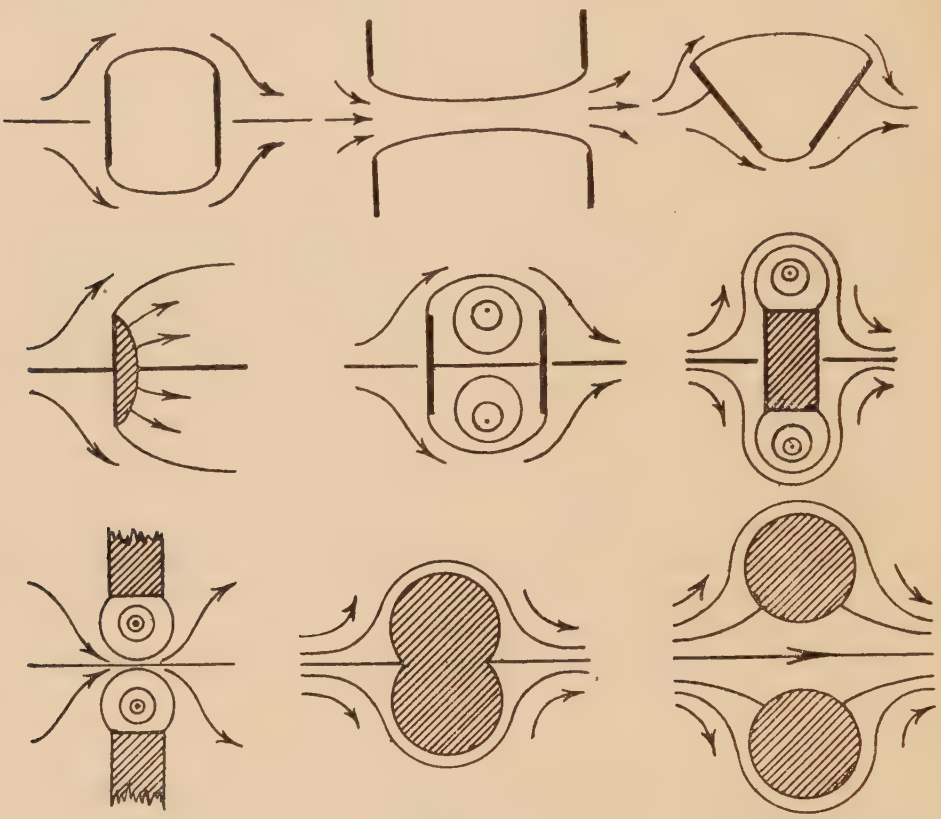

FIG. 1.

the results are not changed if we agree to consider all the points of the axis of symmetry to belong to the half plane which is first taken into account.

3. In a recent paper Mr. Poole* has criticized my solution of the third problem of Fig. 1 on the double ground that the analysis was not exact

* *Proc. London Math. Soc.* (2), 22 (1924), 425–453.

and that the configuration was physically impossible. The former criticism was valid, but the latter rested on the assumption that w must be a uniform function of z.

We have now discussed the problem afresh and the conclusions on which we have agreed are explained in this paper and in one by Mr. Poole, where the scope of the problem is considerably enlarged to include unsymmetrical cases, cases of pure circulation, and those of cyclic motion when both planes are parallel to the stream at infinity.

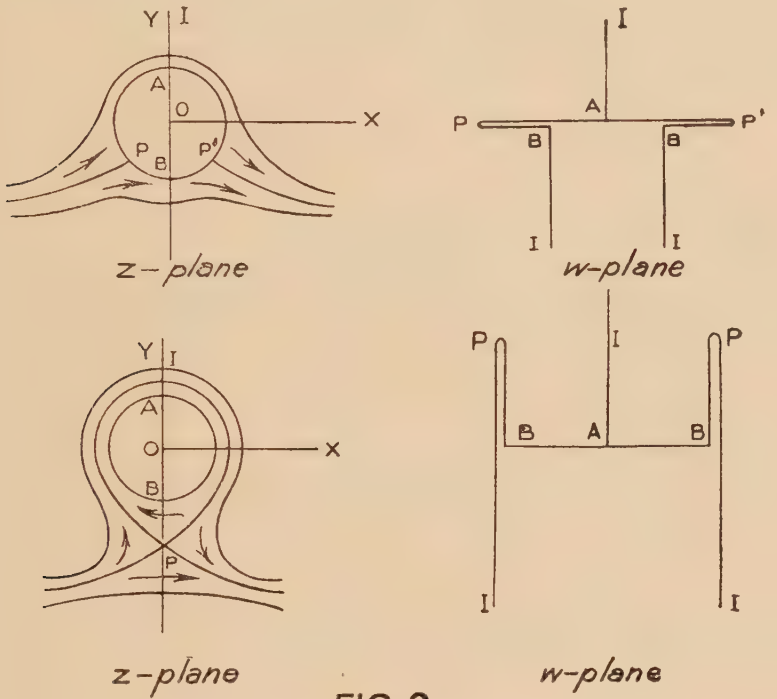

FIG. 2.

4. The subject of this paper will be most easily introduced by a simple example illustrating the principle. An infinitely long circular cylinder of radius a is placed in an infinite stream whose velocity at infinity is U, parallel to the real axis. On this motion is superposed an independent circulation of strength κ. The potential-function and the stream-function are given by the formula

$$w = U\left(z+\frac{a^2}{z}\right)+\frac{i\kappa}{2\pi}\log z+\frac{\kappa}{4},$$

when $w = \phi+i\psi$ and $z = x+iy$. If the ratio κ/Ua is small, the con-

figuration differs little from that of complete symmetry; when the ratio increases the points P, P' at which the stream divides are moved aside from the diameter parallel to the stream at infinity, but remain symmetrical about Oy. When the ratio κ/Ua increases still more, the points P, P' come together at B, on the diameter perpendicular to the general stream, and then leave the surface to form a point P of zero velocity in the fluid on the axis Oy. The lines of flow corresponding to the cases of moderate and strong circulation are shown schematically in Fig. 2.

The function w can be rendered uniform by introducing a barrier along the lower part of the axis of y; the opposite sides of the barrier correspond to equipotentials in the w-plane at a distance κ apart.

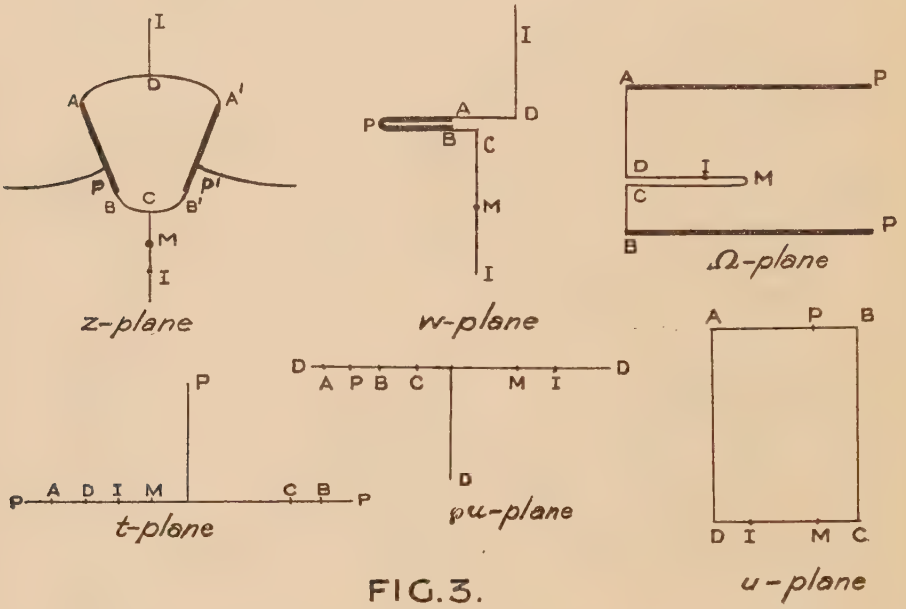

FIG. 3.

5. In the case of two plane obstacles, connected by free stream-lines, there can be also two configurations of flow, of the moderate and strong circulating type respectively.

If the circulation is not so great that the points P, P', at which the stream divides, leave the obstacle, we have the configuration of Fig. 3.

The w- and Ω-regions* shown in the figure correspond to half the fluid region. To connect the w- and Ω-regions with the half-planes of the

* Ω denotes $\log\{q_1(dz/dw)\}$, where q_1 is the speed on the free stream-line.

intermediate variables t and $\wp(u)$ and the rectangle of the variable u, we put

$$(1) \qquad \frac{dw}{dt} = \frac{C_1}{(t-j)^2\sqrt{(t^2-1)}}, \quad \frac{d\Omega}{dt} = \frac{C_2(t-m)}{\sqrt{\{(t^2-1)(t-a)(t-b)\}}},$$

$$(2) \qquad t = \frac{1}{\wp(u)-\wp(\omega_3+\nu)} - 1.$$

The correspondence of points is

	A,	D,	I,	M,	C,	B,	P,
$t =$	a,	-1,	j,	m,	1,	b,	∞,
$u =$	ω_3,	0,	λ,	μ,	ω_1,	$\omega_1+\omega_3$,	$\omega_3+\gamma$.

The circulation is given by integrating dw/dt round the contour, and can be expressed either as $2(w_D - w_C)$, or as $2\pi i \times$[residue of dw/du at $t = j$]. We thus find

$$\kappa = 2\pi C_1 \frac{j}{(1-j^2)^{\frac{3}{2}}}, \quad j = \frac{1}{\wp(\lambda)-\wp(\omega_3+\nu)} - 1.$$

If we eliminate t between (1) and (2), and determine the constant C_2 from the condition that P is a single pole with residue -1 of $d\Omega/du$, we have

$$(3) \qquad \frac{dw}{du} = -\frac{C_1 i}{(1+j)^2\sqrt{2}} \frac{[\wp(u)-\wp(\omega_3+\nu)]\,\wp'(u)}{\{\wp(u)-\wp(\lambda)\}^2\sqrt{\{\wp(u)-e_1\}}},$$

$$(4) \qquad \frac{d\Omega}{du} = -\frac{\wp'(\omega_3+\nu)}{\wp(u)-\wp(\omega_3+\nu)} + \frac{\wp'(\omega_3+\nu)}{\wp(\mu)-\wp(\omega_3+\nu)}.$$

Integrating (4) and observing that at the points D, C respectively, $u = 0$, $\Omega = 0$ and $u = \omega_1$, $\Omega = 0$, we find

$$(5) \qquad q_1 \frac{dz}{dw} = e^{-2[\eta_3+\eta_1(\nu/\omega_1)]u} \frac{\sigma(\omega_3+\nu+u)}{\sigma(\omega_3+\nu-u)} = \frac{\theta(\nu+u)}{\theta(\nu-u)},$$

$$(6) \qquad 2\zeta(\omega_3+\nu) - \frac{\wp'(\omega_3+\nu)}{\wp(\mu)-\wp(\omega_3+\nu)} - 2\eta_3 - 2\eta_1\frac{\nu}{\omega_1} = 0.$$

The last of the formulae is equivalent to the equation

$$\frac{\theta'(\nu+\mu)}{\theta(\nu+\mu)}+\frac{\theta'(\nu-\mu)}{\theta(\nu-\mu)}=0, \tag{6'}$$

which expresses the condition that the speed has a minimum value in M.

At the point A, $u=\omega_3$, $q_1/q=1$, $\theta=\beta$. Substituting these values in (5), we find

$$\beta=\pi\,\frac{\omega_1-\nu}{\omega_1}. \tag{7}$$

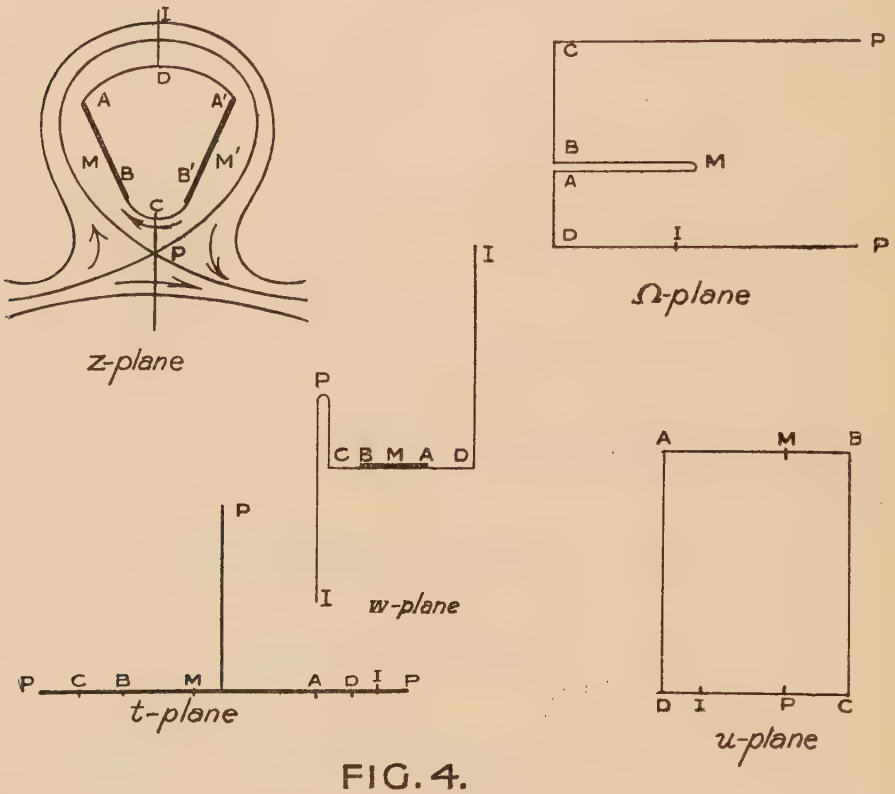

FIG. 4.

The ratios q_1/q_0, q_1/q_m, where q_1 is the speed on the free surface, q_0 the arbitrary velocity of the stream at infinity, and q_m the minimum speed in M, are given by the formulae

$$\frac{q_1}{q_0}=\frac{\theta(\nu+\lambda)}{\theta(\nu-\lambda)}, \qquad \frac{q_1}{q_m}=\frac{\theta(\nu+\mu)}{\theta(\nu-\mu)}.$$

The condition that CI, DI are parts of the same axis will be fulfilled if the residue of (dz/du) at I is zero. This gives

$$\lim_{u\to\lambda}\left[\frac{d}{du}\log\left(\frac{dz}{dw}\frac{dw}{du}\right)+\frac{2}{u-\lambda}\right]=0. \tag{8}$$

The breadth l of the laminae and the distance d between the edges A, A' can be expressed as follows:

$$l=e^{-i\beta}\int_{\omega_3+\lambda}^{\omega_3}\frac{dz}{dw}\frac{dw}{du}\,du+e^{-i\beta}\int_{\omega_3+\omega_1}^{\omega_3+\lambda}\frac{dz}{dw}\frac{dw}{du}\,du, \tag{9}$$

$$d=\text{real part of } 2\int_{\omega_3}^{0}\frac{dz}{dw}\frac{dw}{du}\,du. \tag{10}$$

The five real parameters C_1, $i\omega_3/\omega_1$, λ/ω_1, μ/ω_1, ν/ω_1 are to be determined from the equations (6), (7), (8), (9), (10).

6. If the circulation is strong, we have the configuration of Fig. 4, with a critical stream-line forming a loop about the obstacle. The w-, Ω-, t-, u-planes corresponding to half the fluid region are shown on the figure.

The complete solution of this problem can be obtained by using an argument similar to the one applied in the previous problem.

7. If instead of considering half the region occupied by the fluid, as we did in § 5, we take the entire region, we obtain the areas in the planes w, Ω, shown in Fig. 5. The barriers that we draw are straight lines, which, *by reason of the symmetry of the motion*, pass through the corresponding points in the planes w and Ω.

To transform the w- and Ω-regions into the upper half of the planes t_1 and t respectively, and then to transform these half-planes into the plane $\wp u$, we may write

$$\frac{dw}{dt_1}=\frac{C_1(t_1^2-p_1^2)}{t_1^3\sqrt{(t_1^2-c_1^2)}}\quad\{t_1^2=\wp(u)+n_1\}, \tag{11}$$

$$\frac{d\Omega}{dt}=\frac{Ct^3}{(t^2-p^2)\sqrt{\{(t^2-a^2)(t^2-b^2)(t^2-c^2)\}}}\quad\{t^2=\wp(u)+n\}. \tag{12}$$

These formulae are equivalent to the formulae (3), (4) of § 5.

8. The Ω-region in Fig. 5 being simply connected on the two-sheeted Riemann surface on which it is drawn, we can also transform the Ω-region

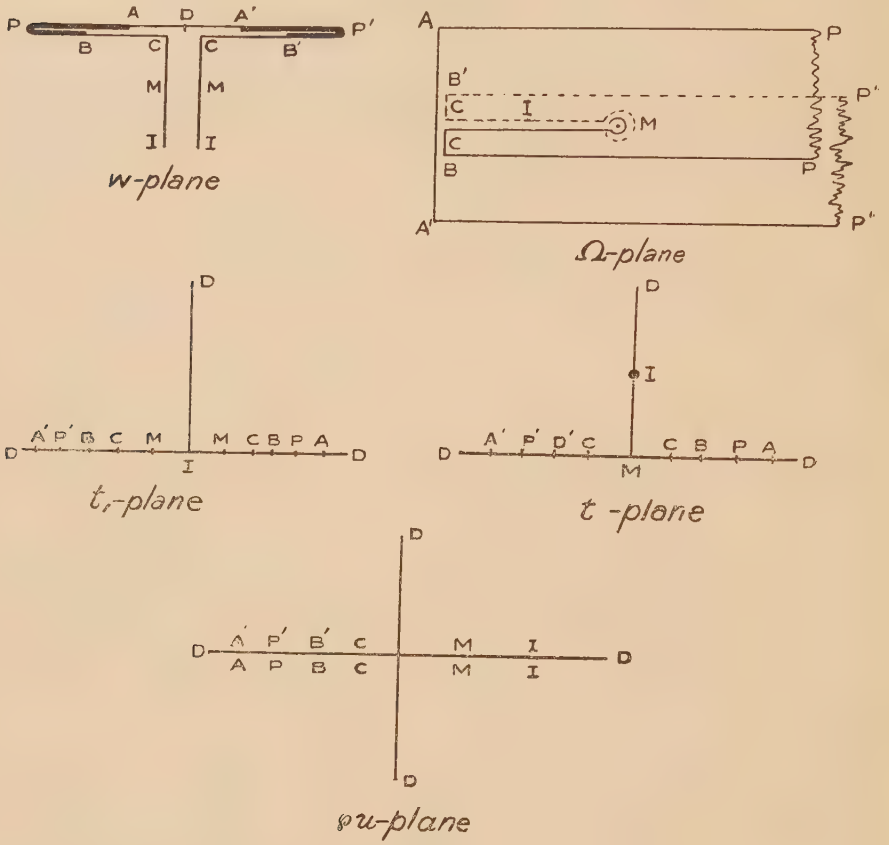

FIG. 5.

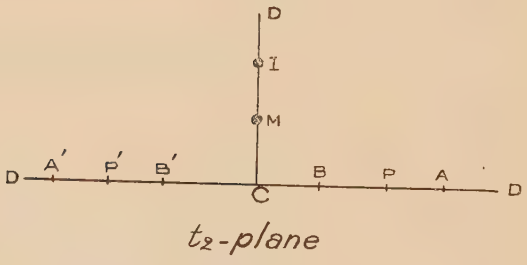

FIG. 6.

into the upper half of the plane t_2 of Fig. 6 without taking into consideration the points of a barrier.

To effect this conformal representation we may write, applying Schläfli's formula,

$$\frac{d\Omega}{dt_2} = \frac{C_2(t_2^2+m_2^2)}{(t_2^2-p_2^2)\sqrt{\{(t_2^2-a_2^2)(t_2^2-b_2^2)\}}}.$$

We can also obtain this equation by transforming the half-plane t of Fig. 5 into the half-plane t_2 so that the points of the barrier should become interior points in the half-plane t_2. To make this transformation, we may write

$$\frac{dt_2}{dt} = \frac{C't}{\sqrt{(t^2-c^2)}} \quad [t_2 = C'\sqrt{(t^2-c^2)}],$$

and substitute this value in (12).

The t_1-plane of Fig. 5 can be transformed in a similar way into the t_2-plane.

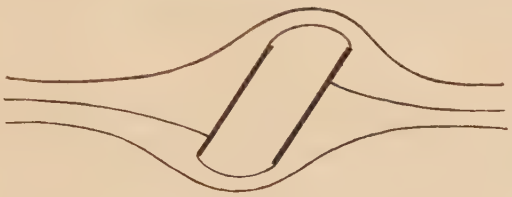

FIG. 7.

The z-region is doubly connected, but, *if the barrier is drawn in such way that the corresponding w-diagram has a symmetrical configuration*, the w-region can be easily transformed into the upper half-plane of the variable t_2, so that the points, corresponding to the barrier, should become interior points in the half-plane t_2 and any two adjacent points, on opposite sides of the trace of the barrier in this half-plane, correspond to two adjacent points on opposite sides of the barrier in the z-plane. It is important to notice that the w-diagram may have a symmetrical configuration even if the Ω-diagram is an unsymmetrical one. The Schläfli-Littlewood formulae allow us then to obtain the complete solution of such problems. It is the principle of the method that Poole* used to find the solution of the problem, shown in Fig. 7, in which the plane obstacles form a parallelogram and the circulation is null.

9. In my paper already mentioned† I calculated the resulting pressure on each of the plane obstacles, by adapting the Joukovsky's theorem.

* *Proc. London Math. Soc.* (2), 22 (1924), 425–453.

† *Comptes rendus du Congrès International des Mathématiciens*, Strasbourg.

This method can be also used in Poole's problem of an unsymmetrical configuration.

Let R_1, R_2 be the resulting pressure on the laminae, and α, β the angles, shown in Fig. 8, which determine their position.

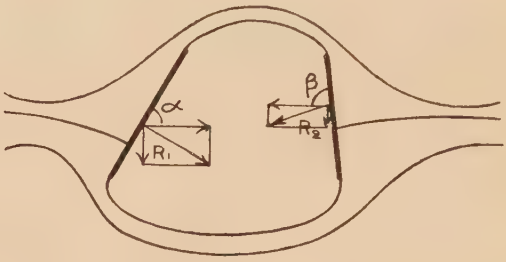

FIG. 8.

Considering the system formed by the obstacles and the dead water, and noticing that the resulting pressure acting on the dead water only is null, we may write

$$R_1 \sin \alpha - R_2 \sin \beta = 0,$$

$$R_1 \cos \alpha + R_2 \cos \beta = \rho \kappa q_0,$$

and therefore

$$R_1 = \frac{\rho \kappa q_0 \sin \beta}{\sin (\alpha + \beta)},$$

$$R_2 = \frac{\rho \kappa q_0 \sin \alpha}{\sin (\alpha + \beta)}.$$

These equations show that, if the circulation is null, the two laminae must be parallel, $\alpha + \beta = \pi$. To calculate the resulting forces in this case, we may transform the equations, putting $\beta = \pi - \alpha - \epsilon$ and passing to the limit for $\epsilon \to 0$. So we find

$$R_1 = R_2 = \rho q_0 \left(\frac{\partial \kappa}{\partial \epsilon}\right)_{\epsilon=0} \sin \alpha.$$

Similarly, to obtain the pressure in case of pure circulation, we may calculate $(\partial q_0 / \partial \epsilon)_{\epsilon=0}$, considering κ as a constant.

The formulae obtained in this section establish an analytical connection between the forces acting on solids due to circulation and those due to the presence of finite or infinite mass of dead water behind the solids. This result is important, as it is generally considered that these two problems are quite independent.

ON A HYDRODYNAMICAL PROBLEM CONSIDERED BY RIABOUCHINSKY

By E. G. C. POOLE.

[Received 29 January, 1925.—Read 12 February, 1925.]

Introduction.

RIABOUCHINSKY'S most interesting paper furnishes an entirely satisfactory and conclusive answer to my criticisms of the hydrodynamical principles underlying his configuration; his lucid explanation will be especially useful to those whose interest lies in the theory of conformal representation rather than in the physical aspect of the problem. After showing that the velocity-potential need not be a uniform function of position, he studies the history of the configuration as the circulation increases from zero to values large compared with the strength of the stream at infinity. He solves analytically the cases where there is an axis of symmetry, which can be dealt with by the Schwarz-Christoffel formula and elliptic functions.

The present paper, which arose out of our discussions, extends the analysis to the unsymmetrical cases, and details are given of the cases of parallel obstacles, where the Kutta-Joukovsky formula suggests some of the results. From that formula we can infer that, when the obstacles are parallel, either the resultant thrust vanishes, or else the obstacles are parallel to the stream at infinity. The former case arises if the velocity at infinity is null, giving pure circulation, or if the circulation is null, a case discussed in my previous paper.* In the latter case, there is a transverse thrust perpendicular to the obstacles and to the stream, as required by the formula.

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* *Proc. London Math. Soc.* (2), 22 (1923), 425. [It may be mentioned that there is an error in sign in formula (59) of this paper, with consequential errors in formulae (62)–(65).]

Note on some Modern Methods.

Consider a perfect fluid under no forces moving in two dimensions and occupying a region free from singularities and bounded by stream-lines, which may correspond to rigid walls or to free surfaces; let the motion be steady. Then we have, with the usual notations,

$$p = p_0 - \tfrac{1}{2}\rho(u^2+v^2) = p_0 - \tfrac{1}{2}\rho\left|\frac{dw}{dz}\right|^2 \quad \text{if} \quad w = \phi + i\psi.$$

Let us evaluate the pressure and couple exerted by the stream on part of the boundary. We shall always choose the sense of integration positively with respect to the stream, *i.e.* leaving the stream on the left. If R_x, R_y are the components of pressure on the arc AB of the boundary, we have the following formulas given by Blasius,* the proof being founded on the discussions by Cisotti† and Faber :‡

$$R_y + iR_x = -\int_A^B p(dx - i\,dy).$$

We may omit the hydrostatic term depending on p_0, and the remaining component will reduce to

$$R_y + iR_x = \frac{\rho}{2}\int_A^B \left(\frac{dw}{dz}\right)\left(\frac{d\bar{w}}{d\bar{z}}\right) d\bar{z}.$$

But along a stream line, $dw = d\bar{w} = d\phi$, so that

$$R_y + iR_x = \frac{\rho}{2}\int_A^B \left(\frac{dw}{dz}\right)^2 dz. \qquad \text{(I)}$$

Again, the moment measured counter-clockwise about the origin of the pressure on AB is

$$G = -\int_A^B p(x\,dx + y\,dy).$$

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* H. Blasius, *Math. Zeitschrift*, 58 (1910), 90-110.

† U. Cisotti, *Idromeccanica Piana*, 1 (1921), 2 (1922) (Milano). There is also an interesting discussion of the two-obstacle problem for obstacles of various shapes in this valuable work, which I have only recently seen [2 (1922), 364–368].

‡ G. Faber, *Münchener Sitzungsberichte* (1920), 49–64.

Neglecting the hydrostatic term, this gives

$$G = \frac{\rho}{4}\int_A^B \left|\frac{dw}{dz}\right|^2 (z\,d\bar{z}+\bar{z}\,dz)$$

$$G = \frac{\rho}{4}\int_A^B \left\{\left(\frac{dw}{dz}\right)^2 z\,dz+\left(\frac{d\bar{w}}{d\bar{z}}\right)^2 \bar{z}\,d\bar{z}\right\},$$

$$G = R\,\frac{\rho}{2}\int_A^B \left(\frac{dw}{dz}\right)^2 z\,dz. \qquad \text{(II)}$$

Let us now consider the special case of an infinite stream surrounding one single simply-connected dead region, bounded by stream-lines, free or fixed. Then Faber deduces from (I) the celebrated Kutta-Joukovsky* formula. For when the boundary is a closed curve, we have a contour integral which can be evaluated by Cauchy's Theorem. Since the region is free from singularities, the contour can be expanded to infinity. Faber makes a conformal representation on the exterior of a circle, but it seems sufficient to consider the form of dw/dz as a function of z directly. We have, if the flow is steady at infinity,

$$\frac{dw}{dz} = U-iV+\frac{a_1}{z}+\frac{a_2}{z^2}+\ldots.$$

The imaginary part of a_1 is $i\kappa/2\pi$ where κ is the circulation about the dead region, measured with the stream on the left. The real part of a_1 gives the strength of the source or sink at infinity; in the present paper this is always zero, so that we write simply $a_1 = i\kappa/2\pi$. The components of velocity at infinity are (U, V). Then from (I)

$$R_y+iR_x = \frac{\rho}{2}\int_\infty \left\{U-iV+\frac{i\kappa}{2\pi z}+\cdots\right\}^2 dz,$$

$$R_y+iR_x = \rho(U-iV)\cdot\frac{i\kappa}{2\pi}\int_\infty \frac{dz}{z}.$$

The integral is to be taken clockwise, so that finally we have

$$R_y = \rho\kappa U, \quad R_x = -\rho\kappa V, \qquad \text{(III)}$$

which are the Kutta-Joukovsky results.

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§ W. M. Kutta, *Illustr. aeronaut. Mitteilungen* (1902), extracts from a dissertation; *Münchener Sitzungsberichte* (1910), 1–58. N. Joukovsky, *Bulletin de l'Institut Aérodynamique de Koutschino* (1906).

Let us now, with the same restrictions, evaluate (II). Here we have

$$G = R\left[\frac{\rho}{2}\int_{\infty}\left(U-iV+\frac{i\kappa}{2\pi z}+\frac{a_2}{z^2}\right)^2 z\,dz\right].$$

The term in κ contributes nothing to the real part, so that

$$G = R\left[\rho(U-iV)\,a_2\int_{\infty}\frac{dz}{z}\right],$$

$$G = -2\pi\rho\,R[(V+iU)\,a_2],$$

And if $a_2 = \alpha+i\beta$,

$$G = 2\pi\rho(U\beta - V\alpha). \qquad \text{(IV)}$$

The Configuration with Branch Points. (Fig. I.)

Let us effect the conformal representation of the regions of z, Ω, and w shown in Fig. I on a rectangle in the plane of u, in such a way that the rigid boundaries correspond to sides parallel to the real axis of u and the free surfaces to sides parallel to the imaginary axis.

In the region considered, z and Ω are uniform functions of position, but w admits a cyclic constant κ corresponding to a description of the boundary with the moving liquid on the left; κ is thus the circulation. If we draw a barrier CI from the boundary to infinity, the function w is rendered uniform, but dw/dz is uniform in the open region without the barrier. The two sides of the barrier correspond to parallel curves at a distance κ apart in the w-plane.

The only critical point in the interior of the region is I, where z has a simple pole in u, Ω is holomorphic, and

$$w \sim (U-iV)z+\frac{i\kappa}{2\pi}\log z \sim \frac{C_1}{u-u'}-\frac{i\kappa}{2\pi}\log(u-u')$$

where (U, V) are the components of velocity of the stream at I. Thus dw/du has a double pole with residue $\kappa/2\pi i$.

From a direct discussion of the case of symmetry, we are led to expect a point of stationary velocity M in the fluid region.

We will now determine dw/du and $d\Omega/du$ as analytic functions of u.

Let us choose the correspondence of points

	A,	P,	B,	B',	P',	A',	I,
$u=$	0,	ϖ,	ω_1,	$\omega_1+\omega_3$,	ϖ',	ω_3,	$\lambda+i\mu$,
$t\equiv\wp(u)=$	∞,	$\wp(\varpi)$,	e_1,	e_2,	$\wp(\varpi')$,	e_3,	$\xi+i\eta$.

At all points of the boundary, both functions are uniform. For at the corners $AB\,A'B'$, the angles in the w-region are double those in the u-region and double those in the Ω-region. Therefore the corners are simple zeros of dw/du and neither poles nor zeros of $d\Omega/du$. Again $d\Omega/du$ has poles with residues -1 at P, P', because the velocity vanishes and has a discontinuity π in direction; dw/du has simple zeros at these points, because angles 2π in the w-region correspond to angles π in the u-region. Hence, at all points of the boundary as well as of the interior, the functions remain uniform.

Along AB, $A'B'$ both functions are real; by Schwarz's principle of symmetric reflexion, they both admit the period $2\omega_3$. Along AA', BB', dw/idu and $d\Omega/du$ are real, so that they also admit the period $2\omega_1$. Hence they are elliptic functions, rationally expressible in terms of $\wp(u)$ and $\wp'(u)$. It is easily shown that $1/\wp'(u)\ dw/du$, and $d\Omega/du$ must be real rational functions of $\wp(u)$ only.

Knowing the singularities on the axis of real values $t=\wp(u)$, and in either the upper or the lower half-plane, we can write down these forms by symmetry. We find

$$(1)\qquad \frac{dw}{du}=\frac{M\{\wp(u)-\wp(\varpi)\}\{\wp(u)-\wp(\varpi')\}\wp'(u)}{[\{\wp(u)-\xi\}^2+\eta^2]^2},\quad M \text{ real},$$

$$(2)\qquad \frac{d\Omega}{du}=-\zeta(u-\varpi)-\zeta(u-\varpi')+\zeta(u+\varpi)+\zeta(u+\varpi')+\text{const.}$$

$$=-\frac{\wp'(\varpi)}{\wp(u)-\wp(\varpi)}-\frac{\wp'(\varpi')}{\wp(u)-\wp(\varpi')}+\text{const.}$$

The former is immediately integrable, and gives

$$(3)\qquad w=-\frac{\kappa}{\pi}\arctan\left(\frac{\wp(u)-\xi}{|\eta|}\right)+\frac{P\wp(u)+Q}{\{\wp(u)-\xi\}^2+\eta^2},$$

where

$$\frac{P+M}{|\eta|}=\frac{2Q-\{\wp(\varpi)+\wp(\varpi')\}M}{2\xi|\eta|}=\frac{M[\{\wp(\varpi)-\xi\}\{\wp(\varpi')-\xi\}+\eta^2]}{|2\eta^3|}=-\frac{\kappa}{\pi}.$$

If U_0, V_0 are the components of velocity at A, where $u=0$, we get

$$(4)\qquad \frac{dz}{dw}=\frac{1}{U_0-iV_0}\frac{\sigma(\varpi+u)\,\sigma(\varpi'+u)}{\sigma(\varpi-u)\,\sigma(\varpi'-u)}\,e^{cu},\quad c=\frac{i\pi}{\omega_1}-\frac{2(\varpi+\varpi')\,\eta_1}{\omega_1},$$

The value of c is given by the relations $\Omega_A=\Omega_B-i\pi$, $2(\eta_1\omega_3-\eta_3\omega_1)=i\pi$. We can now write

$$z=\int\frac{dz}{dw}\cdot\frac{dw}{du}\,.\,du,$$

the integrand is uniform in u and is regular, except at I, where there is a double pole. Since z must itself be uniform, we see by integrating round the boundary and applying Cauchy's Theorem, that the residue of dz/du at I must vanish. Expressing dw/du in sigma functions and differentiating logarithmically, we get the condition

$$\lim_{u\to\lambda+i\mu}\left\{\frac{d\Omega}{du}+\frac{d}{du}\log\frac{dw}{du}+2\zeta(u-\lambda-i\mu)\right\}=0$$

or

$$(5)\quad \zeta(\varpi+\lambda+i\mu)+\zeta(\varpi'+\lambda+i\mu)-\zeta(2\lambda)-\zeta(2i\mu)-\frac{\varpi+\varpi'}{\omega_1}\eta_1+\frac{i\pi}{2}=0.$$

This gives two real conditions connecting the five real parameters ϖ, $\varpi'-\omega_3$, λ, μ, $\omega_3/i\omega_1$, so that the solution has three degrees of freedom.

When the obstacles are not parallel, z cannot usually be found in finite terms. The lengths of the obstacles involve impracticable quadratures. On the other hand, the magnitude of the pressures is known if their resultant is known in magnitude and direction. As this is given by Joukovsky's or Kutta's formula, the individual pressures can be obtained by resolution of forces. We have, in fact, $R_y+iR_x=\kappa\rho(U-iV)$. The formula of Blasius, of which we shall make use in a special case below, enables us to calculate the resultant moment by evaluating a single residue, but another equation would be necessary to find the centres of pressure.

By considering the direction of flow at A, A', the angle between the obstacles is found to be $\{(\omega_3-\varpi-\varpi')/\omega_1\pi+2s\pi\}$. If the obstacles are parallel, dz/dw becomes an elliptic function, and the quadratures can be carried a stage further. It is therefore desirable to study this case more carefully. Now we can predict, by Kutta and Joukovsky's Theorem, that the resultant pressure is perpendicular to the stream at I; since it is also perpendicular to the two obstacles, it follows that either the pressure vanishes or the obstacles are parallel to the stream at infinity.

When there are branch points on the obstacles, the following discussion of (5) will show that the only solution gives $\kappa=0$, which was the case treated in my former paper.

If the configuration is not degenerate, we must have as the condition for parallel obstacles

$$\varpi+\varpi'=\omega_1+\omega_3=-\omega_2 \quad \text{(say)}.$$

Consider the equation $\wp'(u)-2h\{\wp(u)-e_2\}=0$,

where h may be real or complex. This has three zeros, one of which is ω_2,

whose sum is congruent to zero. Hence we may choose h so that ϖ, ϖ' are the zeros of this equation, and, since ϖ is real, h must in this case be real. We have the relations

$$-h=\zeta(\varpi)+\zeta(\varpi')+\eta_2=-\tfrac{1}{2}\frac{\wp'(\varpi)}{\wp(\varpi)-e_2}=-\tfrac{1}{2}\frac{\wp'(\varpi')}{\wp(\varpi')-e_2}=-\tfrac{1}{2}\left(\frac{\wp'(\varpi)-\wp'(\varpi')}{\wp(\varpi)-\wp(\varpi')}\right),$$

and $\wp(\varpi)$, $\wp(\varpi')$ are the roots of

$$f(t)\equiv(t-e_1)(t-e_3)-h^2(t-e_2)=0.$$

We now write (5) in the form

$$\{\zeta(\varpi)+\zeta(\varpi')+\eta_2\}+\tfrac{1}{2}\sum_{\varpi,\,\varpi'}\frac{\wp'(\varpi)-\wp'(\lambda+i\mu)}{\wp(\varpi)-\xi-i\eta}+2\zeta(\lambda+i\mu)-\zeta(2\lambda)-\zeta(2i\mu)=0,$$

$$-h+h\sum_{\varpi,\,\varpi'}\frac{\wp(\varpi)-e_2}{\wp(\varpi)-\xi-i\eta}$$

$$-\wp'(\lambda+i\mu)\left\{\frac{1}{\xi+i\eta-\xi+i\eta}+\tfrac{1}{2}\sum_{\varpi,\,\varpi'}\frac{1}{\wp(\varpi)-\xi-i\eta}\right\}=0,$$

$$h\left\{1-\frac{(\xi+i\eta-e_2)f'(\xi+i\eta)}{f(\xi+i\eta)}\right\}=\frac{\wp'(\lambda+i\mu)}{2i\eta}\left\{1-\frac{i\eta f'(\xi+i\eta)}{f(\xi+i\eta)}\right\}.$$

This finally reduces to

$$\frac{2i\eta h}{\wp'(\lambda+i\mu)}\{(\xi+i\eta-e_2)^2-e_1e_3-2e_2^2\}=\{(\xi-e_1)\xi-e_3)-h^2(\xi-e_2)+\eta^2\}.$$

From the identity

$$\wp(2u)-e_2=\left\{\frac{\{\wp(u)-e_2\}^2-e_1e_3-2e_2^2}{\wp'(u)}\right\}^2,$$

and, from the formula for the circulation, we find

$$h\sqrt{\{\wp(2\lambda+2i\mu)-e_2\}}=\frac{i\eta^2\kappa}{M}. \tag{6}$$

The value of the radical being purely imaginary, we must have $\wp(2\lambda+2i\mu)$ real and less than e_2, so that $2i\mu=\omega_3$. The condition (5) now becomes

$$\zeta\left(\lambda+\varpi+\frac{\omega_3}{2}\right)+\zeta\left(\lambda+\omega_1-\varpi-\frac{\omega_3}{2}\right)-\zeta(2\lambda)-\eta_1=0.$$

But we have

$$\zeta(x)+\zeta(y)+\zeta(z)-\zeta(x+y+z)\equiv\frac{\sigma(y+z)\,\sigma(z+x)\,\sigma(x+y)}{\sigma(x)\,\sigma(y)\,\sigma(z)\,\sigma(x+y+z)}.$$

This gives three alternatives, two of which are excluded because λ, ϖ are real. The only solution admissible is $2\lambda = \omega_1$. Thus

$$\lambda + i\mu = \frac{\omega_1+\omega_3}{2}, \quad \xi + i\eta = e_2 - i\sqrt{\{(e_1-e_2)(e_2-e_3)\}}, \tag{7}$$

and the value of ϖ is arbitrary.

We now find

$$\kappa = -\frac{\pi M}{2\eta^3}\{f(e_2)+\eta^2\} = 0, \quad \eta^2 = (e_1-e_2)(e_2-e_3). \tag{8}$$

The expression dz/dw is now an elliptic function, whose poles are ϖ, ϖ' and zeros $-\varpi$, $-\varpi'$, so that we have

$$(U_0 - iV_0)\frac{dz}{dw} = \frac{\wp'(u)+2h\{\wp(u)-e_2\}}{\wp'(u)-2h\{\wp(u)-e_2\}}$$

$$= \frac{\{\wp(u)-e_1\}\{\wp(u)-e_3\}+h^2\{\wp(u)-e_2\}+h\wp'(u)}{\{\wp(u)-e_1\}\{\wp(u)-e_3\}-h^2\{\wp(u)-e_2\}},$$

$$\left(\frac{U_0-iV_0}{M}\right)\frac{dz}{du} = \frac{[\{\wp(u)-e_1\}\{\wp(u)-e_3\}+h^2\{\wp(u)-e_2\}+h\wp'(u)]}{[\{\wp(u)-e_2\}^2+\eta^2]^2}\wp'(u).$$

But
$$\{\wp(u)-e_1\}\{\wp(u)-e_3\} = \{\wp(u)-e_2\}^2+3e_2\wp(u)-e_2^2+e_1e_3$$
$$= \{\wp(u)-e_2\}^2-\eta^2+3e_2\{\wp(u)-e_2\},$$

and
$$\frac{\wp'^2(u)}{[\{\wp(u)-e_2\}^2+\eta^2]^2} = \frac{1}{\wp(2u)-e_2} = \frac{1}{\eta^2}\{e_2-\wp(2u-\omega_2)\}.$$

Hence, we may integrate at once, and we find

$$\left(\frac{U_0-iV_0}{M}\right)z = -\frac{\{\wp(u)-e_2\}}{\{\wp(u)-e_2\}^2+\eta^2} - \frac{\frac{1}{2}(h^2+3e_2)}{\{\wp(u)-e_2\}^2+\eta^2}$$
$$+\frac{h}{2\eta^2}\{\zeta(2u-\omega_2)+e_2u\}+C. \tag{9}$$

Putting $z = 0$ at A, it is easily verified that

$$\left(\frac{U_0-iV_0}{M}\right)z_B = -\frac{h^2+e_2-2e_3}{2(e_1-e_2)(e_1-e_3)}+\frac{h}{2\eta^2}(2\eta_1+2e_2\omega_1),$$

$$\left(\frac{U_0-iV_0}{M}\right)z_{A'} = -\frac{h^2+e_2-2e_1}{2(e_1-e_3)(e_2-e_3)}+\frac{h}{2\eta^2}(2\eta_3+2e_2\omega_3),$$

$$\left(\frac{U_0-iV_0}{M}\right)z_{B'} = -\frac{(h^2+3e_2)}{2\eta^2}+\frac{h}{2\eta^2}(2\eta_1+2\eta_3+2e_2\omega_1+2e_2\omega_3)$$

$$= \left(\frac{U_0-iV_0}{M}\right)(z_B+z_{A'}).$$

Since $z_A+z_{B'} = z_{A'}+z_B$, the figure is a parallelogram.

The pressure is easily found by a quadrature

$$P_{AB} = \frac{\rho(U_0^2+V_0^2)}{2}\int_A^B \left(1-\frac{1}{U_0^2+V_0^2}\left|\frac{dw}{dz}\right|^2\right)|dz|$$

$$= \frac{\rho(U_0^2+V_0^2)}{2}\int_A^B \left\{1-\left(\frac{\wp'(u)-2h\{\wp(u)-e_2\}}{\wp'(u)+2h\{\wp(u)-e_2\}}\right)^2\right\}|dz|$$

$$= \rho(U_0^2+V_0^2)\int_A^B \frac{h\wp'(u)}{[\{\wp(u)-e_1\}\{\wp(u)-e_3\}+h^2\{\wp(u)-e_2\}+h\wp'(u)]}\left|\frac{dz}{du}\right| du$$

$$= \rho(U_0^2+V_0^2)^{\frac{1}{2}}\left|Mh\int_0^{\omega_1}\frac{\wp'^2\,u\,du}{[\{\wp(u)-e_2\}^2+\eta^2]^2}\right|.$$

The integral is merely

$$\int_0^{\omega_1}\frac{du}{\wp(2u)-e_2} = \frac{1}{\eta^2}\int_0^{\omega_1}\left(e_2-\wp(2u-\omega_2)\right)du.$$

Hence

$$P_{AB} = \rho(U_0^2+V_0^2)^{\frac{1}{2}}|Mh(e_2\omega_1+\eta_1)| \div (e_1-e_2)(e_2-e_3), \tag{10}$$

on inserting the value of η^2.

Since the configuration is symmetrical about a point, the centres of pressure will be known when we have found the resultant moment, $G = 2\pi\rho(U\beta-V\alpha)$, where

$$\frac{dw}{dz} = U-iV+\frac{i\kappa}{2\pi z}+\frac{\alpha+i\beta}{z^2}+\ldots.$$

Let h' be the complex number

$$h' = \frac{\wp'(\frac{1}{2}\omega_2)}{2\{\wp(\frac{1}{2}\omega_2)-e_2\}} = \frac{-\wp'(\frac{1}{2}\omega_2)}{2i\eta}. \tag{11}$$

Then the equation $\wp'(u) = 2h'\{\wp(u)-e_2\}$ has a simple zero $u=\omega_2$ and a double zero $-\frac{1}{2}\omega_2 = \frac{1}{2}\omega_1+\frac{1}{2}\omega_3$. Let us also put

$$\varpi+\tfrac{1}{2}\omega_2 = \chi, \quad u+\tfrac{1}{2}\omega_2 = v.$$

In the neighbourhood of I we shall have

$$\frac{dw}{dz} = (U_0-iV_0)\left\{\frac{\wp'(u)-2h\{\wp(u)-e_2\}}{\wp'(u)+2h\{\wp(u)-e_2\}}\right\} = (U_0-iV_0)\left(\frac{h'-h}{h'+h}\right)\left\{\frac{\wp(v)-\wp(\chi)}{\wp(v)-\wp(\chi+\omega_2)}\right\}.$$

Thus $$U-iV=(U_0-iV_0)\left(\frac{h'-h}{h'+h}\right),$$

$$\frac{dw}{dz}=(U-iV)[1+\{\wp(\chi+\omega_2)-\wp(\chi)\}v^2+O(v^4)],$$

$$\wp(\chi+\omega_2)-\wp(\chi)=\frac{\wp'(\varpi)\,\wp'(\frac{1}{2}\omega_2)}{\{\wp(\varpi)-\wp(\frac{1}{2}\omega_2)\}^2}$$

$$=\frac{\wp'(\varpi)\,\wp'(\frac{1}{2}\omega_2)}{\{\wp(\varpi)-e_2\}^2-2i\eta\{\wp(\varpi)-e_2\}-(e_1-e_2)(e_2-e_3)}$$

$$=\frac{\wp'(\varpi)\,\wp'(\frac{1}{2}\omega_2)}{\{\wp(\varpi)-e_1\}\{\wp(\varpi)-e_3\}-3e_2\{\wp(\varpi)-e_2\}-2i\eta\{\wp(\varpi)-e_2\}}$$

$$=\frac{2h\,\wp'(\frac{1}{2}\omega_2)}{h^2-3e_2-2i\eta}=\frac{4hh'i\eta}{h'^2-h^2},$$

since $h'^2=3e_2+2i\eta$.

Again, near I we have

$$z\sim\frac{M}{U_0-iV_0}\left(\frac{h}{4\eta^2}-\frac{1}{4i\eta h'}+\frac{h^2+3e_2}{8h'\eta^2}\right)\frac{1}{v},$$

$$z\sim\frac{M}{U_0-iV_0}\cdot\frac{(h+h')^2}{8h'\eta^2}\cdot\frac{1}{v}.$$

Hence $$\alpha+i\beta=\frac{ihM^2(h+h')^2}{16h'\eta^3(U_0-iV_0)},$$

$$(U-iV)(\alpha+i\beta)=\frac{ihM^2(h'^2-h^2)}{16h'\eta^3},$$

and finally we have

(12) $$G=R\left\{\frac{\rho\pi h(h^2-h'^2)M^2}{4\eta^2\,\wp'(\frac{1}{2}\omega_2)}\right\}.$$

The distance between the lines of action of the resultant pressures is, of course, G/P.

The Configuration without Branch Points. (Fig. II.)

The nature of the configuration is known from the case of symmetry. Instead of two branch points P, P' on the obstacles and a point of steady velocity M in the interior, we have a point of zero velocity P in the interior and two points of minimum velocity M, M' on the obstacles.

Analytically, P is a simple zero of dw/dz or dw/du, and is a simple pole with residue -1 of $d\Omega/du$; it is an ordinary point of $z(u)$. If we draw in the z-plane a curve which does not intersect itself from a point C on the boundary, through P, to I, and if the curvature of this curve is continuous at P, then it is found that w is rendered uniform by the com-

plete barrier CPI and Ω by the portion of the barrier CP. The Ω-region is an open strip bounded on two sides by parallel curves, the corresponding points of which have affixes differing by $2\pi i$, and corresponding to points on opposite sides of the barrier CP. Thus $d\Omega/du$ is uniform in the open z-region without any barrier. The w-region is part of a Riemann surface bounded by two parallel curves corresponding to opposite sides of the barrier CPI, corresponding points having affixes which differ by κ, and by portions of the real axis; these curves have cusps at P, since this is a zero of dw/dz. The functions dw/dz or dw/du are uniform in the open region.

We choose the correspondence of points

$$\begin{array}{cccccc} A, & B, & B', & A', & P, & I, \\ u=0, & \omega_1, & \omega_1+\omega_3, & \omega_3, & \varpi=\alpha+i\beta, & \lambda+i\mu, \\ \wp(u)=\infty, & e_1, & e_2, & e_3, & \gamma+i\delta, & \xi+i\eta. \end{array}$$

[No confusion need arise with the former meaning of α, β.]

By the arguments already employed, dw/du and $d\Omega/du$ reduce respectively to the form $\wp'(u)R_1\{\wp(u)\}$ and $R_2\{\wp(u)\}$, where R_1, R_2 are real rat functions of $\wp(u)$. We find

$$\frac{dw}{du}=\frac{M[\{\wp(u)-\gamma\}^2+\delta^2]\wp'(u)}{[\{\wp(u)-\xi\}^2+\eta^2]^2}, \tag{13}$$

$$w=-\frac{\kappa}{\pi}\arctan\left(\frac{\wp(u)-\xi}{|\eta|}\right)+\frac{P\wp(u)+Q}{\{\wp(u)-\xi\}^2+\eta^2}, \tag{14}$$

where $$P+M=\frac{\gamma M-Q}{\xi}=\frac{M\{(\gamma-\xi)^2+\eta^2+\delta^2\}}{2\eta^2}=-|\eta|\frac{\kappa}{\pi},$$

$$\frac{d\Omega}{du}=\frac{-\wp'(\varpi)}{\wp(u)-\wp(\varpi)}-\frac{\wp'(\bar{\varpi})}{\wp(u)-\wp(\bar{\varpi})}+\text{const.},$$

where $$\varpi=\alpha+i\beta, \quad \bar{\varpi}=\alpha-i\beta.$$

The first term is given by the residue at P and the second by the Schwarzian principle of symmetry. The equation transforms into

$$\frac{d\Omega}{du}=\Sigma\zeta(a\pm i\beta\pm u)+\text{const.}$$

$$\frac{dz}{dw}=e^{\Omega}=\frac{e^{-(4\eta_1\alpha\pi)/\omega_1}}{U_0-iV_0}\cdot\frac{\sigma(\varpi+u)\,\sigma(\bar{\varpi}+u)}{\sigma(\varpi-u)\,\sigma(\bar{\varpi}-u)}. \tag{15}$$

The indeterminate constant is given by the condition $\Omega_A=\Omega_B$ and (U_0, V_0) are the components of velocity at $u=0$ or A. Since z must be

uniform within the fundamental region, we find that the residue of dz/du at I must vanish, so that

$$\zeta(\varpi+\lambda+i\mu)+\zeta(\bar{\varpi}+\lambda+i\mu)-\zeta(2\lambda)-\zeta(2i\mu)-\frac{2\eta_1\alpha}{\omega_1}=0, \tag{16}$$

so that the five real parameters α, β, λ, μ, $\omega_3/i\omega_1$ are connected by two real relations. The angle between the obstacles is given by the change in the direction of flow, and

$$\theta_{A'}-\theta_A=2\pi\left(1-\frac{\alpha}{\omega_1}\right).$$

Important simplifications occur if the obstacles are parallel—*i.e.* $\alpha=\frac{1}{2}\omega_1$. Part of the results are rendered intuitive by the Kutta-Joukovsky formula; for since the resultant thrust is perpendicular to the stream at infinity, and also obviously to the obstacles, it follows that either the obstacles are parallel to the stream at infinity or the thrust is null. The latter requires that either $\kappa=0$, the case already discussed, or that there should be rest at infinity, corresponding to a pure circulation. In addition to confirming these results, a detailed analysis shows that there is one axis of symmetry in all cases of parallel obstacles with $\kappa\neq 0$, and complete symmetry about both axes, if there is rest at infinity.

For since $\alpha=\frac{1}{2}\omega_1$, we can find a real number h, such that the zeros of

$$\wp'(u)-2h\{\wp(u)-e_1\}=0$$

are congruent to ϖ, $\bar{\varpi}$, and ω_1. Hence, $\gamma\pm i\delta$ satisfy

$$f(x)\equiv(x-e_2)(x-e_3)-h^2(x-e_1)=0.$$

Replacing e_2 by e_1 in the previous discussion, we find from (16)

$$\sqrt{\{\wp(2\lambda+2i\mu)-e_1\}}=\frac{\xi^2+\eta^2+\xi(e_1-h^2)+e_2e_3+h^2e_1}{-2i\eta h}=\frac{i\eta^2\kappa}{M\pi h}.$$

Hence $\wp(2\lambda+2iu)$ is real and less than e_1, which requires either $2\lambda=\omega_1$ or $2i\mu=\omega_3$, in view of the fact that I is an interior point of the fundamental region.

If $2i\mu=\omega_3$, we have

$$\zeta(\varpi+\lambda+\tfrac{1}{2}\omega_3)+\zeta(\bar{\varpi}+\lambda+\tfrac{1}{2}\omega_3)-\zeta(2\lambda)+\zeta(\omega_1+\omega_3)=0$$

or $$\sigma(\omega_1+\omega_3+2\lambda)\,\sigma(i\beta+\tfrac{1}{2}\omega_1+\tfrac{1}{2}\omega_3-\lambda)\,\sigma(i\beta-\tfrac{1}{2}\omega_1-\tfrac{1}{2}\omega_3+\lambda)=0,$$

and the only acceptable solution is $\lambda = \frac{1}{2}\omega_1$, $i\beta = \frac{1}{2}\omega_3$. Thus I and P coincide and we have the case of pure circulation, the liquid being at rest at infinity.

Since, in all cases, when the obstacles are parallel $\lambda = \frac{1}{2}\omega_1$, we can find a real number h' such that $\lambda \pm i\mu$ and ω_1 are the zeros of

$$\wp'(u) = 2h'\{\wp(u)-e_1\},$$

and $\xi \pm i\eta$ are the roots of

$$\phi(x) \equiv (x-e_2)(x-e_3)-h'^2(x-e_1) = 0.$$

Let us now take the axis of real values of z parallel to the obstacles. The equation (15) reduces to the simpler forms

$$\frac{dz}{dw} = \frac{1}{U_0}\frac{\wp'(u)+2h\{\wp(u)-e_1\}}{\wp'(u)-2h\{\wp(u)-e_1\}}$$

$$= \frac{1}{U_0}\cdot\frac{\{\wp(u)-e_2\}\{\wp(u)-e_3\}+h^2\{\wp(u)-e_1\}+h\wp'(u)}{f\{\wp(u)\}},$$

hence $$\frac{dz}{du} = \frac{M}{U_0}\left\{\frac{\{\wp(u)-e_2\}\{\wp(u)-e_3\}+h^2\{\wp(u)-e_1\}+h\wp'(u)}{\phi^2\{\wp(u)\}}\right\}\wp'(u),$$

a doubly periodic function with double poles at $u = \pm\frac{1}{2}\omega_1 \pm i\mu$, the residues at $u = \frac{1}{2}\omega_1 \pm i\mu$ being zero. Resolving into simple elements and using the forms in h' to simplify the algebra, we find at once

$$(17)\quad \frac{dz}{du} = A\Sigma\wp(u-\tfrac{1}{2}\omega_1 \pm i\mu)+B\Sigma\wp(u+\tfrac{1}{2}\omega_1 \pm i\mu)$$
$$+iC\{\zeta(u+\tfrac{1}{2}\omega_1+i\mu)-\zeta(u+\tfrac{1}{2}\omega_1-i\mu)\}+D,$$

where $$A = -\frac{(h+h')^2M}{8\eta^2h'U_0}, \quad B = \frac{(h-h')^2M}{8\eta^2h'U_0}.$$

The residues at the two remaining poles, being conjugate, and their sum being zero, are purely imaginary, and we find by logarithmic differentiation and expansion near $u = -\frac{1}{2}\omega_1 - i\mu$ the value

$$\frac{iC}{2B} = \zeta(\varpi-\tfrac{1}{2}\omega_1-i\mu)+\zeta(\bar{\varpi}-\tfrac{1}{2}\omega_1-i\mu)+\zeta(2i\mu).$$

From (16),

$$\zeta(\varpi+\tfrac{1}{2}\omega_1+i\mu)+\zeta(\bar{\varpi}+\tfrac{1}{2}\omega_1+i\mu)-\zeta(2i\mu)-2\eta_1 = 0.$$

Hence $$\frac{iC}{2B} = 2\{\zeta(\varpi)+\zeta(\bar{\varpi})-\eta_1\} + \sum_{\varpi,\bar{\varpi}} \frac{\wp'(\varpi)}{\wp(\varpi)-\xi-i\eta}$$

$$= -2h+2h\,\Sigma\,\frac{\wp(\varpi)-e_1}{\wp(\varpi)-\xi-i\eta}$$

$$= 2h-2h\,\frac{(\xi+i\eta-e_1)f'(\xi+i\eta)}{f(\xi+i\eta)}$$

$$= 2h+\frac{2h}{h^2-h'^2}(2\xi+2i\eta+e_1-h^2)$$

$$= \frac{4i\eta h}{h^2-h'^2},$$

$$C = \frac{h(h-h')M}{h'(h+h')U_0},$$

and, putting $u=0$, we find

$$D = -\frac{h\xi M}{\eta^2 U_0}+iC\{\zeta(\tfrac{1}{2}\omega_1-i\mu)-\zeta(\tfrac{1}{2}\omega_1+i\mu)\}.$$

The relation (16) is expressible algebraically in the form

$$\frac{(\xi-\eta)^2+\eta^2+\delta^2}{-2i\eta h} = \surd\{\wp(\omega_1+2i\mu)-e_1\}$$

$$= \frac{(\wp-e_1)^2-(e_1-e_2)(e_1-e_3)}{\wp'}$$

$$= \frac{(2\wp+e_1)(\wp-e_1)-(\wp-e_2)(\wp-e_3)}{\wp'}$$

$$= \frac{2\wp+e_1-h'^2}{2h'}$$

$$= \frac{i\eta}{h'},$$

where the arguments of the elliptic functions are all $\lambda+i\mu$.

Hence $$2h\eta^2 = h'\{f(\xi)+\eta^2\},$$

$$2(h-h')\eta^2 = h'(h^2-h'^2)(e_1-\xi).$$

If $h = h'$, we shall prove below that we have the case of pure circulation; the other alternative gives

$$2\eta^2 + h'(h+h')(\xi - e_1) = 0. \tag{16A}$$

Integrating (17) and writing $v = u - (\omega_1/2)$, we have

$$z = -A\Sigma\zeta(v \pm iu) - B\Sigma\zeta(v \pm \overline{iu+\omega_1}) + Dv + \frac{iC}{2}\log\left[\frac{\sigma^2(v+iu)\{\wp(v+iu)-e_1\}}{\sigma^2(v-iu)\{\wp(v-iu)-e_1\}}\right], \tag{18}$$

the additive constants being chosen to make $z = 0$ when $v = 0$. When $v = \omega_3$, we have

$$z(\omega_3) = -2(A+B)\eta_3 + D\omega_3 - 4C\mu\eta_3,$$

while

$$z(v) + z(-v) \equiv 0,$$

$$z(v) + z(2\omega_3 - v) - 2z(\omega_3) \equiv 0.$$

Thus there is an axis of symmetry perpendicular to the two obstacles, crossing them where $v = 0,\ \omega_3$. The lengths of the obstacles and the coordinates of their extremities are given by substituting $v = \pm(\omega_1/2)$, $\omega_3 \pm (\omega_1/2)$ in the expression (18). The resultant pressure is given by the Kutta-Joukovsky formula $R_y = \rho U\kappa$, $R_x = 0$, where U is the velocity at infinity, which is parallel to the real axis

$$U = \frac{dw}{dz} = \frac{h'-h}{h'+h}\,U_0. \tag{19}$$

The pressures on the two obstacles act along the axis of symmetry, and one of them must be evaluated directly to complete the solution.

The pressure on the obstacles AB is

$$\begin{aligned} P_{AB} &= \frac{\rho}{2}\int_A^B \left(U_0^2 - \left|\frac{dw}{dz}\right|^2\right)|\,dz| \\ &= \frac{\rho U_0^2}{2}\int_0^{\omega_1}\left(\left|\frac{dz}{dw}\right| - U_0^{\frac{1}{2}}\left|\frac{dw}{dz}\right|\right)\left|\frac{dw}{du}\right| du \\ &= \frac{M\rho U_0}{2}\int_0^{\omega_1}\left[\frac{\wp'(u)+2h\{\wp(u)-e_1\}}{\wp'(u)-2h\{\wp(u)-e_1\}}\right. \\ &\qquad \left. - \frac{\wp'(u)-2h\{\wp(u)-e_1\}}{\wp'(u)+2h\{\wp(u)-e_1\}}\right]\frac{f\{\wp(u)\}}{\phi^2\{\wp(u)\}}\,|\,\wp'(u)\,|\,du. \end{aligned}$$

Following the general idea of Thiry's method* in similar cases, we observe that

$$P_{AB} = \frac{M\rho U_0}{2} \int_{-\omega_1}^{\omega_1} \frac{\wp'(u) - 2h\{\wp(u) - e_1\}}{\wp'(u) + 2h\{\wp(u) - e_1\}} \cdot \frac{f\{\wp(u)\}}{\phi^2\{\wp(u)\}} \wp'(u)\, du,$$

since $\wp' u$ is negative along the line of integration. This integrand is the same elliptic function as occurs in dz/du, with the sign of u changed. Hence we may write symbolically

$$P_{AB} = \frac{\rho U_0^2}{2} \left| \left[z(u) \right]_{u=-\omega_1}^{u=\omega_1} \right|,$$

the point indicated by the lower limit being imaginary; and a straightforward calculation gives

$$P_{AB} = \frac{\rho U_0^2}{2} \left| 4(A+B)\eta_1 + (4\eta_1\mu - 2\pi)C - 2D\omega_1 \right|,$$

$$P_{AB} = \rho U_0 M \left| \frac{h}{\eta^2}\eta_1 + \frac{h(h-h')}{h'(h+h')}(\pi - 2\eta_1\mu) + \frac{U_0 D\omega_1}{M} \right|. \tag{20}$$

The Case of Pure Circulation.

If the liquid is at rest at infinity, the dead point P must be identified with I. Since the Kutta-Joukovsky formula shows that the thrust is null, the obstacles must be parallel. We shall verify this analytically, and incidentally it will appear that the configuration is completely symmetrical about both axes. The fundamental condition (16) is now

$$\zeta(2\lambda + 2i\mu) - \zeta(2i\mu) - \frac{2\eta_1\lambda}{\omega_1} = 0. \tag{21}$$

Taking the imaginary part, we must have $\wp'(2i\mu) = 0$, and the only admissible solution is $2i\mu = \omega_3$, so that

$$F(\lambda) \equiv \zeta(2\lambda + \omega_3) - \eta_3 - \frac{2\eta_1\lambda}{\omega_1} = 0. \tag{21A}$$

* Thiry, *Annales Sci. de l'Ecole Normale* (3), 38 (1921), 35–36.

It is easily shown that the only solution such that $0<\lambda<\omega_1$ is $\lambda=\frac{1}{2}\omega_1$. For, if $0<\lambda<\frac{1}{2}\omega_1$, we have

$$F'(\lambda)\equiv -2\wp(2\lambda+\omega_3)-\frac{2\eta_1}{\omega_1} \quad \text{and} \quad F''(\lambda)\equiv -4\wp'(2\lambda+\omega_3)<0,$$

Thus $F'(\lambda)$ can only change sign once between $\lambda=0$, $\lambda=\frac{1}{2}\omega_1$, and so $F(\lambda)=0$ has no other root between $\lambda=0$ and $\lambda=\frac{1}{2}\omega_1$.

Hence finally we have

(21B) $$\lambda+i\mu=\tfrac{1}{2}(\omega_1+\omega_3)=\alpha+i\beta,$$

we have also $$h=h', \quad h^2=(e_2-e_3),$$

(22) $$\begin{cases} f(x)\equiv\phi(x)\equiv(x-e_2)(x-e_3)-(e_2-e_3)(x-e_1)\equiv(x-e_2)^2-2e_2^2-e_1e_3, \\ \gamma=\xi=e_2, \quad \delta=\eta=-\sqrt{\{(e_1-e_2)(e_2-e_3)\}}, \\ A=-\dfrac{hM}{2\eta^2U_0}, \quad B=C=0, \quad D=-\dfrac{he_2M}{\eta^2U_0}, \end{cases}$$

and the singularities of z at $-\frac{1}{2}\omega_1\pm\frac{1}{2}\omega_3$ disappear.

The corresponding expressions for w and z reduce to

(23) $$w=-\frac{\kappa}{\pi}\arctan\frac{\{\wp(u)-e_2\}}{|\eta|}, \quad \kappa=-\frac{M\pi}{|\eta|},$$

(24) $$z=\frac{hM}{2\eta^2U_0}\Sigma\xi(u-\tfrac{1}{2}\omega_1\pm\tfrac{1}{2}\omega_3)-\frac{he_2M}{\eta^2U_0}(u-\tfrac{1}{2}\omega_1)$$

or $$z=\frac{hM}{\eta^2U_0}\left\{\xi(v)+\tfrac{1}{2}\frac{\wp'(v)}{\wp(v)-\wp(\frac{1}{2}\omega_3)}-e_2v\right\},$$

where $v=u-\frac{1}{2}\omega_1$.

The length of either obstacle is

(25) $$l=\left|\frac{hM}{\eta^2U_0}(h+e_2\omega_1-\eta_1)\right|,$$

and the pressure on each is

(26) $$P=\frac{\rho U_0M}{\eta^2}|\eta_1-e_2\omega_1|.$$

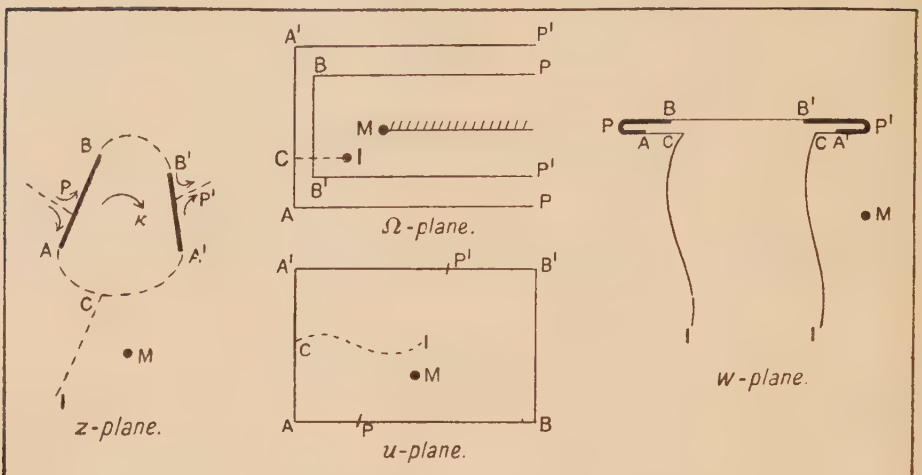


FIG. I. The Configuration with Branch Points.

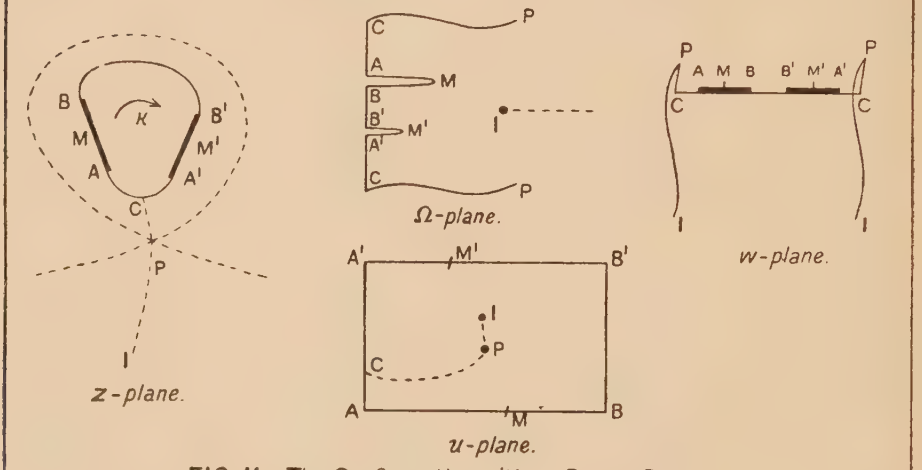
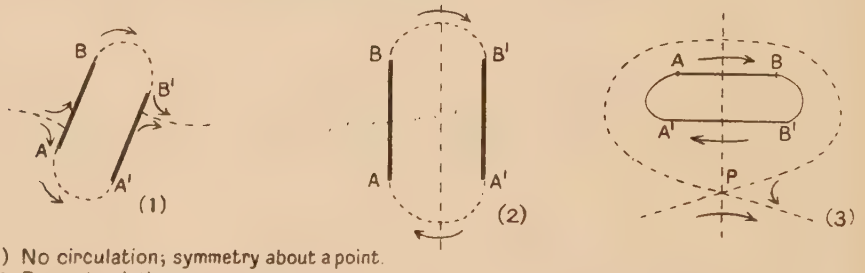


FIG. II. The Configuration without Branch Points.



(1) No circulation; symmetry about a point.
(2) Pure circulation; complete symmetry.
(3) Obstacles parallel to the stream

FIG. III. Parallel Obstacles.

ON INTEGRALS WITH A NUCLEUS

By J. M. WHITTAKER.

(*Communicated by* E. T. WHITTAKER.)

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1. The integrals with which we are concerned are of the form

$$\int_a^b f(x)\, g_1(x)\, dx$$

in which $g_1(x)$ is integrable, in the sense of Riemann, in (a, b). It is shown that, under certain quite general conditions, we can transform an integral of this type into the more manageable form of an integral with respect to the function $g(x)$ defined by

$$g(x) = \int_a^x g_1(t)\, dt,$$

that is to say, what has been called by Prof. Young a "Riemann-Stieltjes integral". These definitions of the integral are followed throughout the paper.

THEOREM.—*Let* $g_1(x) \geqslant 0,$

and let this function be bounded and integrable in (a, b). *Let* $f(x)$ *be bounded in this interval. Then the equation*

$$\int_a^b f(x)\, g_1(x)\, dx = \int_a^b f(x)\, dg(x)$$

is true whenever either side exists.

The proof falls into two parts. In the first* it is shown that the integrals are equal whenever both exist, and that this is the case whenever $\int f\, dg$ exists.

* Suggested by Mr. Pollard.

2. In fact, with the usual notation,

$$S = \sum_{\nu=1}^{n} M_\nu \{g(x_\nu) - g(x_{\nu-1})\},$$

$$s = \sum_{\nu=1}^{n} m_\nu \{g(x_\nu) - g(x_{\nu-1})\},$$

and by the first mean-value theorem

$$m_\nu \int_{x_{\nu-1}}^{x_\nu} g_1(x)\,dx \leqslant \underline{\int}_{x_{\nu-1}}^{x_\nu} f(x)\,g_1(x)\,dx \leqslant \overline{\int}_{x_{\nu-1}}^{x_\nu} f(x)\,g_1(x)\,dx \leqslant M_\nu \int_{x_{\nu-1}}^{x_\nu} g_1(x)\,dx.$$

Thus, by addition,

$$s \leqslant \underline{\int}_a^b f(x)\,g_1(x)\,dx \leqslant \overline{\int}_a^b f(x)\,g_1(x)\,dx \leqslant S.$$

Proceeding to bounds, we have

$$\underline{\int}_a^b f(x)\,dg(x) \leqslant \underline{\int}_a^b f(x)\,g_1(x)\,dx \leqslant \overline{\int}_a^b f(x)\,g_1(x)\,dx \leqslant \overline{\int}_a^b f(x)\,dg(x),$$

from which the required result follows immediately.

3. It remains to prove that the existence of $\int f\,dg$ follows from that of $\int f g_1\,dx$. The argument is longer than that in § 2, and of a different character.

Suppose, to start with, $f(x) \geqslant 0$.

By hypothesis, the functions $g_1(x)$, $f(x)\,g_1(x)$ are integrable in (a, b). The set of points of discontinuity of each of these functions is therefore of measure zero. The same is true of the sum of these sets, which can thus be enclosed in an open set O of measure less than any positive number ϵ given in advance.

At the points of the set G complementary to O with respect to the range (a, b) of integration $g_1(x)$ and $f(x)g_1(x)$ are continuous. We do not know, however, that $f(x)$ by itself is continuous. If this were so we should have nothing more to prove, as $f(x)$ would then certainly be integrable with respect to $g(x)$.

In general it is not the case that $f(x)$ is continuous. A difficulty is introduced, which we overcome by means of the following series of lemmas.

4. LEMMA.—*$g_1(x) = 0$ at every point of $E+E'$, the set E consisting of those points, if any, of G at which $f(x)$ is discontinuous.*

For let ζ be a point of E. Then, since $f(x)$ is discontinuous at ζ, there exists a positive number h such that

$$|f(x_1)-f(\zeta)| > h \tag{4.1}$$

for at least one point x, in every neighbourhood of ζ.

Suppose, if possible, that

$$g_1(\zeta) = a > h.$$

Then, since $g_1(x)$ is continuous at ζ, corresponding to each positive number ω, we can assign η such that

$$a+\omega > g_1(x) > a-\omega,$$

when

$$\zeta-\eta \leqslant x \leqslant \zeta+\eta.$$

Let x_1 be a point in this neighbourhood for which (4.1) is satisfied. Then

$$f(x_1) > f(\zeta) \quad \text{or} \quad f(x_1) < f(\zeta).$$

Take the first alternative. Then

$$\begin{aligned} f(x_1)\,g_1(x_1)-f(\zeta)\,g_1(\zeta) &> \{h+f(\zeta)\}(a-\omega)-f(\zeta)\,a \\ &= h(a-\omega)-\omega f(\zeta) \\ &> \tfrac{1}{2}ha \end{aligned}$$

by a suitable choice of ω. If

$$f(x_1) < f(\zeta),$$

we find similarly $\quad f(\zeta)\,g_1(\zeta)-f(x_1)\,g_1(x_1) > \tfrac{1}{2}ha.$

Thus we have found a positive number $\tfrac{1}{2}h\alpha$, which is such that

$$|f(x_1)\,g_1(x_1)-f(\zeta)\,g_1(\zeta)| > \tfrac{1}{2}ha$$

for at least one point x_1 in every neighbourhood of ζ.

As this contradicts the hypothesis that $f(x)\,g_1(x)$ is continuous at ζ, we must have $\alpha = 0$ or

$$g_1(\zeta) = 0.$$

Moreover, $g_1(x)$ is continuous at all points of G, which is closed and which contains E. Hence

$$g_1(x) = 0$$

at every point of E'.

LEMMA 2.—*If* $\{\Delta_n\}$ *is the system of nets such that the end-points of the meshes of* Δ_n *are the points* $0, 1/2^n, 2/2^n, \ldots, (2^n-1)/2^n, 1$, *there is an integer* ν *such that*

$$0 \leqslant g_1(x) < \epsilon$$

in every mesh of Δ_ν *which includes a point of* $E+E'$.

For the set $E+E'$ is closed, and is a subset of G at the points of which $g_1(x)$ is continuous. It follows by the usual bisection argument that we can find ν such that the oscillation of $g_1(x)$ in a mesh of Δ_ν, including a point of $E+E'$, is less than ϵ.

But, by Lemma 1,

$$g_1(x) = 0 \text{ at every point of } E+E'.$$

Hence

$$0 \leqslant g_1(x) < \epsilon$$

in every mesh of Δ_ν which has in its interior, or as one of its end-points, a point of $E+E'$.

LEMMA 3.—*If* α, β *are the end-points of a mesh of* Δ_ν *as obtained above*,

$$g(\beta)-g(\alpha) < \epsilon(\beta-\alpha). \tag{4.2}$$

For

$$g(\beta)-g(\alpha) = \int_\alpha^\beta g_1(t)\,dt < \epsilon \int_\alpha^\beta dt = \epsilon(\beta-\alpha).$$

5.* Now let Λ denote the set of meshes of Δ_ν which include points of $E+E'$. The complement of Λ contains no points of discontinuity of $f(x)$, except possibly points interior to the open set O. Let H_ϵ denote the set of those of them at which the oscillation of $f(x)$ is greater than or equal to ϵ.

H_ϵ is closed. If, then, we associate with each point of H_ϵ the interval of O to which it is interior (cutting off from this interval any parts of it which lie in Λ), the Heine-Borel theorem shows that there is a finite set O_ϵ of intervals of O having all the points of H_ϵ in its interior.

We have now subtracted from (a, b) two finite sets of intervals, Λ and O_ϵ. The set which remains consists of a finite number of closed† intervals and is such that at every point of it the oscillation of $f(x)$ is

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* The argument of this section is similar to that used to establish Prof. Young's necessary and sufficient condition that an R.S.-integral exists. Cf. Hobson, *Functions of a Real Variable*, 1 (2 ed.), 509. I owe this remark to the referee.

† An end-point of an interval Λ or O_ϵ is taken as a point of N if it is on the other side an end-point of an interval of N. Otherwise it is taken as a point of $\Lambda + O_\epsilon$.

less than ϵ. It can, therefore, be subdivided further into a finite set N of sub-intervals, in each of which the oscillation of $f(x)$ is less than 2ϵ.

6. Let $x_0 = a,\ x_1,\ x_2,\ \ldots,\ x_\nu,\ \ldots,\ x_n = b$ be in order the finite set of points each of which is an end-point of an interval of one of the sets $\Lambda,\ O_\epsilon,\ N$.

Then, if $H,\ K$ denote the upper bounds of $f(x),\ g_1(x)$ in (a, b) and $H^\nu_{\nu-1},\ h^\nu_{\nu-1}$ the upper and lower bounds of $f(x)$ in $(x_{\nu-1},\ x_\nu)$

$$\sum_{\nu=1}^{n} \{H^\nu_{\nu-1} - h^\nu_{\nu-1}\} \{g(x_\nu) - g(x_{\nu-1})\} < H(b-a)\epsilon + HK\epsilon + K(b-a) < 2\epsilon.$$

The first term on the right arises from the intervals Λ. For these, by (4 . 2),

$$\Sigma_\Lambda \{g(x_\nu) - g(x_{\nu-1})\} < \epsilon \Sigma_\Lambda (x_\nu - x_{\nu-1}) \leqslant \epsilon(b-a).$$

The second term arises from the intervals O_ϵ. For these

$$\Sigma_{O_\epsilon} \{g(x_\nu) - g(x_{\nu-1})\} < K \Sigma_{O_\epsilon} (x_\nu - x_{\nu-1}) < K\epsilon$$

since $$mO_\epsilon \leqslant mO < \epsilon.$$

Finally, for the intervals N, which give rise to the last term

$$H^\nu_{\nu-1} - h^\nu_{\nu-1} < 2\epsilon,$$

and $$\Sigma_N \{g(x_\nu) - g(x_{\nu-1})\} < \sum_{\nu=1}^{n} \{g(x_\nu) - g(x_{\nu-1})\} = g(x_n) \leqslant K(b-a).$$

Now ϵ is arbitrary. Hence, with the stated conditions and with

$$f(x) \geqslant 0$$

we have proved the existence of

$$\int_a^b f(x)\, dg(x).$$

7. In the general case we take

$$\begin{aligned} f_+(x) &= f(x) && \text{when} \quad f(x) \geqslant 0, \\ &= 0 && \text{when} \quad f(x) < 0, \\ f_-(x) &= -f(x) && \text{when} \quad f(x) \leqslant 0, \\ &= 0 && \text{when} \quad f(x) > 0, \end{aligned}$$

so that $$f(x) = f_+(x) - f_-(x).$$

We have now to prove that if the functions $f(x)\,g_1(x)$, $g_1(x)$ are integrable, then the functions $f_+(x)\,g_1(x)$, $f_-(x)\,g_1(x)$ are integrable.*

LEMMA 4.—*At a point of discontinuity of* $f_\pm(x)\,g_1(x)$ *at least one of the functions* $f(x)\,g_1(x)$, $g_1(x)$ *is discontinuous.*

For let $f_+(x)\,g_1(x)$ be discontinuous at ζ, and let $g_1(x)$ be continuous at this point. Then, if

$$g_1(\zeta) > 0,$$

there is a neighbourhood of ζ within which $g_1(x)$ is positive. If x_1 be a point within this neighbourhood,

$$|f_+(x_1)\,g_1(x_1) - f_+(\zeta)\,g_1(\zeta)| \leqslant |f(x_1)\,g_1(x_1) - f(\zeta)\,g_1(\zeta)|.$$

This is easily seen to be true in each of the possible cases :—(*a*) $f(x_1)$, $f(\zeta)$ both positive or zero; (*b*) $f(x_1)$, $f(\zeta)$ both negative, (*c*) $f(x_1)$ positive or zero, and $f(\zeta)$ negative, or *vice versa*. The inequality also holds if $g_1(\zeta)$ is negative. It is now obvious that if

$$g_1(\zeta) > 0 \quad \text{or} \quad g_1(\zeta) < 0$$

$f(x)\,g_1(x)$ is discontinuous at ζ.

The case

$$g_1(\zeta) = 0$$

is excluded because $f_+(x)\,g_1(x)$ would then be continuous at ζ.

The case of $f_-(x)\,g_1(x)$ is treated in the same way.

The sets of discontinuities of $f_\pm(x)\,g_1(x)$ are thus both contained in the set of points at which one or other of the functions $f(x)\,g_1(x)$, $g_1(x)$ is discontinuous. This latter set is, by hypothesis, of measure zero, and so the functions $f_\pm(x)\,g_1(x)$ are integrable, and

$$\int_a^b f_+(x)\,g_1(x)\,dx - \int_a^b f_-(x)\,g_1(x)\,dx$$

$$= \int_a^b \{f_+(x) - f_-(x)\}\,g_1(x)\,dx = \int_a^b f(x)\,g_1(x)\,dx.$$

Transforming each of the integrals on the left side by the theorem of § 6, we deduce the final result :

$$\int_a^b f(x)\,g_1(x)\,dx = \int_a^b f_+(x)\,g_1(x)\,dx - \int_a^b f_-(x)\,g_1(x)\,dx$$

$$= \int_a^b f_+(x)\,dg(x) - \int_a^b f_-(x)\,dg(x)$$

$$= \int_a^b f(x)\,dg(x).$$

* The proof of this was added 19 May, 1925.

A FURTHER NOTE ON THE CONVERSE OF ABEL'S THEOREM

By G. H. HARDY *and* J. E. LITTLEWOOD.

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1. *Introduction.*

1.1. In this note we prove a theorem conjectured by Littlewood in a paper published in these *Proceedings* in 1910.*

The main result of Littlewood's paper was as follows. *Suppose that*

$$0 < \lambda_{n-1} < \lambda_n \quad (n = 2, 3, \ldots), \tag{1.11}$$

$$\lambda_n \to \infty, \tag{1.12}$$

$$\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \to 0; \tag{1.13}$$

and that

$$a_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right), \tag{1.14}$$

so that the series

$$f(t) = \sum_1^\infty a_n e^{-\lambda_n t} \tag{1.15}$$

is convergent for $t > 0$. *Then the existence of the limit*

$$\lim_{t \to 0} f(t) = A \tag{1.16}$$

implies the convergence of the series Σa_n *to sum* A.

* J. E. Littlewood, "The converse of Abel's theorem on power series", *Proc. London Math. Soc.* (2), 9 (1910), 434–448.

It was observed by Littlewood that the condition (1 . 13) is not essential to the truth of the theorem, but that it embodies, none the less, an important distinction. Suppose, first, that (1 . 13) is true, as in the standard case when $\lambda_n = n$. Then (1 . 14), in this particular case

$$a_n = O\left(\frac{1}{n}\right), \tag{1 . 17}$$

is not only a sufficient but also a "best possible" condition, which cannot be replaced by any less restrictive condition of the type $a_n = O(\phi_n)$. This ceases to be true when (1 . 13) is false, for example when $\lambda_n = 2^n$. In this case (1 . 14) becomes

$$a_n = O(1), \tag{1 . 18}$$

which is, in fact, a sufficient condition. But (1 . 18) is in no sense a best possible condition, and the conjecture to which we have referred, and which we now propose to substantiate, is this, that if

$$\frac{\lambda_n}{\lambda_{n-1}} \geqslant \theta > 1,$$

so that (1 . 14) becomes (1 . 18), or, in other words, if λ_n increases with sufficient rapidity and regularity, then the conclusion of the theorem is true *without any restriction whatever on* a_n.

1 . 2. Our main result, then, will be

THEOREM 1.—*If*

$$\lambda_1 > 0, \quad \frac{\lambda_{n+1}}{\lambda_n} \geqslant \theta > 1 \quad (n = 1, 2, \ldots), \tag{1 . 21}$$

and

$$f(t) = \sum_1^\infty a_n e^{-\lambda_n t} \to s, \tag{1 . 22}$$

when $t \to 0$, *then* Σa_n *converges to* s.

It should be observed that (1 . 22) is to be interpreted as implying the convergence of the series for $t > 0$. The condition for this is that

$$a_n = O(e^{\delta\lambda_n}) \tag{1 . 23}$$

for every positive δ.

The theorem at once shows that such functions as

$$e^{-\alpha t} - e^{-\alpha^2 t} + e^{-\alpha^3 t} - \ldots \quad (\alpha > 1), \qquad e^{-1!\,t} - e^{-2!\,t} + e^{-3!\,t} - \ldots$$

cannot tend to limits when $t \to 0$. Such special oscillating functions have been considered by various writers.*

The proof of the theorem divides sharply into two stages. In § 2 we prove it when $a_n = O(1)$, the present form of (1 . 14). In §§ 3-4 we show that the general case may be reduced to that of § 2, and thus complete the proof. In § 5 we show how our analysis may be applied to the construction of non-differentiable functions.

It might seem natural that we should go further. We might suppose that $f(t)$ satisfies an asymptotic formula of the type

$$f(t) \sim A t^{-\alpha} \left(\log \frac{1}{t}\right)^{\beta} \ldots,$$

and attempt to deduce corresponding asymptotic formulae for

$$s_n = a_1 + a_2 + \ldots + a_n,$$

on the lines of our corresponding researches for series subject to (1 . 13).† We leave such questions aside, because of the recent discovery by R. Schmidt of an alternative line of attack on these Tauberian theorems.‡ The method of Schmidt, involving, as it does, not only our machinery of repeated differentiation, but also the general theory of the "moment problem" of Stieltjes, is much less direct and elementary than ours, and the most important results should be proved in the simplest manner possible. But in view of the great power and elegance of Schmidt's methods, and the very comprehensive character of his results, we may well be content to leave our own researches in this field where they stand at present.

We must express our thanks to Prof. G. Pólya, who read this paper in manuscript, and made a series of suggestions which have led to a great simplification of the proof. A number of the lemmas, and, in particular, Lemmas 1, 2, 5, 6, and 10, in their present form, are due to him. We

* See, for example, G. H. Hardy, "On certain oscillating series", *Quart. J. of Math.*, 38 (1907), 268–288, and "Some theorems concerning infinite series", *Math. Annalen*, 64 (1907), 77–94; J. Belifante, "On power-series of the form $x^{p_0} - x^{p_1} + x^{p_2} - \ldots$", *Proc. Acad. Sc. Amsterdam*, 26 (1924), 456–462.

† G. H. Hardy and J. E. Littlewood, "Some theorems concerning Dirichlet's series", *Messenger of Math.*, 43 (1914), 134–147. See also "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.* (2), 13 (1913), 174–191.

‡ R. Schmidt, "Über divergente Reihen und lineare Mittelbildungen", *Math. Zeitschrift*, 22 (1925), 89–152. See also "Die Umkehrsätze des Borelschen Summierungsverfahrens", *Schriften d. Königsberger Gelehrten Gesellschaft*, 1 (1925), 205–256.

have also adopted his suggestion of enunciating Theorem 2 (§ 4 . 5) as an explicit theorem.

2. *Proof of Theorem* 1 *when* $a_n = O(1)$.

2 . 1. In what follows the letter A denotes always a positive number which is a function of θ only. O's and o's refer to the passage of n to infinity or t to zero, and are not, in general, uniform in the parameters of the analysis.* We suppose, as we may, that $|a_n|<1$. We attribute to λ_0 the value 0.

LEMMA 1.—*If* $0<\alpha<1<\beta$, $r=2, 3, \ldots$, *then*

$$(2.11)\quad \int_0^{\alpha r} x^r e^{-x}\,dx < \left(\frac{\alpha}{1-\alpha}\right)^2 \frac{r!}{r-1}, \quad \int_{\beta r}^{\infty} x^r e^{-x}\,dx < \left(\frac{1}{\beta-1}\right)^2 \frac{r!\,(r+2)}{r^2}.$$

We have

$$\int_0^{\infty} \left(\frac{r}{x}-1\right)^2 x^r e^{-x}\,dx = \frac{r!}{r-1},$$

$$\left(\frac{1}{\alpha}-1\right)^2 \int_0^{\alpha r} x^r e^{-x}\,dx < \int_0^{\alpha r} \left(\frac{r}{x}-1\right)^2 x^r e^{-x}\,dx$$

$$< \int_0^{\infty} \left(\frac{r}{x}-1\right)^2 x^r e^{-x}\,dx = \frac{r!}{r-1},$$

which proves the first of the inequalities (2 . 11). The second follows similarly from the equation

$$\int_0^{\infty} (x-r)^2 x^r e^{-x}\,dx = r!\,(r+2).$$

2 . 2. LEMMA 2.—*If*

$$(2.21)\quad t = \frac{r}{\sqrt{(\lambda_\nu \lambda_{\nu+1})}} \quad (\nu = 1, 2, \ldots),$$

$$(2.22)\quad w_n = w_n(r, \nu) = \frac{1}{r!} \int_{\lambda_n t}^{\lambda_{n+1} t} x^r e^{-x}\,dx \quad (n = 0, 1, 2, \ldots),$$

then

$$(2.23)\quad \nu w_0 + (\nu-1) w_1 + \ldots + 2w_{\nu-2} + w_{\nu-1} + w_{\nu+1} + 2w_{\nu+2} + \ldots < \frac{A}{r}.$$

---

* Thus $f = O(\phi)$, where f and ϕ are functions of t and (say) θ and k, means $|f| < C\phi$, where C is independent of t but a function of θ and k.

We write

$$u_m = w_{\nu-m}+w_{\nu-m-1}+\ldots+w_0 = \frac{1}{r!}\int_0^{\lambda_{\nu-m+1}t} x^r e^{-x}\,dx,$$

$$v_m = w_{\nu+m}+w_{\nu+m+1}+\ldots = \frac{1}{r!}\int_{\lambda_{\nu+m}t}^{\infty} x^r e^{-x}\,dx.$$

Then
$$u_m = \frac{1}{r!}\int_0^{\alpha r} x^r e^{-x}\,dx,$$

where
$$\alpha = \frac{\lambda_{\nu-m+1}}{\lambda_\nu}\sqrt{\left(\frac{\lambda_\nu}{\lambda_{\nu+1}}\right)} < \theta^{\frac{1}{2}-m};$$

and so, by Lemma 1,

(2.24)
$$u_m < \frac{1}{r-1}\left(\frac{1}{\theta^{m-\frac{1}{2}}-1}\right)^2.$$

Similarly
$$v_m = \frac{1}{r!}\int_{\beta r}^{\infty} x^r e^{-x}\,dx,$$

where
$$\beta = \frac{\lambda_{\nu+m}}{\lambda_{\nu+1}}\sqrt{\left(\frac{\lambda_{\nu+1}}{\lambda_\nu}\right)} > \theta^{m-\frac{1}{2}};$$

and so

(2.25)
$$v_m < \frac{r+2}{r^2}\left(\frac{1}{\theta^{m-\frac{1}{2}}-1}\right)^2.$$

From (2.24) and (2.25) it follows that

$$\nu w_0+(\nu-1)w_1+\ldots+w_{\nu-1}+w_{\nu+1}+\ldots = u_1+u_2+\ldots+u_m+v_1+v_2+\ldots$$
$$< \left(\frac{1}{r-1}+\frac{r+2}{r^2}\right)\sum_{m=1}^{\infty}\left(\frac{1}{\theta^{m-\frac{1}{2}}-1}\right)^2 < \frac{A}{r}.$$

2.3. LEMMA 3.—*If* $r>0$, *then*

(2.31)
$$\sum_1^{\infty}\lambda_n^r e^{-\lambda_n t} = O(t^{-r}).$$

Suppose that
$$\lambda = r/t, \quad \lambda_\mu \leqslant \lambda < \lambda_{\mu+1}.$$

Then
$$t^r\sum_1^{\infty}\lambda_n^r e^{-\lambda_n t} = r^r e^{-r}\sum_1^{\infty}\left\{\frac{\lambda_n}{\lambda}\exp\left(1-\frac{\lambda_n}{\lambda}\right)\right\}^r = r^r e^{-r}S,$$

say. The function xe^{1-x} increases from $x=0$ to $x=1$, when it attains the maximum 1, and then decreases, so that no term of S exceeds 1. If $n \leqslant \mu-1$, $n=\mu-m$, where $1 \leqslant m \leqslant \mu-1$, and

$$\frac{\lambda_n}{\lambda} = \frac{\lambda_{\mu-m}}{\lambda} \leqslant \frac{\lambda_{\mu-m}}{\lambda_\mu} < \theta^{-m} < 1;$$

and if $n \geqslant \mu+2$, $n = \mu+m+1$, where $m \geqslant 1$, and

$$\frac{\lambda_n}{\lambda} = \frac{\lambda_{\mu+m+1}}{\lambda} > \frac{\lambda_{\mu+m+1}}{\lambda_{\mu+1}} > \theta^m > 1.$$

Hence

$$S < 2+\sum_{m=1}^{\mu-1}\left\{\frac{\lambda_{\mu-m}}{\lambda_\mu}\exp\left(1-\frac{\lambda_{\mu-m}}{\lambda_\mu}\right)\right\}^r + \sum_{m=1}^{\infty}\left\{\frac{\lambda_{\mu+m+1}}{\lambda_\mu}\exp\left(1-\frac{\lambda_{\mu+m+1}}{\lambda_\mu}\right)\right\}^r *$$

$$< 2+\sum_{m=1}^{\infty}(\theta^{-m}e^{1-\theta^{-m}})^r + \sum_{m=1}^{\infty}(\theta^m e^{1-\theta^m})^r.$$

These series are greatest when $r=1$, so that $S<A$, which proves the lemma.

LEMMA 4.—*If* $f(t)\to s$ *when* $t\to 0$, *and*

$$t^r f^{(r)}(t) = O(1)$$

for every r, *then* $t^r f^{(r)}(t) = o(1)$

for every r.†

2.4. We can now prove Theorem 1 when $a_n = O(1)$. We may obviously suppose without loss of generality that $s=0$. We have then $f(t)=o(1)$, and

$$t^r f^{(r)}(t) = (-1)^r t^r \sum_1^\infty a_n \lambda_n^r e^{-\lambda_n t} = O\left(t^r \sum_1^\infty \lambda_n^r e^{-\lambda_n t}\right) = O(1),$$

by Lemma 3, and therefore

$$t^r f^{(r)}(t) = o(1),$$

by Lemma 4.

Now

$$f(t) = \sum_1^\infty a_n e^{-\lambda_n t} = \sum_1^\infty s_n(e^{-\lambda_n t}-e^{-\lambda_{n+1}t}) = t\sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} e^{-ut}\,du,$$

$$(-t)^r f^{(r)}(t) = t^{r+1}\sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} u^r e^{-ut}\,du - rt^r \sum_1^\infty s_n \int_{\lambda_n}^{\lambda_{n+1}} u^{r-1}e^{-ut}\,du$$

$$= V_r - rV_{r-1},$$

say. Hence

$$(2.41)\quad V_r = rV_{r-1}+o(1) = r(r-1)\,V_{r-2}+o(1) = \ldots = r!\,V_0+o(1)$$

$$= r!f(t)+o(1) = o(1).$$

* The first of these sums is absent if $\mu < 2$.

† J. E. Littlewood, *loc. cit.*, 438.

Now $$r!\, s_\nu = s_\nu \int_0^\infty x^r e^{-x} dx = r!\, s_\nu \sum_0^\infty w_n;$$

and so, if $s_0 = 0$, $$\overline{\lim_{\nu\to\infty}} |s_\nu| = \overline{\lim_{\nu\to\infty}} \left| \sum_{n=0}^{\infty} s_\nu w_n \right|$$

$$\leqslant \overline{\lim_{\nu\to\infty}} \left| \sum_0^\infty s_n w_n \right| + \overline{\lim_{\nu\to\infty}} \left| \sum_0^\infty (s_n - s_\nu) w_n \right|.$$

But $$r! \sum_0^\infty s_n w_n = \sum_1^\infty s_n \int_{\lambda_n t}^{\lambda_{n+1} t} x^r e^{-x} dx = V_r = o(1),$$

by (2 . 41). Hence, since $|a_n| < 1$, we have

$$\overline{\lim_{\nu\to\infty}} |s_\nu| \leqslant \overline{\lim_{\nu\to\infty}} \left| \sum_0^\infty (s_n - s_\nu) w_n \right|$$

$$\leqslant \nu w_0 + (\nu-1) w_1 + \ldots + w_{\nu-1} + w_{\nu+1} + 2w_{\nu+2} + \ldots < \frac{A}{r},$$

by Lemma 2. This is only possible if $s_\nu \to 0$, which proves the theorem.

2 . 5. So far we have *assumed* the truth of (1 . 18). The remainder of the proof of Theorem 1 consists in showing that its hypotheses *imply* (1 . 18). For this, it is plainly sufficient to show that

(2 . 51) $$f(t) = O(1)$$

implies (1 . 18). We suppose, therefore, that

(2 . 52) $$a_n \neq O(1),$$

and we show that (2 . 51) is then impossible.

We write $$\overline{\lim} \frac{\log |a_n|}{\log \lambda_n} = \kappa,$$

so that $0 \leqslant \kappa \leqslant \infty$; and we may distinguish three cases: (i) $\kappa = \infty$, (ii) $0 < \kappa < \infty$, and (iii) $\kappa = 0$.

3. *Discussion of case* (i) : $\kappa = \infty$.

3 . 1. The discussion of case (i) does not require the machinery of differentiation. We use two additional lemmas.

LEMMA 5.—*There is an* $R = R(\theta)$* *such that*

(3 . 11) $$\lambda_\nu^r e^{-\lambda_\nu t_\nu} > 2 \left(\sum_{n=1}^{\nu-1} + \sum_{n=\nu+1}^{\infty} \right) \lambda_n^r e^{-\lambda_n t_\nu}$$

* So that R is an A.

when

$$r \geqslant R, \quad t_\nu = r/\lambda_\nu \quad (\nu = 1, 2, \ldots). \tag{3.12}$$

We have

$$S = \lambda_\nu^{-r} e^{\lambda_\nu t_\nu} \left(\sum_{n=1}^{\nu-1} + \sum_{n=\nu+1}^{\infty} \right) \lambda_n^r e^{-\lambda_n t_\nu}$$

$$= \sum_{m=1}^{\nu-1} \left\{ \frac{\lambda_{\nu-m}}{\lambda_\nu} \exp\left(1 - \frac{\lambda_{\nu-m}}{\lambda_\nu}\right) \right\}^r + \sum_{m=1}^{\infty} \left\{ \frac{\lambda_{\nu+m}}{\lambda_\nu} \exp\left(1 - \frac{\lambda_{\nu+m}}{\lambda_\nu}\right) \right\}^r$$

$$< \sum_{m=1}^{\infty} (\theta^{-m} e^{1-\theta^{-m}})^r + \sum_{m=1}^{\infty} (\theta^m e^{1-\theta^m})^r,$$

as in the proof of Lemma 3. These series are convergent when $r = 1$, and their terms decrease when r increases, so that they are uniformly convergent for $r \geqslant 1$. Finally, each term tends to zero when $r \to \infty$, so that $S \to 0$, which proves the lemma. It is plain that the 2 of the enunciation might be replaced by any larger constant.

3.2. LEMMA 6.—*Suppose that $r \geqslant R$, where R is defined as in Lemma 5; that the series*

$$g(t) = \sum_1^{\infty} b_n e^{-\lambda_n t} \tag{3.21}$$

is convergent for $t > 0$, and that

$$\overline{\lim_{n \to \infty}} \, \lambda_n^{-r} | b_n | = \infty. \tag{3.22}$$

Then

$$\overline{\lim_{t \to 0}} \, t^r | g(t) | = \infty. \tag{3.23}$$

We denote by $\nu = \nu(t)$ the index and by $\mu = \mu(t)$ the value of the maximum term of the series $\Sigma | b_n | \lambda_n^{-r} e^{-\lambda_n t}$, so that

$$\mu(t) = \operatorname*{Max}_{1 \leqslant n < \infty} | b_n | \lambda_n^{-r} e^{-\lambda_n t} = | b_\nu | \lambda_\nu^{-r} e^{-\lambda_\nu t}.$$

If several terms have the same maximum value, ν is the highest of their indices. It is plain that μ increases steadily when t decreases. Also

$$\mu \geqslant | b_n | \lambda_n^{-r} e^{-\lambda_n t}$$

for every n. If H is given we can, by (3.22), determine n so that $| b_n | \lambda_n^{-r} > 2H$, and then choose t so that $e^{-\lambda_n t} > \frac{1}{2}$, $\mu > H$. It

follows that $\mu \to \infty$ when $t \to 0$, and from this that $\nu \to \infty$, since every term of the series is bounded. We can therefore, given any pair of positive numbers ϵ, g, choose τ so that

$$(3.24) \quad 0 < \tau < \epsilon, \quad \nu = \nu(\tau) > g, \quad \mu = \mu(\tau) = |b_\nu| \lambda_\nu^{-r} e^{-\lambda_\nu \tau} > g.$$

We now take

$$(3.25) \quad t = \tau + t_\nu = \tau + \frac{r}{\lambda_\nu}$$

(so that t_ν is defined as in Lemma 5). Then t tends to 0 with ϵ, since $\nu \to \infty$. Since ν defines the maximum term, we have

$$|b_n| \lambda_n^{-r} e^{-\lambda_n \tau} \leqslant |b_\nu| \lambda_\nu^{-r} e^{-\lambda_\nu \tau}$$

for all values of n, and so

$$\frac{|b_n| e^{-\lambda_n t}}{|b_\nu| e^{-\lambda_\nu t}} = \frac{|b_n| e^{-\lambda_n (\tau + t_\nu)}}{|b_\nu| e^{-\lambda_\nu (\tau + t_\nu)}} \leqslant \frac{\lambda_n^r e^{-\lambda_n t_\nu}}{\lambda_\nu^r e^{-\lambda_\nu t_\nu}}.$$

Hence, by Lemma 5,

$$\frac{1}{|b_\nu| e^{-\lambda_\nu t}} \left\{ \left(\sum_1^{\nu-1} + \sum_{\nu+1}^{\infty} \right) |b_n| e^{-\lambda_n t} \right\} \leqslant \frac{1}{\lambda_\nu^r e^{-\lambda_\nu t_\nu}} \left\{ \left(\sum_1^{\nu-1} + \sum_{\nu+1}^{\infty} \right) \lambda_n^r e^{-\lambda_n t_\nu} \right\} < \tfrac{1}{2};$$

and

$$(3.26) \quad |g(t)| \geqslant |b_\nu| e^{-\lambda_\nu t} - \left(\sum_1^{\nu-1} + \sum_{\nu+1}^{\infty} \right) |b_n| e^{-\lambda_n t} > \tfrac{1}{2} |b_\nu| e^{-\lambda_\nu t}$$

$$= \tfrac{1}{2} \lambda_\nu^r e^{-\lambda_\nu t_\nu} . |b_\nu| \lambda_\nu^{-r} e^{-\lambda_\nu \tau} = \tfrac{1}{2} r^r e^{-r} t_\nu^{-r} \mu(\tau)$$

$$> \tfrac{1}{2} r^r e^{-r} t^{-r} \mu(t),$$

since τ and t_ν are each less than t. It is plain that (3.26) contains the result of Lemma 6, and more.

3.3. We can now dispose of case (i) of Theorem 1. If $\kappa = \infty$,

$$\overline{\lim}\, \lambda_n^{-r} |a_n| = \infty$$

for every r. We may therefore take $b_n = a_n$, $f(t) = g(t)$ in Lemma 6, and it appears at once that $f(t)$ is unbounded, and indeed that

$$\overline{\lim}\, t^r |f(t)| = \infty$$

for every r.

4. *Discussion of cases* (ii) *and* (iii).

4.1. The remaining cases, which we can discuss together, are rather more delicate. We require the machinery of repeated differentiation, and in particular the following lemma.

LEMMA 7.—*If* $\phi(t)$ *increases* as* t *decreases,*

$$f(t) = o(\phi)$$

when $t \to 0$, *and*

$$t^r f^{(r)}(t) = O(\phi)$$

for every r, *then*

$$t^r f^{(r)}(t) = o(\phi)$$

for every r.†

4.2. We suppose now that

$$\overline{\lim_{n\to\infty}} |a_n| = \infty, \tag{4.21}$$

$$\overline{\lim} \frac{\log |a_n|}{\log \lambda_n} = \kappa < \infty, \tag{4.22}$$

and we have to prove that $f(t)$ is unbounded. We write

$$b_n = a_n \lambda_n^r, \quad g(t) = \Sigma b_n e^{-\lambda_n t} = (-1)^r f^{(r)}(t), \tag{4.23}$$

$$\mu(t) = |a_\nu| e^{-\lambda_\nu t} = \text{Max}\,(|a_n| e^{-\lambda_n t}) = \text{Max}\,(|b_n| \lambda_n^{-r} e^{-\lambda_n t}). \tag{4.24}$$

The definitions of $\mu(t)$ and $\nu(t)$ then agree with those of Lemma 6.

LEMMA 8.—*If* $\kappa < l$, *then* $t^l \mu(t) \to 0$.

For

$$t^l \mu(t) = |a_\nu| \lambda_\nu^{-l} (t\lambda_\nu)^l e^{-\lambda_\nu t} \leqslant l^l e^{-l} |a_\nu| \lambda_\nu^{-l},$$

which tends to zero, since $\nu \to \infty$ and $\kappa < l$.

LEMMA 9.—*There are arbitrarily small values of* t *for which*

$$\mu(\tfrac{1}{2}t) < 2^l \mu(t).$$

For otherwise there would be a positive δ such that

$$\mu(2^{-n}\delta) \geqslant 2^l \mu(2^{-n+1}\delta) \geqslant \ldots \geqslant 2^{nl} \mu(\delta),$$

$$(2^{-n}\delta)^l \mu(2^{-n}\delta) \geqslant \delta^l \mu(\delta),$$

for every n, and this would contradict Lemma 8.

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* In the wide sense.

† G. H. Hardy and J. E. Littlewood, "Contributions to the arithmetic theory of series", *Proc. London Math. Soc.* (2), 11 (1912), 411–478 (Theorem 6). We take this opportunity of observing that the proof given in this memoir of Theorem 2 is superfluous, since Theorem 2 can be deduced from Theorem 3 (which is independent of it) by making r tend to infinity.

4.3. LEMMA 10.—*We have*

$$\overline{\lim_{t\to\infty}} \frac{t^r\,|f^{(r)}(t)|}{\mu(\frac{1}{2}t)} < \infty \tag{4.31}$$

for every r, *and*

$$\overline{\lim_{t\to\infty}} \frac{t^r\,|f^{(r)}(t)|}{\mu(\frac{1}{2}t)} > 0 \tag{4.32}$$

for all sufficiently large values of r.

In the first place

$$|f^{(r)}(t)| \leqslant \Sigma(|a_n|\,e^{-\frac{1}{2}\lambda_n t}\,.\,\lambda_n^r e^{-\frac{1}{2}\lambda_n t}) \leqslant \mu(\tfrac{1}{2}t)\,\Sigma\,\lambda_n^r e^{-\frac{1}{2}\lambda_n t}.$$

The last series is $O(t^{-r})$, by Lemma 3, and this proves (4.31).

Next, we choose τ, as we may do after Lemma 9, so that $0 < \tau < \epsilon$ and

$$\mu(\tfrac{1}{2}\tau) < 2^l \mu(\tau).$$

We take this τ as the τ of Lemma 6, and then choose $t = \tau + t_\nu$ as there. We have then

$$|f^{(r)}(t)| > C_r \mu(\tau)\, t^{-r},$$

by (3.26), C_r being a positive number depending only on r. But

$$\mu(\tau) > 2^{-l}\mu(\tfrac{1}{2}\tau) > 2^{-l}\mu(\tfrac{1}{2}t),$$

and so

$$|f^{(r)}(t)| > C_r \mu(\tfrac{1}{2}t)\, t^{-r};$$

which proves (4.32).

4.4. We can now complete the proof of the theorem. In fact, if $f(t) = O(1)$, we have *a fortiori*

$$f(t) = o\{\mu(\tfrac{1}{2}t)\},$$

and

$$t^r f^{(r)}(t) = O\{\mu(\tfrac{1}{2}t)\},$$

for every r, by (4.31). Hence, taking $\phi = \psi = \mu(\frac{1}{2}t)$ in Lemma 7, we have

$$t^r f^{(r)}(t) = o\{\mu(\tfrac{1}{2}t)\}$$

for every r. This plainly contradicts (4.32). Hence $f(t)$ cannot be bounded.

4 . 5. Our primary object in §§ 3–4 has been to prove that $f(t)$ cannot be bounded unless $|a_n|$ is bounded. It may be convenient that we should collect here, as an explicit theorem, and in a somewhat sharpened form, the principal results of our discussion, as these results have an interest independent of their application to the proof of Theorem 1.

THEOREM 2.—*Suppose that*

$$\lambda_1 > 0, \quad \theta > 1, \quad \lambda_{n+1} \geqslant \theta\lambda_n;$$

that $f(t) = \Sigma a_n e^{-\lambda_n t}$ *is convergent for* $t > 0$, *and that*

$$\mu(t) = \operatorname{Max}(|a_n| e^{-\lambda_n t}) \quad (n = 1, 2, \ldots).$$

Then, if

$$\overline{\lim_{n\to\infty}} |a_n| = \infty, \tag{4.51}$$

we have

$$\lim_{t\to 0} \mu(t) = \infty, \tag{4.521}$$

$$\overline{\lim_{t\to 0}} |f(t)| = \infty, \tag{4.522}$$

$$\overline{\lim_{t\to 0}} \frac{|f(t)|}{\mu(t)} > 0. \tag{4.523}$$

And if

$$\overline{\lim_{n\to\infty}} \frac{\log|a_n|}{\log\lambda_n} = \infty, \tag{4.53}$$

we have

$$\overline{\lim_{t\to 0}} \frac{\log\mu(t)}{\log(1/t)} = \infty, \tag{4.541}$$

$$\overline{\lim_{t\to 0}} \frac{\log|f(t)|}{\log(1/t)} = \infty, \tag{4.542}$$

$$\overline{\lim_{t\to 0}} \frac{|f(t)|}{\mu(t)} \geqslant 1. \tag{4.543}$$

Of the propositions (4 . 52), (4 . 521) was proved incidentally, and is in any case almost obvious*; (4 . 522) was our principal conclusion, and we deduced (4 . 522) from (4 . 523).†

* Write a_n for $b_n\lambda_n^{-r}$ in the first sentences of the proof of Lemma 6.

† Strictly, from a slightly stronger proposition (with $\frac{1}{2}t$ for t).

The proofs of the propositions (4 . 54) are contained, in substance, in the proof of Lemma 6. Take there $b_n = a_n$, $g(t) = f(t)$; it will be observed that $\mu(t)$ has then no longer its meaning of § 3 . 2. In virtue of (4 . 53), we have $|a_n| > \lambda_n^r$ for every r and an appropriate sequence of values of n, and

$$t^r \mu(t) \geqslant (\lambda_n t)^r e^{-\lambda_n t} > e^{-1}$$

if $\lambda_n t = 1$. This proves (4 . 541), and (4 . 542) is equivalent to the main result of Lemma 6. Finally

$$|f(t)| \geqslant \tfrac{1}{2}\mu(t) \tag{4 . 57}$$

for an appropriate sequence of values of t, by (3 . 26).* This would give $\frac{1}{2}$ instead of the 1 of (4 . 543). But the $\frac{1}{2}$ arises only from the 2 of Lemma 5, and this 2 could obviously be replaced by any larger constant, so that the $\frac{1}{2}$ of (4 . 57) could be replaced by any number less than 1. This remark completes the proof of Theorem 2, which could naturally have been arranged more elegantly had this theorem been our goal.

5. *Examples of non-differentiable functions.*

5 . 1. We conclude by applying Theorem 1 to the construction of non-differentiable functions. We require two further lemmas.

LEMMA 11.—*Suppose that* $f(s) = f(\sigma+it)$ *is regular for* $\sigma > 0$, *continuous for* $\sigma \geqslant 0$, *uniformly in* t, *and tends uniformly to zero when* $\sigma \to \infty$; *and that* $f(iu) = \phi(u)$. *Then*

$$f(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma\phi(u)}{\sigma^2+(u-t)^2}\,du. \tag{5 . 11}$$

This is a very special case of well known results in the general theory of functions. All the conditions are satisfied if, as we shall suppose,

$$f(s) = \sum_1^\infty a_n e^{-\lambda_n s},$$

where λ_n satisfies (1 . 21) and Σa_n is absolutely convergent.

We write

$$f(s) = G(\sigma, t)+iH(\sigma, t), \quad \phi(t) = g(t)+ih(t).$$

* First line [noting the altered meaning of $\mu(t)$].

LEMMA 12.—*If $\phi(t)$ has a finite derivative $\phi'(t)$ for a particular value of t, then*

$$f'(s) \to \phi'(t) \tag{5.12}$$

when t is fixed and $\sigma \to 0$. If $g(t)$ has a finite derivative $g'(t)$, then

$$\frac{\partial G}{\partial t} \to g'(t); \tag{5.13}$$

and similarly for H and h.

It is plainly sufficient to prove (5.13); and we may suppose, without loss of generality, that the particular value of t in question, and the value of $g'(t)$, are both zero. We have then, on differentiating (5.11) and putting $t = 0$,

$$\frac{\partial G}{\partial t} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sigma u g(u)}{(\sigma^2+u^2)^2}\,du.$$

We can choose δ so that $|g(u)| < \epsilon|u|$ in $(-\delta, \delta)$. The integral outside these limits plainly tends to zero, and

$$\left|\frac{2}{\pi}\int_{-\delta}^{\delta} \frac{\sigma u g(u)}{(\sigma^2+u^2)^2}\,du\right| < \frac{4\epsilon}{\pi}\int_0^{\delta} \frac{\sigma u^2 du}{(\sigma^2+u^2)^2} < \frac{4\epsilon}{\pi}\int_0^{\infty} \frac{w^2 dw}{(1+w^2)^2} < A\epsilon;$$

which proves the lemma.

5.2. THEOREM 3.—*Suppose that λ_n satisfies (1.21) and that $\Sigma|a_n|$ is convergent. If then the function*

$$\phi(t) = \Sigma a_n e^{-\lambda_n it}$$

has a finite derivative for a particular value of t, the series

$$\Sigma a_n \lambda_n e^{-\lambda_n it} \tag{5.21}$$

is convergent, to sum $i\phi'(t)$. If a_n is real, and $g(t) = \Sigma a_n \cos \lambda_n t$ or $h(t) = \Sigma a_n \sin \lambda_n t$ has a finite derivative, then the corresponding one of the series

$$\Sigma a_n \lambda_n \sin \lambda_n t, \quad \Sigma a_n \lambda_n \cos \lambda_n t \tag{5.22}$$

is convergent, to sum $-g'(t)$ or $h'(t)$.

In other words, if the derivative exists, it is given by term-by-term differentiation.

To prove Theorem 3 we have only to observe that, by Lemma 12,

$$\Sigma a_n \lambda_n e^{-\lambda_n(\sigma+it)}$$

(or the corresponding real series) tends to a limit when $\sigma \to 0$. The result then follows from Theorem 1.

5 . 3. We can now construct very general classes of continuous non-differentiable functions.

In the first place, the series (5 . 21) cannot converge, for any t, if

$$a_n \lambda_n \neq o(1);$$

and in these circumstances the (complex) function $\phi(t)$ cannot be differentiable for any t. If $|a_n \lambda_n|$ has a positive lower bound, the real series are non-differentiable almost everywhere, since the set of points for which $\sin \lambda_n t$ or $\cos \lambda_n t$ tends to zero is in any case of measure zero;* but it is necessary to particularize further if we wish for results valid without exception.

Suppose that

$$a_n = \lambda_n^{-\alpha} b_n, \tag{5 . 31}$$

where $0 < \alpha \leqslant 1$ and $|b_n|$ has positive bounds. In these circumstances we can prove that the real series have no finite derivatives for any t. This is, however, not an immediate deduction from Theorem 1, and we need an additional lemma.

LEMMA 13.—*If $g(t)$ has a finite derivative $g'(t)$, then*

$$\frac{\partial^2 G}{\partial t^2} = o\left(\frac{1}{\sigma}\right).$$

The proof is similar to that of Lemma 12. Differentiating (5 . 11) twice, and making the same simplifications, we have

$$\frac{\partial^2 G}{\partial t^2} = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sigma(u^2-\sigma^2)}{(\sigma^2+u^2)^3} g(u)\, du.$$

The part of the integral outside $(-\delta, \delta)$ again tends to zero, and

$$\left| \frac{2}{\pi} \int_{-\delta}^{\delta} \frac{\sigma(u^2-\sigma^2)}{(\sigma^2+u^2)^3} g(u)\, du \right| < \frac{4\epsilon}{\pi} \int_0^{\delta} \frac{\sigma u(u^2-\sigma^2)}{(\sigma^2+u^2)^3}\, du < \frac{4\epsilon}{\pi\sigma} \int_0^{\infty} \frac{w(w^2-1)}{(1+w^2)^3}\, dw,$$

which proves the lemma.

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* See F. Bernstein, "Über eine Anwendung der Mengenlehre auf ein aus der Theorie der säkularen Störungen herrührendes Problem", *Math. Annalen*, 71 (1912), 417–439 (421); G. H. Hardy and J. E. Littlewood, "Some problems of Diophantine approximation", *Acta Math.*, 37 (1914), 155–190 (180).

5 . 4. If now

$$\psi(\sigma) = \frac{\partial G}{\partial t} = -\Sigma a_n \lambda_n \sin \lambda_n t e^{-\lambda_n \sigma},$$

$$\chi(\sigma) = \frac{\partial^2 G}{\partial t^2} = -\Sigma a_n \lambda_n^2 \cos \lambda_n t e^{-\lambda_n \sigma},$$

and $g'(t)$ exists, we have

$$\psi(\sigma) - g'(t) = o(1) = o(\sigma^{\alpha-1}), \tag{5 . 41}$$

by Lemma 12, and

$$\chi(\sigma) = o\left(\frac{1}{\sigma}\right) = o(\sigma^{\alpha-2}), \tag{5 . 42}$$

by Lemma 13. Also

$$\begin{aligned} \sigma^r \psi^{(r)}(\sigma) &= (-1)^{r+1} \sigma^r \Sigma a_n \lambda_n^{r+1} \sin \lambda_n t e^{-\lambda_n \sigma} \\ &= O(\sigma^r \Sigma \lambda_n^{r+1-\alpha} e^{-\lambda_n \sigma}) = O(\sigma^{\alpha-1}), \end{aligned} \tag{5 . 43}$$

by (5 . 31) and Lemma 3, and

$$\begin{aligned} \sigma^{r-1} \chi^{(r-1)}(\sigma) &= (-1)^r \sigma^{r-1} \Sigma a_n \lambda_n^{r+1} \cos \lambda_n t e^{-\lambda_n \sigma} \\ &= O(\sigma^{r-1} \Sigma \lambda_n^{r+1-\alpha} e^{-\lambda_n \sigma}) = O(\sigma^{\alpha-2}), \end{aligned} \tag{5 . 44}$$

for the same reasons. Applying Lemma 7 to (5 . 41) and (5 . 43), we obtain

$$\sigma^r \psi^{(r)}(\sigma) = o(\sigma^{\alpha-1}); \tag{5 . 45}$$

and similarly, from (5 . 42) and (5 . 44),

$$\sigma^{r-1} \chi^{(r-1)}(\sigma) = o(\sigma^{\alpha-2}). \tag{5 . 46}$$

From (5 . 45) and (5 . 46) we deduce

$$\begin{aligned} f^{(r+1)}(s) &= (-1)^{r+1} \Sigma a_n \lambda_n^{r+1} e^{-\lambda_n(\sigma+it)} \\ &= -\chi^{(r-1)}(\sigma) + i\psi^{(r)}(\sigma) = o(\sigma^{\alpha-r-1}). \end{aligned} \tag{5 . 47}$$

But this is impossible. For, by (4 . 523),

$$\overline{\lim} \frac{|f^{(r+1)}(s)|}{\mathbf{m}(\sigma)} > 0,$$

where

$$\mathbf{m}(\sigma) = \text{Max}\,(|a_n| \lambda_n^{r+1} e^{-\lambda_n \sigma}).$$

This is greater than a constant multiple of

$$\text{Max}\,(\lambda_n^{r+1-\alpha} e^{-\lambda_n \sigma}),$$

and therefore than a constant multiple of $\sigma^{\alpha-r-1}$, which contradicts (5 . 47). It follows that $g'(t)$ cannot exist for any value of t.

5 . 5. An interesting case is that in which λ_{n+1}/λ_n is a bounded integer μ_{n+1}. Consider, for example, the cosine series $g(t)$. If $g(t)$ has a finite derivative, the first series (5 . 22) is convergent. If also $|a_n\lambda_n|$ has a positive lower bound, $\sin \lambda_n t$ must tend to zero. If then $t = \pi u$, we have

$$\lambda_n u = m_n + \epsilon_n,$$

where m_n is integral and $\epsilon_n \to 0$. Hence

$$(\mu_{n+1} m_n - m_{n+1}) + (\mu_{n+1}\epsilon_n - \epsilon_{n+1}) = 0,$$

which is only possible, for large n, if the two terms are separately zero. It follows that

$$\epsilon_{n+1} = \mu_{n+1}\epsilon_n \quad (n > n_0),$$

$$\epsilon_{n+k} = \mu_{n+1}\mu_{n+2} \ldots \mu_{n+k}\epsilon_n \quad (n > n_0).$$

As $\epsilon_{n+k} \to 0$ when $k \to \infty$, ϵ_n must be zero, and u is of the form m_n/λ_n. Thus $g(t)$ cannot possess a finite derivative except perhaps for values of t belonging to a certain enumerable set, viz. the set

$$t = \frac{M\pi}{\lambda_j} \quad (j = 1, 2, \ldots)$$

where M is integral.

Suppose, in particular, that a_n is positive, and that t has the value (5 . 51). We may suppose, without real loss of generality, that $j = 1$. Then if $g'(t)$ exists it is zero, by Theorem 3. But

$$\frac{g(t+h)-g(t)}{h} = -\frac{1}{h}\Sigma a_n(1-\cos\lambda_n h) = -\frac{2}{h}\Sigma a_n \sin^2 \tfrac{1}{2}\lambda_n h.$$

The right-hand side has the sign opposite to that of h, and is numerically greater than a constant multiple of

$$\frac{1}{h}\sum_{\lambda_n h \leqslant 1}\frac{1}{\lambda_n}(\lambda_n h)^2 = \sum_{\lambda_n h \leqslant 1}\lambda_n h,$$

which is not less than 1 if h is of the form $1/\lambda_r$. Hence $g'(t)$ cannot be zero.

We may sum up the results of §§ 5 . 3–5 . 5 in

THEOREM 4.—*The continuous complex function $\phi(t)$ cannot possess a finite derivative for any value of t in any case in which $a_n\lambda_n$ does not tend to zero. If a_n is real, and $|a_n\lambda_n|$ has a positive lower bound, the continuous functions $g(t)$ and $h(t)$ can at most have finite derivatives at a set of points of measure zero. If a_n is of the form $\lambda_n^{-\alpha}b_n$, where $0<\alpha<1$ and $|b_n|$ has positive upper and lower bounds, then neither $g(t)$ nor $h(t)$ can have a finite derivative for any value of t. If λ_{n+1}/λ_n is a bounded integer, and $|a_n\lambda_n|$ has a positive lower bound, then $g(t)$ cannot have a finite derivative except perhaps when t is of the form $M\pi/\lambda_j$, where M is an integer. If in addition a_n is positive, then $g(t)$ cannot have a finite derivative for any value of t.*

For example, the series

$$\Sigma(-1)^{p_n}b^{-n\alpha}\cos b^n t, \quad \Sigma b^{-n\alpha}p_n\cos b^n t, \quad \Sigma(-1)^{p_n}(n!)^{-\alpha}\cos n!\, t$$

have never finite derivatives. Here $0<\alpha\leqslant 1$, p_n is an arbitrary positive integer, and b is integral in the second series but not necessarily in the first.

A particular case of the theorem is that Weierstrass's function $\Sigma a^n\cos b^n t$ has no finite derivative for any value of t when $ab\geqslant 1$.*

* G. H. Hardy, "Weierstrass's non-differentiable function", *Trans. Amer. Math. Soc.*, 17 (1916), 301-325 (303).

ON PARSEVAL'S THEOREM

By S. POLLARD.

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1. The particular result with which we are concerned runs to the effect that, *if $f(x)$ is absolutely integrable in $(-\pi, \pi)$, then, no matter what may be the behaviour of the square of $f(x)$,*

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = \tfrac{1}{2}a_0^2 + \sum_{m=1}^{\infty}(a_m^2+b_m^2), \tag{1}$$

where

$$\left.\begin{aligned} a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx \end{aligned}\right\}. \tag{2}$$

It is supposed that real variables and functions only are considered throughout.

It is not assumed that $f^2(x)$ has an integral whose value is finite. If the value of its integral is infinite then (1) still holds, the sum of the series being infinite also.

2. It may be observed that this last mentioned case, in which the value of the integral is infinite, is somewhat elusive, the majority of the methods employed to establish (1) when the value of the integral is finite failing to apply. Thus the respective proofs given by Young* and Lebesgue† for the finite case both break down in the infinite case.

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* Young, "On a class of parametric integrals and their application to the theory of Fourier series", *Proc. Royal Soc.* (A), 85 (1911), 401–414 (412).

† Lebesgue, "Sur les intégrales singulières", *Annales de Toulouse* (3), 1 (1909), 25–117 (107).

This seems to be due to the fact that, in the infinite case, the theorem borders on that of Riesz-Fischer,* and requires to be dealt with by methods which are appropriate to this theorem, to which, indeed, the infinite case, when established, eventually leads.

3. Conversely, the Riesz-Fischer theorem, when established, leads to the infinite case of Parseval's theorem. For, if in any instance the series has a finite sum when the value of the integral is infinite, a contradiction can be obtained by means of Lebesgue's result (3) below.

The Riesz-Fischer theorem contributes the fact that, when the sum of the series is finite, there exists a solution $g(x)$ of the system of integral equations (2) such that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g^2(x)\,dx = \tfrac{1}{2}a_0^2 + \sum_{m=1}^{\infty}(a_m^2+b_m^2).$$

By (3) below $g(x)$ coincides with $f(x)$ almost everywhere, *i.e.* except at most, at the points of a set of measure zero, so that

$$\int_{-\pi}^{\pi} g^2(x)\,dx = \int_{-\pi}^{\pi} f^2(x)\,dx,$$

which is contrary to the assumption that the value of the integral in (1) is infinite.

It must be recognized, however, that the Riesz-Fischer theorem alone does not suffice, as it only discusses such solutions of the system (2) as have squares the integrals of which are finite in value.

4. What we are about to develop is an argument which deals immediately with the infinite case of Parseval's theorem, and at the same time overcomes most of the difficulties inherent in the finite case, to yield which it only requires to be supplemented by a well known inequality which is very easy to obtain.

It may be described as a modernized version of the extremely obvious method of proof in which the Fourier series of $f(x)$ is squared and then integrated term by term through the interval $(-\pi, \pi)$. This method, as it stands, is devoid of rigour and leads to nothing. It only requires, however, one or two modifications to become quite sound.

* F. Riesz, "Über orthogonale Funktionensysteme", *Göttinger Nachrichten* (1907), 116–122.

To avoid trivial but troublesome complications the usual assumption is made to the effect that $f(\pi) = f(-\pi)$ and that $f(x)$ is continued outside the interval $(-\pi, \pi)$ by making it periodic in 2π. This causes no loss of generality.

5. If $$s_n(x) = \tfrac{1}{2}a_0 + \sum_{m=1}^{n} (a_m \cos mx + b_m \sin mx),$$

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + \ldots + s_{n-1}(x)}{n}$$

$$= \tfrac{1}{2}a_0 + \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right)(a_m \cos mx + b_m \sin mx),$$

then, as Lebesgue has shown,

$$\sigma_n(x) \to f(x) \tag{3}$$

almost everywhere in $(-\pi, \pi)$.

6. From Euler's formulae:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos px \cos qx\, dx = 0 \quad (p \neq q)$$

$$= 1 \quad (p = q),$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin px \sin qx\, dx = 0 \quad (p \neq q)$$

$$= 1 \quad (p = q),$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos px \sin qx\, dx = 0,$$

where p and q are non-negative integers which do not both vanish, it follows that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \sigma_n^2(x)\, dx = \tfrac{1}{2}a_0^2 + \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right)^2 (a_m^2 + b_m^2). \tag{4}$$

7. Since, as is readily shown, as $n \to \infty$,

$$\tfrac{1}{2}a_0^2 + \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right)^2 (a_m^2 + b_m^2) \to \tfrac{1}{2}a_0^2 + \sum_{m=1}^{\infty} (a_m^2 + b_m^2),$$

the right-hand side being either finite or infinite, it follows from (3) that (1) is equivalent to

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sigma_n^2(x)\, dx = \int_{-\pi}^{\pi} \lim_{n\to\infty} \sigma_n^2(x)\, dx.$$

8. This forms the basis of a proof given by Lebesgue in his *Leçons sur les séries trigonométriques.*

Lebesgue observes in effect that, in virtue of a theorem due to Fejér, whenever $f(x)$ is bounded, the sums $\sigma_n(x)$ lie, all of them, between the bounds of $f(x)$, this being a consequence of the formulae

$$\sigma_n(x) = \frac{1}{n\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f(x+2\theta)\left(\frac{\sin n\theta}{\sin\theta}\right)^2 d\theta,$$

$$\frac{1}{n\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\left(\frac{\sin n\theta}{\sin\theta}\right)^2 d\theta = 1.$$

The above passage to the limit now follows at once, by means of a well known result.

9. When $f(x)$ is not bounded, Fejér's result is no longer available and Lebesgue's theorem on passage to the limit under the integral sign no longer applicable as it stands. This difficulty we overcome by transforming Lebesgue's condition so that the formal expression is different, although the underlying principle remains the same. We prove, in fact, the following, which is a completed form of a result communicated by Lebesgue to Fatou.*

It should be observed that the particular form of what is about to be proved is due to the adoption of de la Vallée Poussin's method of treating integrals of unbounded functions, according to which, when the function to be integrated is throughout of the same sign, there is no objection to the admission of an integral whose value is infinite. This avoids unnecessary restrictions.

10. THEOREM.—*If* $\phi_n(x) \to \phi(x)$ *almost everywhere in a measurable set* E, *not necessarily of finite measure, the functions* $\phi_n(x)$ *being measurable and never negative, then*

$$\lim\int_E \phi_n(x)\,dx \geqslant \int_E \phi(x)\,dx, \tag{5}$$

the expression on the left being understood as referring to all the limits of the sequence, which need not be convergent, of integrals.

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* Fatou, "Séries trigonométriques et séries de Taylor", *Acta Math.*, 30 (1906), 335–400 (375–376).

Suppose, to begin with, that the measure of E is finite. Take any positive integer p and define auxiliary functions as follows:

$$\psi_{n,p}(x) = \phi_n(x) \quad \text{if} \quad \phi_n(x) < p$$

$$= p \quad \text{otherwise,}$$

$$\psi_p(x) = \phi(x) \quad \text{if} \quad \phi(x) < p$$

$$= p \quad \text{otherwise.}$$

It is plain that $$\psi_{n,p}(x) \to \psi_p(x)$$

almost everywhere in E. Also

$$0 \leqslant \psi_{n,p}(x) \leqslant p,$$

so that, by the usual theorem,

$$\int_E \psi_{n,p}(x)\,dx \to \int_E \psi_p(x)\,dx.$$

Since $\phi_n(x)$ is never negative,

$$\phi_n(x) \geqslant \psi_{n,p}(x),$$

so that $$\int_E \phi_n(x)\,dx \geqslant \int_E \psi_{n,p}(x)\,dx,$$

which gives

$$\operatorname{limm} \int_E \phi_n(x)\,dx \geqslant \lim \int_E \psi_{n,p}(x)\,dx = \int_E \psi_p(x)\,dx. \tag{6}$$

Now, if $\phi(x)$ is bounded, $\psi_p(x)$ coincides with $\phi(x)$ for all sufficiently large values of p, so that, as $p \to \infty$,

$$\int_E \psi_p(x)\,dx \to \int_E \phi(x)\,dx.$$

If $\phi(x)$ is not bounded this is still the case, being simply the definition of the right-hand integral.

Thus, when the measure of E is finite, (5) follows from (6) when $p \to \infty$.

When the measure of E is infinite, by definition

$$\int_E \phi(x)\,dx = \lim_{\nu \to \infty} \int_{E_\nu} \phi(x)\,dx,$$

where E_ν, of finite measure, increases to E as $\nu \to \infty$.

Since $\phi_n(x)$ is never negative,

$$\int_E \phi_n(x)\,dx \geqslant \int_{E_\nu} \phi_n(x)\,dx.$$

By the above, E_ν being of finite measure,

$$\operatorname{limm} \int_E \phi_n(x)\,dx \geqslant \operatorname{limm} \int_{E_\nu} \phi_n(x)\,dx$$

$$\geqslant \int_{E_\nu} \phi(x)\,dx$$

for each ν.

Making $\nu \to \infty$, we have (5).

In effect, the integral of a non-negative function, being defined by means of a sequence tending to the function from below, inclines to the small side, and this is expressed by (5).

11. The application to Parseval's theorem is immediate. Since $\sigma_n^2(x)$ is never negative, we have, by (3),

$$\operatorname*{limm}_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma_n^2(x)\,dx \geqslant \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx,$$

i.e.
$$\operatorname*{limm}_{n\to\infty} \left[\tfrac{1}{2}a_0^2 + \sum_{m=1}^{n-1} \left(1 - \frac{m}{n}\right)^2 (a_m^2 + b_m^2) \right] \geqslant \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx$$

or
$$\tfrac{1}{2}a_0^2 + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \geqslant \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx. \tag{7}$$

This gives at once the infinite case of Parseval's theorem.

For the finite case we still require the complementary inequality

$$\tfrac{1}{2}a_0^2 + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x)\,dx. \tag{8}$$

This, however, is simply Bessel's inequality, an expanded form of

$$0 \leqslant \lim_{n\to\infty} \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x) - s_n(x)]^2\,dx.$$

12. Apart from Young, who insists on it a good deal, writers do not seem to make much use of the type of result given in § 10. This is a little surprising, since the result is quite powerful and often enables one

to avoid a tedious argument involving the selection of endless sets of points.

It applies, for instance, very effectively to Fischer's theorem concerning convergence on the mean.

This runs to the effect that if, corresponding to each positive number ϵ, there is a positive integer ν such that

$$\int_E [f_\nu(x)-f_n(x)]^2\,dx < \epsilon$$

whenever $n \geqslant \nu$, then there exists a function $f(x)$ towards which $f_n(x)$ converges on the mean in E, *i.e.* such that

$$\int_E [f(x)-f_n(x)]^2\,dx \to 0. \tag{9}$$

The set E is supposed measurable, but not necessarily of finite measure.

Take a series $$\epsilon_1+\epsilon_2+\ldots+\epsilon_n+\ldots$$

of positive terms with a finite sum.

Choose, as is easily shown to be possible, a sub-sequence $\{f_{n_r}(x)\}$ of the functions $f_n(x)$ such that

$$\int_E u_r^2(x)\,dx < \epsilon_r^2$$

where $$u_r(x) = f_{n_{r+1}}(x)-f_{n_r}(x).$$

By the inequality of Schwarz,

$$\int_e |\,u_r(x)\,|\,dx \leqslant \epsilon_r$$

for every subset e of E of unit measure, and so

$$\sum_{r=1}^{\infty} \int_e |\,u_r(x)\,|\,dx \leqslant \sum_{r=1}^{\infty} \epsilon_r.$$

A first application of the theorem gives

$$\int_e \sum_{r=1}^{\infty} |\,u_r(x)\,|\,dx \leqslant \sum_{r=1}^{\infty} \epsilon_r,$$

so that the sum $\sum_{r=1}^{\infty} |\,u_r(x)\,|$ is finite almost everywhere in e, and therefore the sequence $\{f_{n_r}(x)\}$ converges to a finite limit almost everywhere in e, and so, since E can be exhibited as the sum of a countable aggregate of sets e, almost everywhere in E.

Writing $f(x)$ for the limit function, considered where it exists, we have, by a second application of the theorem,

$$\int_E [f(x)-f_n(x)]^2\,dx = \int_E \lim_{r\to\infty} [f_{n_r}(x)-f_n(x)]^2\,dx$$

$$\leqslant \limsup_{r\to\infty} \int_E [f_{n_r}(x)-f_n(x)]^2\,dx$$

$$< \epsilon$$

if $n \geqslant \nu$, which is (9).

13. We should observe that the property of the trigonometrical orthogonal system, in virtue of which (1) remains true when the integral of $f^2(x)$ is infinite in value, is not a general property of orthogonal systems. For Banach has proved*, in the case of positive integrable functions, that, if

$$\int_a^b f^2(x)\,dx = \infty,$$

then it is possible to construct a set of normalized functions orthogonal in (a, b) such that, if

$$c_n = \int_a^b f(x)\,\phi_n(x)\,dx,$$

then

$$\sum_{n=1}^{\infty} c_n^2 = 0,$$

which is contrary to the analogue of (1).

14. We should also observe that Parseval's theorem, in common with most theorems involving a passage to the limit under the sign of Lebesgue integration, has never been proved apart from the so-called *Principle of Selection*, the validity of which appears to be independent of the usual assumptions of mathematics and logic.

There is, however, rather more than this to be said. For one of the peculiarities of this troublesome principle is its ability to divide convenient identities into two inequalities on different footings. One of them is independent of the principle and the other not.

Thus the fundamental identity of measure,

$$mE = mE_1 + mE_2 + \ldots + mE_n + \ldots,$$

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* Banach, "An example of an orthogonal development whose sum is everywhere different from the developed function", *Proc. London Math. Soc.* (2), 21 (1922), 95-97 (95).

whenever the sets E_n are measurable and mutually exclusive, and

$$E = E_1 + E_2 + \ldots + E_n + \ldots$$

divides into the inequalities

$$m^*E \leqslant mE_1 + mE_2 + \ldots + mE_n + \ldots,$$

$$m_*E \geqslant mE_1 + mE_2 + \ldots + mE_n + \ldots,$$

of which the first requires the principle while the second does not.

This is precisely what occurs in the present instance, the identity (1) dividing into the inequalities (7) and (8), of which (7) requires the principle while (8) does not.

15. If $f(x)$ is not absolutely integrable in $(-\pi, \pi)$, then only the infinite case of Parseval's theorem is possible. For, by the inequality of Schwarz,

$$\left\{\int_{-\pi}^{\pi} |f(x)|\, dx\right\}^2 \leqslant 2\pi \int_{-\pi}^{\pi} f^2(x)\, dx,$$

which shows at once that

$$\int_{-\pi}^{\pi} f^2(x)\, dx = \infty. \tag{10}$$

To fix the ideas, let us assume that $f(x)$ is integrable in the sense of Denjoy-Perron. Then, since $\cos nx$, $\sin nx$ are functions of bounded variation, the integrals in (2) are definite in value and define what may be called the generalized Fourier constants of $f(x)$.

In order to prove what remains of Parseval's theorem under the new hypotheses we have only to show that

$$\tfrac{1}{2}a_0^2 + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) = \infty. \tag{11}$$

This may be accomplished in a variety of ways, among which perhaps the most simple is that which follows the theorem of Riesz-Fischer.

If (11) does not hold, then there exists the function $g(x)$ of integrable square such that

$$\left.\begin{aligned} a_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos mx\, dx \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin mx\, dx \end{aligned}\right\}. \tag{12}$$

Write
$$F(x) = \int_{-\pi}^{x} \{f(t) - \tfrac{1}{2}a_0\}\, dt,$$
$$G(x) = \int_{-\pi}^{x} \{g(t) - \tfrac{1}{2}a_0\}\, dt.$$

Then $F(x)$, $G(x)$ are continuous functions which vanish at π and $-\pi$, so that, as follows from the formula for integration by parts taken in conjunction with (2) and (12),

$$\int_{-\pi}^{\pi} F(x) \cos mx\, dx = \int_{-\pi}^{\pi} G(x) \cos mx\, dx,$$
$$\int_{-\pi}^{\pi} F(x) \sin mx\, dx = \int_{-\pi}^{\pi} G(x) \sin mx\, dx,$$

i.e. $F(x)$ and $G(x)$ have the same Fourier constants. Being continuous, they must, by Fejér's theorem, be identical.

By theorems respectively due to Denjoy and Lebesgue, however,

$$F'(x) = f(x) - \tfrac{1}{2}a_0, \qquad G'(x) = g(x) - \tfrac{1}{2}a_0,$$

except at most in a set of points of measure zero. It follows that $f(x)$ and $g(x)$, and therefore $f^2(x)$ and $g^2(x)$ coincide almost everywhere in $(-\pi, \pi)$. This contradicts (10), since $g(x)$ is of integrable square.

The only possibility is (11), which we set out to prove.

This result also, of course, is subject to the Principle of Selection.

THE FIRST AND SECOND VARIATIONS OF THE LENGTH-INTEGRAL IN RIEMANNIAN SPACE

By J. L. SYNGE.

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Synopsis: 1. Introduction. 2. Notation. 3. First form of the first variation. 4. Second form of the first variation. 5. First form of the second variation. 6. Second form of the second variation. 7. Introduction of the Riemannian curvature. 8. Second variation when the curve is geodesic and the end-points are fixed. 9. Expression for the second variation using the unit normal variation vector. 10. Conjugate points.

1. *Introduction.*

The problem of the variations of the length-integral in Riemannian space is a special case of the general Lagrangian problem*. The present paper is, however, developed independently of the existing general theory, the methods being based essentially on the theory of tensors. Extensive use is made of the contravariant form associated with the parallel propagation of Levi-Civita†.

We consider a manifold V_N in which there exist a coordinate system $(x^1, x^2, \ldots, x^N)$ and a line-element ds defined by

$$ds^2 = g_{mn}\,dx^m\,dx^n, \tag{1.1}$$

where the right-hand side is a positive definite form and $g_{mn}\,(=g_{nm})$ are functions of the coordinates only, possessing continuous partial deriva-

* O. Bolza, *Vorlesungen über Variationsrechnung* (1909), 6, and ch. 11 and 12.

† See L. Bianchi, *Lezioni di Geometria Differenziale*, 2_2 (1924), 790.

tives of the second order with respect to the coordinates at all points of a region S of V_N.

Let C be a curve joining the points P_1 and P_2 and lying wholly in S. Let C be defined by the equations

$$x^i = x^i(u), \tag{1.2}$$

where $u = u_1$ at P_1 and $u = u_2$ at P_2, u_2 being greater than u_1. The length of this curve is by definition

$$L = \int_{u_1}^{u_2} \left(g_{mn} \frac{dx^m}{du} \frac{dx^n}{du} \right)^{\frac{1}{2}} du. \tag{1.3}$$

We shall consider only those curves for which d^2x^i/du^2 are continuous $(u_1 \leqslant u \leqslant u_2)$ and for which dx^i/du do not all vanish simultaneously for any value of u in this range.

The equations

$$x^i = x^i(u, v) \tag{1.4}$$

define a two-space V_2. Let

$$x^i(u, 0) \equiv x^i(u) \quad (u_1 \leqslant u \leqslant u_2), \tag{1.5}$$

so that the curve $v = 0$ of V_2 coincides with C. We shall call the portions of the parametric lines of u intercepted between the curves $u = u_1$ and $u = u_2$ the *u-curves* or *varied positions of* C, and the parametric lines of v the *v-curves* or *curves of variation*. The curves of variation of P_1 and P_2 have the equations $u = u_1$ and $u = u_2$ respectively. We shall assume the functions in (1.4) to be such that

$$\frac{\partial^2 x^i}{\partial u^2}, \quad \frac{\partial^2 x^i}{\partial u \partial v}, \quad \frac{\partial^2 x^i}{\partial v^2}, \quad \frac{\partial^3 x^i}{\partial u^2 \partial v}, \quad \frac{\partial^3 x^i}{\partial u \partial v^2}$$

are continuous functions of u and v for $u_1 \leqslant u \leqslant u_2$ and for a small range of values of v on either side of zero. We shall further assume that $(\partial x^i/\partial v)_{v=0}$ do not all vanish simultaneously for more than a finite number of values of u between u_1 and u_2. The length of a u-curve is a function of v only, since we are not considering variations of u_1 and u_2; we shall write this length $L(v)$. The parameter u on C will be chosen equal to the length of the arc of C measured from some definite point, so that $du/ds = 1$ on C. This choice of parameter is important, since it produces considerable simplification in the formulae.

2. *Notation.*

We shall indicate the partial derivative with respect to u of any function of u and v by a superposed point,

$$\frac{\partial X}{\partial u} = \dot{X}, \tag{2.01}$$

and the partial derivative with respect to v by a dash,

$$\frac{\partial X}{\partial v} = X'. \tag{2.02}$$

We shall write $\xi^i = x^i$, $\eta^i = x^{i\prime}$, so that

$$\dot{\xi}^{i\prime} = \dot{\eta}^i. \tag{2.10}$$

We shall call η^i the *variation vector* and $\eta^i \delta v$ the *infinitesimal variation vector*. The *magnitude* of a vector X^i will be denoted by X, so that

$$X^2 = g_{mn} X^m X^n \quad (X \geqslant 0). \tag{2.11}$$

A *unit vector* being one whose magnitude is unity, we shall call μ^i the *unit variation vector* where

$$\eta^i = \eta \mu^i, \tag{2.12}$$

μ^i being therefore codirectional with η^i. The angle between two vectors X^i and Y^i will be denoted by $\theta(\mathbf{X}, \mathbf{Y})$, so that

$$XY \cos \theta(\mathbf{X}, \mathbf{Y}) = g_{mn} X^m Y^n. \tag{2.125}$$

The length of any u-curve is

$$L(v) = \int_{u_1}^{u_2} \{ g_{mn} \xi^m \xi^n \}^{\frac{1}{2}} du, \tag{2.13}$$

or, writing

$$F^2 = g_{mn} \dot{\xi}^m \dot{\xi}^n, \tag{2.14}$$

$$L(v) = \int_{u_1}^{u_2} F \, du. \tag{2.15}$$

We note that $F = 1$ along any u-curve for which $du/ds = 1$. We shall denote the partial derivatives of F with respect to x^i and ξ^i in the following manner :—

$$(2.16)\qquad \begin{cases} \dfrac{\partial F}{\partial x^i} = F_{i|}, \quad \dfrac{\partial F}{\partial \xi^i} = F_{|i}, \\ \dfrac{\partial^2 F}{\partial x^i \partial x^j} = F_{ij|}, \quad \dfrac{\partial^2 F}{\partial x^i \partial \xi^j} = F_{i|j}, \quad \dfrac{\partial^2 F}{\partial \xi^i \partial \xi^j} = F_{|ij}. \end{cases}$$

It is to be noted that only those expressions which have all their subscripts to the right of the vertical bar are tensors*. We shall now write the explicit expressions for the partial derivatives in the case where $F = 1$ on the u-curve in question. We find

$$(2.17)\qquad F_{i|} = \frac{1}{2F}\frac{\partial (F^2)}{\partial x^i} = \tfrac{1}{2}\frac{\partial g_{jk}}{\partial x^i}\xi^j \xi^k,$$

$$(2.18)\qquad F_{|i} = \frac{1}{2F}\frac{\partial (F^2)}{\partial \xi^i} = g_{ij}\xi^j,$$

$$(2.19)\qquad F_{ij|} = \frac{\partial}{\partial x^i}\left(\frac{1}{2F}\frac{\partial (F^2)}{\partial x^j}\right) = \frac{1}{2F}\frac{\partial^2 (F^2)}{\partial x^i \partial x^j} - \frac{1}{4F^3}\frac{\partial (F^2)}{\partial x^i}\frac{\partial (F^2)}{\partial x^j}$$

$$= \tfrac{1}{2}\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}\xi^k \xi^l - \tfrac{1}{4}\frac{\partial g_{kl}}{\partial x^i}\xi^k \xi^l \frac{\partial g_{mn}}{\partial x^j}\xi^m \xi^n,$$

$$(2.20)\qquad F_{i|j} = \frac{\partial}{\partial x^i}\left(\frac{1}{2F}\frac{\partial (F^2)}{\partial \xi^j}\right) = \frac{1}{2F}\frac{\partial^2 (F^2)}{\partial x^i \partial \xi^j} - \frac{1}{4F^3}\frac{\partial (F^2)}{\partial x^i}\frac{\partial (F^2)}{\partial \xi^j}$$

$$= \frac{\partial g_{jk}}{\partial x^i}\xi^k - \tfrac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\xi^k \xi^l g_{jm}\xi^m,$$

$$(2.21)\qquad F_{|ij} = \frac{\partial}{\partial \xi^i}\left(\frac{1}{2F}\frac{\partial (F^2)}{\partial \xi^j}\right) = \frac{1}{2F}\frac{\partial^2 (F^2)}{\partial \xi^i \partial \xi^j} - \frac{1}{4F^3}\frac{\partial (F^2)}{\partial \xi^i}\frac{\partial (F^2)}{\partial \xi^j}$$

$$= g_{ij} - g_{ik}\xi^k g_{jl}\xi^l.$$

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* Cf. "A generalization of the Riemannian line element", *Trans. Amer. Math. Soc.* 27 (1925), 61. What was denoted by F in that paper is here denoted by F^2.

If X^i be a contravariant vector given along a curve $x^i = x^i(t)$, then, as is well known*,

$$\frac{dX^i}{dt} + \left\{ \begin{matrix} m & n \\ i & \end{matrix} \right\} X^m \frac{dx^n}{dt} \tag{2.22}$$

is also a contravariant vector: it may conveniently be termed the *contravariant derivative of X^i with respect to the given curve.* We shall denote the contravariant derivative of a vector with respect to a u-curve by a superposed bar,

$$\bar{X}^i = \dot{X}^i + \left\{ \begin{matrix} m & n \\ i & \end{matrix} \right\} X^m \xi^n, \tag{2.23}$$

and with respect to a v-curve by a superposed circumflex accent,

$$\hat{X}^i = X^{i\prime} + \left\{ \begin{matrix} m & n \\ i & \end{matrix} \right\} X^m \eta^n. \tag{2.24}$$

When two of these operational symbols occur together, the order is to be read from the top downwards. By (2.10) it is easily seen that

$$\hat{\bar{\xi}}^i = \bar{\eta}^i. \tag{2.25}$$

The equations for parallel propagation of X^i along the u-curves and the v-curves are respectively $\bar{X}^i = 0$ and $\hat{X}^i = 0$.

3. *First form of the first variation.*

The first variation of the length of a u-curve for the displacement arising from an infinitesimal increment δv is by definition $\delta L = L'(v)\delta v$. We have from (2.15)

$$L'(v) = \int_{u_1}^{u_2} F' \, du. \tag{3.1}$$

Now

$$\begin{aligned} F' &= F_{|i}\,\dot{\xi}^{i\prime} + F_{i|}\,x^{i\prime} \\ &= F_{|i}\,\dot{\eta}^i + F_{i|}\,\eta^i \\ &= F_{|i}\left(\bar{\eta}^i - \left\{ \begin{matrix} j & k \\ i & \end{matrix} \right\} \eta^j \xi^k\right) + F_{i|}\,\eta^i \\ &= F_{|i}\,\bar{\eta}^i + \Psi_i\,\eta^i, \end{aligned} \tag{3.11}$$

where

$$\Psi_i = F_{i|} - F_{|l} \left\{ \begin{matrix} i & k \\ l & \end{matrix} \right\} \xi^k. \tag{3.12}$$

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* Cf. Bianchi, *loc. cit.*, 790.

Inspection of (3 . 11) shows that Ψ_i is a covariant vector. But if we introduce a system of Riemannian coordinates* at the point in question, so that $\partial g_{jk}/\partial x^l$ all vanish at the point, we see that all the components of Ψ_i vanish. Therefore, from the covariant character, they must also vanish for any coordinate system. Thus we have

$$F' = F_{|i}\bar{\eta}^i. \tag{3 . 2}$$

Hence, for any value of v,

$$L'(v) = \int_{u_1}^{u_2} F_{|i}\bar{\eta}^i\, du. \tag{3 . 3}$$

To evaluate this expression for the curve C, we may apply (2 . 18) and obtain the equivalent forms for the first variation,

$$\delta L = \delta v \int_{u_1}^{u_2} g_{ij}\,\xi^i\,\bar{\eta}^j\, du, \tag{3 . 4}$$

$$\delta L = \delta v \int_{u_1}^{u_2} \bar{\eta}\cos\theta(\xi, \bar{\eta})\, du. \tag{3 . 5}$$

We shall call either of these expressions the *first form of the first variation.*

The following theorems result directly from (3 . 5) :

THEOREM I.—*The first variation of the length of a curve C is zero when the variation vector is propagated parallelly along C.*

In this case C and the varied curve cannot have a common point, for in parallel propagation the magnitude of a vector remains constant. Thus if the magnitude of the variation vector vanishes at one point of C, it vanishes at all points, and there is no infinitesimal displacement.

THEOREM II.—*The first variation of the length of a curve C is zero if the contravariant derivative of the variation vector with respect to C is normal to C at every point.*

THEOREM III.—*The first variation of the length of a curve C has the same sign as (the opposite sign to) δv, if the contravariant derivative of the variation vector with respect to C makes an acute (obtuse) angle with C at every point.*

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* Cf. Laue, *Relativitätstheorie*, 2 (1921), 75.

4. *Second form of the first variation.*

Since

$$\frac{\partial}{\partial u}(g_{ij}\xi^i\eta^j) = g_{ij}\bar{\xi}^i\eta^j + g_{ij}\xi^i\bar{\eta}^j, \tag{4.1}$$

we find at once from (3.4) for the curve C

$$\delta L = \delta v\left([g_{ij}\xi^i\eta^j]_{u_1}^{u_2} - \int_{u_1}^{u_2} g_{ij}\bar{\xi}^i\eta^j\,du\right), \tag{4.2}$$

or

$$\delta L = \delta v\left([\eta\cos\theta(\boldsymbol{\xi},\boldsymbol{\eta})]_{u_1}^{u_2} - \int_{u_1}^{u_2}\bar{\xi}\eta\cos\theta(\bar{\boldsymbol{\xi}},\boldsymbol{\eta})\,du\right). \tag{4.3}$$

We shall call either of these expressions the *second form of the first variation.*

Noting that the vector $\bar{\xi}^i$ defines the principal normal of C and that $\bar{\xi}$ is the principal curvature*, and also that $\bar{\xi}$ vanishes if C is geodesic, the following theorems result directly, Theorem IV being well known:

THEOREM IV.—*The first variation of the length of a curve C is zero if C is geodesic and the variation vector either vanishes or is normal to C at the end points.*

THEOREM V.—*The first variation of the length of a curve C is zero if the variation vector is perpendicular to the principal normal of C at every point and either vanishes or is normal to C at the end points.*

THEOREM VI.—*The first variation of the length of a curve C has the same sign as (the opposite sign to) δv if the variation vector is perpendicular to the principal normal of C at every point and makes with the direction of C at the end points angles which are obtuse and acute (acute and obtuse) in order.*

The principal normal of a curve has an intrinsic positive sense defined by $\bar{\xi}^i$, since a reversal of the sense of the curve in which u increases does

* Cf. Bianchi, *loc. cit.*, 456.

not reverse $\bar{\xi}^i$. We may state as a direct deduction from (4 . 2) the following theorem :

THEOREM VII.—*The first variation of the length of a curve C is negative for an infinitesimal variation in the positive direction of the principal normal of C at every point.*

5. *First form of the second variation.*

The second variation of the length of a u-curve for the displacement arising from an infinitesimal increment δv is by definition

$$\delta^2 L = \tfrac{1}{2} L''(v)\,\delta v^2.$$

To find $L''(v)$ we differentiate (3 . 1) with respect to v, obtaining

$$L''(v) = \int_{u_1}^{u_2} F''\,du, \tag{5 . 10}$$

where, by (3 . 2),

$$\begin{aligned} F'' &= \frac{\partial}{\partial v}\,(F_{|i}\,\bar{\eta}^i) \qquad (5 . 11)\\ &= (F_{|ij}\,\xi^{j\prime} + F_{j|i}\,\eta^j)\,\bar{\eta}^i + F_{|i}\left(\hat{\bar{\eta}}^i - \left\{\begin{matrix} j & k \\ i & \end{matrix}\right\}\bar{\eta}^j\,\eta^k\right). \end{aligned}$$

Hence, since

$$\xi^{j\prime} = \dot{\eta}^j = \bar{\eta}^j - \left\{\begin{matrix} k & l \\ j & \end{matrix}\right\}\eta^k\,\xi^l, \tag{5 . 12}$$

we find

$$F'' = F_{|i}\,\hat{\bar{\eta}}^i + F_{|ij}\,\bar{\eta}^i\,\bar{\eta}^j + \Phi_{ij}\,\bar{\eta}^i\,\eta^j, \tag{5 . 13}$$

where

$$\Phi_{ij} = -F_{|ik}\left\{\begin{matrix} j & l \\ k & \end{matrix}\right\}\xi^l + F_{j|i} - F_{|k}\left\{\begin{matrix} i & j \\ k & \end{matrix}\right\}. \tag{5 . 14}$$

Inspection of (5 . 13) shows that Φ_{ij} is a covariant tensor. But if we introduce a system of Riemannian coordinates at the point in question, it is easy to see that all the components of Φ_{ij} vanish. Therefore from the covariant character they must also vanish for every coordinate system, and we have

$$F'' = F_{|i}\,\hat{\bar{\eta}}^i + F_{|ij}\,\bar{\eta}^i\,\bar{\eta}^j. \tag{5 . 15}$$

Substitution in (5 . 10) gives

$$L''(v) = \int_{u_1}^{u_2} (F_{|i}\,\hat{\bar{\eta}}^i + F_{|ij}\,\bar{\eta}^i\,\bar{\eta}^j)\,du. \tag{5.16}$$

For the curve C we find, using (2 . 18) and (2 . 21),

$$F_{|i}\,\hat{\bar{\eta}}^i + F_{|ij}\,\bar{\eta}^i\,\bar{\eta}^j = g_{ij}\,\xi^i\,\hat{\bar{\eta}}^j + g_{ij}\,\bar{\eta}^i\,\bar{\eta}^j - g_{ik}\,\xi^k\,g_{jl}\,\xi^l\,\bar{\eta}^i\,\bar{\eta}^j$$

$$= g_{ij}\,\xi^i\,\hat{\bar{\eta}}^j + g_{ij}\,\bar{\eta}^i\,\bar{\eta}^j - (g_{ij}\,\xi^i\,\bar{\eta}^j)^2 \tag{5.17}$$

$$= \hat{\bar{\eta}}\cos\theta(\boldsymbol{\xi}, \hat{\bar{\boldsymbol{\eta}}}) + \bar{\eta}^2 \sin^2\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}}). \tag{5.18}$$

Substitution in (5 . 16) gives the equivalent forms

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [g_{ij}\,\xi^i\,\hat{\bar{\eta}}^j + g_{ij}\,\bar{\eta}^i\,\bar{\eta}^j - (g_{ij}\,\xi^i\,\bar{\eta}^j)^2]\,du, \tag{5.19}$$

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\hat{\bar{\eta}}\cos\theta(\boldsymbol{\xi}, \hat{\bar{\boldsymbol{\eta}}}) + \bar{\eta}^2 \sin^2\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}})]\,du. \tag{5.20}$$

Either of these forms we shall call the *first form of the second variation*. They exist whether the first variation vanishes or not.

There are evidently some critical conditions governing the sign of the second variation. These we proceed to write down.

CONDITION (α).—$\hat{\bar{\eta}} = 0$ *at every point of* C.

If the vector $\bar{\eta}^i$ is propagated parallelly along the v-curves, this condition is satisfied.

CONDITION $\begin{Bmatrix}(\beta_1)\\(\beta_2)\\(\beta_3)\end{Bmatrix}$.—$\theta(\boldsymbol{\xi}, \hat{\bar{\boldsymbol{\eta}}})$ *is* $\begin{Bmatrix}\text{acute}\\\text{a right angle}\\\text{obtuse}\end{Bmatrix}$ *at every point of* C.

CONDITION (γ).—$\bar{\eta} = 0$ *at every point of* C.

This is a necessary and sufficient condition that the variation vector should be propagated parallelly along C.

CONDITION (δ).—*The vector $\bar{\eta}^i$ is codirectional with C wherever it does not vanish.*

The following theorems result directly from (5 . 20) :

THEOREM VIII.—*The second variation is positive if (α) is not true and (β_1) is true.*

THEOREM IX.—*The second variation is positive if one of the three conditions (α), (β_1), (β_2) is true and neither (γ) nor (δ) is true.*

THEOREM X.—*The second variation is zero if either (α) or (β_2) is true and either (γ) or (δ) is true.*

THEOREM XI.—*The second variation is negative if (β_3) and either (γ) or (δ) are true and (α) is not true.*

6. *Second form of the second variation.*

We shall now obtain a second form of the second variation by application of the method of integration by parts and by use of the following fact. If X^i be any contravariant vector given as a function of u and v, it is easily seen by direct computation that

$$\hat{\bar{X}}^i - \bar{\hat{X}}^i = -G^i_{jkl} X^j \xi^k \eta^l, \tag{6 . 10}$$

where

$$G^i_{jkl} = \frac{\partial}{\partial x^k}\left\{\begin{matrix} j & l \\ i & \end{matrix}\right\} - \frac{\partial}{\partial x^l}\left\{\begin{matrix} j & k \\ i & \end{matrix}\right\} + \left\{\begin{matrix} j & l \\ m & \end{matrix}\right\}\left\{\begin{matrix} m & k \\ i & \end{matrix}\right\} - \left\{\begin{matrix} j & k \\ m & \end{matrix}\right\}\left\{\begin{matrix} m & l \\ i & \end{matrix}\right\}, \tag{6 . 11}$$

the curvature tensor of V_N*. Thus

$$\hat{\bar{\eta}}^i - \bar{\hat{\eta}}^i = -G^i_{jkl} \eta^j \xi^k \eta^l, \tag{6 . 12}$$

and therefore

$$\int_{u_1}^{u_2} F_{|i} \hat{\bar{\eta}}^i \, du = \int_{u_1}^{u_2} F_{|i} \bar{\hat{\eta}}^i \, du - \int_{u_1}^{u_2} F_{|i} G^i_{jkl} \eta^j \xi^k \eta^l \, du. \tag{6 . 13}$$

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* Cf. J. A. Schouten, *Der Ricci-Kalkül* (1924), 83, where the differential notation is employed. Schouten's arrangement of indices is, however, different. The present notation follows Einstein, *Die Grundlage der allgemeinen Relativitätstheorie* (1916), 39, and Weyl, *Raum, Zeit, Materie* (1921), 107.

Substituting in (5 . 16) and using (2 . 18), we obtain for the curve C

$$L''(0) = \int_{u_1}^{u_2} g_{ij}\,\xi^i \bar{\hat{\eta}}^j\,du + \int_{u_1}^{u_2} [g_{ij}\,\bar{\eta}^i\,\bar{\eta}^j - (g_{ij}\,\xi^i\,\bar{\eta}^j)^2 - G_{ijkl}\,\xi^i\,\eta^j\,\xi^k\,\eta^l]\,du, \tag{6 . 14}$$

where

$$G_{ijkl} = g_{in}\,G^n_{jkl}. \tag{6 . 15}$$

Now

$$\frac{\partial}{\partial u}(g_{ij}\,\xi^i\,\hat{\eta}^j) = g_{ij}\,\bar{\xi}^i\,\hat{\eta}^j + g_{ij}\,\xi^i\,\bar{\hat{\eta}}^j, \tag{6 . 16}$$

and therefore

$$\int_{u_1}^{u_2} g_{ij}\,\xi^i\,\bar{\hat{\eta}}^j\,du = [g_{ij}\,\xi^i\,\hat{\eta}^j]_{u_1}^{u_2} - \int_{u_1}^{u_2} g_{ij}\,\bar{\xi}^i\,\hat{\eta}^j\,du. \tag{6 . 17}$$

Substituting in (6 . 14), we see that the second variation may be written

$$\delta^2 L = \tfrac{1}{2}\delta v^2 (T - I_1 + I_2), \tag{6 . 18}$$

where

$$T = [\hat{\eta}\cos\theta(\boldsymbol{\xi}, \hat{\boldsymbol{\eta}})]_{u_1}^{u_2}, \tag{6 . 19}$$

$$I_1 = \int_{u_1}^{u_2} \bar{\xi}\,\hat{\eta}\,\cos\theta(\bar{\boldsymbol{\xi}}, \hat{\boldsymbol{\eta}})\,du, \tag{6 . 20}$$

$$I_2 = \int_{u_1}^{u_2} [\bar{\eta}^2 \sin^2\theta(\boldsymbol{\xi}, \boldsymbol{\eta}) - G_{ijkl}\,\xi^i\,\eta^j\,\xi^k\,\eta^l]\,du. \tag{6 . 21}$$

We shall call (6 . 18) the *second form of the second variation*. It exists whether the first variation vanishes or not.

7. *Introduction of the Riemannian Curvature.*

We shall now modify the expression for I_2 by the introduction of the Riemannian curvature.

If, at any point of V_N, W_M ($M \leqslant N$) is formed from all the geodesics of V_N emanating from the point in directions lying in an assigned

M-element, and if the equations of W_M are expressed parametrically as

$$x^i = x^i(\overset{0}{x}{}^1, \overset{0}{x}{}^2, \ldots, \overset{0}{x}{}^M), \tag{7.1}$$

then

$$\overset{0}{G}_{\mu\nu\sigma\tau} = G_{mnst} \frac{\partial x^m}{\partial \overset{0}{x}{}^\mu} \frac{\partial x^n}{\partial \overset{0}{x}{}^\nu} \frac{\partial x^s}{\partial \overset{0}{x}{}^\sigma} \frac{\partial x^t}{\partial \overset{0}{x}{}^\tau} \quad (\mu, \nu, \sigma, \tau = 1, \ldots, M), \tag{7.2}$$

where the left-hand side is the curvature tensor of W_M for the coordinate system $(\overset{0}{x}{}^1, \ldots, \overset{0}{x}{}^M)$. This result is known* and admits of a very simple proof by the use of Riemannian coordinates. Now if, at any point P of C, a two-space W_2 is constructed from all the geodesics of V_N tangential to V_2 at P, and if there is taken in W_2 a coordinate system $(\overset{0}{x}{}^1, \overset{0}{x}{}^2)$ such that at P

$$\xi^i \equiv \frac{\partial x^i}{\partial u} = \frac{\partial x^i}{\partial \overset{0}{x}{}^1}, \quad \eta^i \equiv \frac{\partial x^i}{\partial v} = \frac{\partial x^i}{\partial \overset{0}{x}{}^2}, \tag{7.3}$$

then†

$$\overset{0}{G}_{1212} = G_{mnst}\, \xi^m \eta^n \xi^s \eta^t. \tag{7.4}$$

If K denotes the Gaussian curvature of W_2 at P (or, in other words, the Riemannian curvature of V_N for the directions ξ^i, η^i), then

$$K\overset{0}{g} = \overset{0}{G}_{1212}, \tag{7.5}$$

where

$$\overset{0}{g} = \begin{vmatrix} \overset{0}{g}_{11} & \overset{0}{g}_{12} \\ \overset{0}{g}_{21} & \overset{0}{g}_{22} \end{vmatrix} \tag{7.6}$$

and

$$\overset{0}{g}_{11}(dx^1)^2 + 2\overset{0}{g}_{12}(dx^1)(dx^2) + \overset{0}{g}_{22}(dx^2)^2 \tag{7.7}$$

is the square of the line-element of W_2 at P‡. By virtue of (7.3) it is evident that

$$\overset{0}{g}_{11}\, du^2 + 2\overset{0}{g}_{12}\, du\, dv + \overset{0}{g}_{22}\, dv^2 \tag{7.8}$$

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* Schouten, *loc. cit.*, 198.

† Cf. Bianchi, *loc. cit.*, 430.

‡ *Ibid.*, 428.

is the square of the line-element at V_2 at P. Therefore we may write

$$I_2 = \int_{u_1}^{u_2} [\bar{\eta}^2 \sin^2 \theta(\xi, \bar{\eta}) - K\underset{o}{g}]\, du, \tag{7.9}$$

where K is the Riemannian curvature of V_N for the directions ξ^i, η^i and $\underset{o}{g}$ is the determinant formed from the fundamental tensor of V_2 for the coordinate system (u, v).

8. *Second variation when the curve is geodesic and the end points are fixed.*

Turning to the second form for the second variation given in (6.18), we see that I_1 is zero if C is geodesic, since then $\bar{\xi}$ is zero. If the end points are fixed, η vanishes at $u = u_1$ and $u = u_2$ for all values of v under consideration, and therefore $\hat{\eta}$ vanishes at these points. Thus T is zero. Under these circumstances the second variation is

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\bar{\eta}^2 \sin^2 \theta(\xi, \bar{\eta}) - K\underset{o}{g}]\, du, \tag{8.1}$$

using the form of I_2 given in (7.9). Thus the second variation depends only on the direction and magnitude of the variation vector along C and in no other way on the nature of V_2.

We shall now assume that the variation vector is normal to C at every point, so that on C

$$g_{ij}\xi^i\eta^j = 0. \tag{8.2}$$

This condition implies no restriction on the variations, the end points being fixed. It implies, however, a restricted choice of surface coordinates in V_2. Being given V_2 and an ∞^1 family of varied positions of C in V_2, defined as the parametric lines of u, it is only necessary to take new coordinates (u^*, v^*) such that the parametric lines of u^* coincide with the parametric lines of u and the parametric lines of v^* are the orthogonal trajectories of the parametric lines of u, together with the condition that $u^* = u$ on C. Then, dropping the asterisks, we know that (8.2) is true, and all results previously established are equally true for the new (u, v) coordinate system now adopted.

Differentiating (8 . 2) with respect to u, we obtain

$$g_{ij}\,\bar{\xi}^i\,\eta^j + g_{ij}\,\xi^i\,\bar{\eta}^j = 0, \tag{8 . 3}$$

and hence, since C is geodesic, we have on C

$$\bar{\eta}\cos\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}}) = 0. \tag{8 . 4}$$

Thus

$$\bar{\eta}^2 \sin^2\theta(\boldsymbol{\xi}, \bar{\boldsymbol{\eta}}) = \bar{\eta}^2. \tag{8 . 5}$$

Furthermore, on C we have $\underset{0}{g_{11}}$ equal to unity, while $\underset{0}{g_{12}}$ vanishes. Thus we have

$$\underset{0}{g} = \underset{0}{g_{22}} = g_{ij}\,\eta^i\,\eta^j = \eta^2. \tag{8 . 6}$$

Substitution from (8 . 5) and (8 . 6) in (8 . 1) gives

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\bar{\eta}^2 - K\eta^2]\,du. \tag{8 . 7}$$

The following theorem results directly :

THEOREM XII.—*A curve C being geodesic, the second variation of its length is positive for all variations with fixed end points included in the type given in § 1 (and its length is therefore a relative minimum with respect to variations of that type), if the Riemannian curvature corresponding to every two-dimensional plane element containing the direction of C is zero or negative*. The equivalent sufficient analytical condition is that $G_{ijkl}\,\xi^i\,\eta^j\,\xi^k\,\eta^l$ should be zero or negative at every point of C for arbitrary values of the components of η^m.*

9. *Expression for the second variation using the unit normal variation vector.*

Let us now further transform the expression for the second variation in the case where C is geodesic, the end points are fixed, and the v-curves cut C orthogonally, by the use of the *unit normal variation vector* μ^i defined by

$$\eta^i = \eta\mu^i. \tag{9 . 10}$$

* For the case $N = 2$, see Bolza, *loc. cit.*, 228.

We have then

$$g_{ij}\mu^i\mu^j = 1, \tag{9.11}$$

and

$$g_{ij}\xi^i\mu^j = 0. \tag{9.12}$$

Differentiation of these two equations with respect to u gives respectively

$$g_{ij}\mu^i\bar{\mu}^j = 0, \tag{9.13}$$

and, since C is geodesic,

$$g_{ij}\xi^i\bar{\mu}^j = 0. \tag{9.14}$$

Differentiation of (9.10) with respect to u gives

$$\bar{\eta}^i = \dot{\eta}\mu^i + \eta\bar{\mu}^i, \tag{9.15}$$

and therefore, by (9.11) and (9.13),

$$\bar{\eta}^2 = g_{ij}\bar{\eta}^i\bar{\eta}^j = \dot{\eta}^2 + \eta^2\bar{\mu}^2. \tag{9.16}$$

Introducing this expression into (8.7), we obtain the equation

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\dot{\eta}^2 + (\bar{\mu}^2 - K)\eta^2]\, du. \tag{9.17}$$

An interesting geometrical result can easily be deduced from this equation. Let us at present consider only variations in some definite V_2, so that μ^i has definite values along C. We may treat V_2 as the fundamental manifold. Taking any coordinate system (x^1, x^2) in V_2, all our previous arguments apply, and (9.17) is the expression for the second variation with fixed end points, C being geodesic in V_2 if it is geodesic in V_N*.

In calculating (9.17) when V_2 is the fundamental manifold, η and η have the same values as in the case where V_N is fundamental; but $\bar{\mu}^i$ is the contravariant derivative calculated for the coordinate system in V_2, and K is the Gaussian curvature of V_2 itself (say Γ). But, by (9.13) and (9.14), the vector $\bar{\mu}^i$ is perpendicular to two mutually perpendicular

* Cf. Bianchi, *loc. cit.*, 422.

vectors, namely μ^i and ξ^i. This is impossible in a space of two dimensions, and therefore $\bar{\mu}$ taken with respect to V_2 vanishes. Thus we have

$$\delta^2 L = \tfrac{1}{2}\delta v^2 \int_{u_1}^{u_2} [\dot{\eta}^2 - \Gamma\eta^2]\, du. \tag{9.18}$$

Comparison with (9.17) gives

$$\Gamma = K - \bar{\mu}^2, \tag{9.19}$$

where $\bar{\mu}$ is to be calculated with respect to V_N, and K is the Riemannian curvature of V_N corresponding to V_2. If $\bar{\mu}$ is zero,

$$\Gamma = K. \tag{9.20}$$

We may express these results in the following form :—

THEOREM XIII.—*At any point of a geodesic C of V_N the Gaussian curvature of any two-space V_2 containing C is equal to the excess of the Riemannian curvature of V_N corresponding to the element of V_2 over the square of the magnitude of the contravariant derivative with respect to C of the unit vector lying in V_2 and normal to C, calculated for the manifold V_N. The Gaussian curvature of V_2 is equal to the corresponding Riemannian curvature if, and only if, the unit normal vector is propagated parallelly along C.*

The following restricted theorem is of interest :—

THEOREM XIV.—*In a space V_N of constant and isotropic Riemannian curvature K there exists no two-space containing a geodesic of V_N and having a Gaussian curvature greater than K.*

We have already seen in Theorem XII that the second variation of the length-integral is positive if the Riemannian curvature corresponding to every element containing the direction of C is zero or negative. We shall now show that, if $N > 2$, it is always possible to construct a V_2 containing C, such that for variations in this V_2 the second variation is positive, on the sole assumption that K has a finite upper bound on C. The satisfaction of this condition is, in fact, a consequence of our postulates in § 1 concerning the fundamental tensor of V_N. From (9.17) we see that a sufficient condition for the second variation to be positive is

$$\bar{\mu}^2 \geqslant K \tag{9.21}$$

at all points of C. Let ϕ^i, ψ^i be two mutually perpendicular unit vectors, normal to C and propagated parallelly along C, and let α, β be two scalar functions of u. Write

$$\mu^i = \alpha\phi^i + \beta\psi^i. \tag{9.22}$$

Differentiation gives

$$\bar{\mu}^i = \dot{\alpha}\phi^i + \dot{\beta}\psi^i. \tag{9.23}$$

The vector μ^i defined by (9.22) is normal to C and is a unit vector if

$$\alpha^2 + \beta^2 = 1. \tag{9.24}$$

The value of $\bar{\mu}^2$ is

$$\bar{\mu}^2 = g_{ij}\bar{\mu}^i\bar{\mu}^j = \dot{\alpha}^2 + \dot{\beta}^2. \tag{9.25}$$

Now if we choose $\alpha = \cos Au$, $\beta = \sin Au$, where A is any constant greater than the upper bound of $\sqrt{K}$, (9.24) is satisfied, and (9.25) gives

$$\bar{\mu}^2 = A^2. \tag{9.26}$$

Thus (9.21) is satisfied, and we have the following result :—

THEOREM XV.—*Being given a geodesic C it is possible to find infinitely many two-spaces V_2 containing C, such that the second variation of the length of C is positive for all variations in V_2 between fixed end points.*

By (9.19) the equivalent geometrical statement is that it is always possible to draw infinitely many two-spaces through a geodesic having negative Gaussian curvature at every point of the geodesic. We can, in fact, draw such two-spaces with negative curvature exceeding in absolute value any preassigned positive number, however large.

Equation (9.17) expresses the second variation of the length-integral when C is geodesic and the curves of variation cut C orthogonally, the end points being fixed. In this equation $\eta\,|\delta v|$ is the magnitude of the infinitesimal variation vector, that is, the distance through which a point of C is displaced, and $\dot{\eta}\,|\delta v|$ is the rate of change of the magnitude of this displacement as we move along C. The magnitude of the displacement is arbitrary, except for the condition that it should

vanish at the end points. If we take two two-spaces, V_2 and V_2^*, through C, we obtain an expression (9 . 17) for each of them. If η is the same function of u for V_2 and V_2^*, we shall say that the variations are *similar*. When this is the case, (9 . 17) enables us to compare the lengths of such varied curves adjacent to C lying in V_2 and V_2^* respectively. We may state the following theorem as a direct deduction from (9 . 17) :—

THEOREM XVI.—*If, at each point of a geodesic C, V_N has the same Riemannian curvature for all elements containing the direction of C, then, considering similar variations with fixed end points in all the two-spaces V_2 containing C, the second variation has a common value for all the V_2 formed by parallel propagation of a unit normal vector along C, and this common value is less than the second variation for any V_2 not generated in this manner.* Or, *for an infinitesimal normal variation of assigned magnitude at every point, the length of the varied curve is less when the unit normal vector is propagated parallelly than when it is not.*

10. *Conjugate points.*

From (9 . 17) we see that the Jacobian differential equation* for determination of the conjugate points for variations in any definite V_2 containing the geodesic C is

$$\frac{d^2y}{du^2} + (K - \bar{\mu}^2)\, y = 0, \tag{10 . 1}$$

the conjugate points being consecutive zeros of any solution of this equation. Making use of Sturm's theorem†, the following theorem results :—

THEOREM XVII.—*If the Riemannian curvature corresponding to every element containing the direction of a geodesic C is less than a positive number A, the distance between a pair of conjugate points on C cannot be less than $\pi/\sqrt{A}$; while, if the Riemannian curvature is always greater than a positive number B, the distance between a pair of conjugate points for variations in a V_2 generated by parallel propagation of a unit normal vector is less than $\pi/\sqrt{B}$.*

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* Cf. Bolza, *loc. cit.*, 60.

† Darboux, *Leçons sur la théorie générale des surfaces*, 3 (1894), 100.

THE MAXIMUM OF A CERTAIN BILINEAR FORM

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1. The form in question is

$$S(x, y) = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{r-s} x_r y_s, \tag{1.1}$$

where

$$x_r \geqslant 0, \quad y_s \geqslant 0, \tag{1.2}$$

$$a_0 \geqslant a_1 \geqslant a_2 \geqslant \ldots \geqslant 0, \quad a_\nu = a_{-\nu}. \tag{1.3}$$

If the coefficients are given, and the x's and y's are given in all but arrangement, the form $S(x, y)$ assumes a finite number of values corresponding to the different arrangements possible. Among these values there is a maximum, and our first problem is to determine an arrangement of the variables which yields this maximum value. This problem is solved by Theorem 1, which shows (this being indeed the kernel of the theorem) that the arrangement depends only on the order of magnitude of the x's and y's.

In § 3 we suppose that

$$A = \sum_{r=1}^{n} x_r^p, \quad B = \sum_{s=1}^{n} y_s^q, \tag{1.4}$$

where

$$p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} > 1, \tag{1.41}$$

are given, and that

$$a_{r-s} = |r-s|^{-\lambda} \ (r \neq s), \quad a_0 = 1, \tag{1.5}$$

where

$$\lambda = 2 - \frac{1}{p} - \frac{1}{q}; \tag{1.51}$$

and we prove that

$$S \leqslant C_{p,q} A^{1/p} B^{1/q}, \tag{1.6}$$

where $C_{p,q}$ is a certain function of p and q only. In § 4 we apply this result to the proof of a number of theorems concerning the convergence of series of positive terms. All these theorems are not only very curious in themselves, but are important for a more extensive series of researches which two of us have undertaken in the theory of "Riemann-Liouville integrals" of non-integral order*.

The form $S(x, y)$ resembles in many respects the form

$$T(x, y) = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{r+s} x_r y_s, \tag{1.7}$$

but the theory of $S(x, y)$ is a good deal more difficult than that of $T(x, y)$. We use the latter form to illustrate the most interesting points in the theory of the former as they arise.

The particular form

$$T_0 = \sum_{r=1}^{n} \sum_{s=1}^{n} \frac{x_r y_s}{r+s}$$

was first considered by Hilbert, who proved that

$$T_0 < A (\Sigma x_r^2)^{\frac{1}{2}} (\Sigma y_s^2)^{\frac{1}{2}},$$

where A is an absolute constant, and this theorem may be regarded as the starting point of all researches in this field.

2.1. THEOREM 1.—*Suppose that the coefficients and variables satisfy the conditions* (1.2) *and* (1.3), *the coefficients being given, and the*

* A preliminary account of some of some of these researches has already been printed in the *Proceedings*: see G. H. Hardy and J. E. Littlewood, "Some properties of integrals of fractional order", *Records of Proceedings* for 12 March, 1925.

variables being given in every respect except arrangement. Then among the arrangements for which S assumes its maximum value there is one in which* (i)

$$x_\mu \leqslant x_{\mu'}, \quad y_\nu \leqslant y_{\nu'}, \tag{2.1}$$

if

$$|\mu'-\tfrac{1}{2}n| < |\mu-\tfrac{1}{2}n|, \quad |\nu'-\tfrac{1}{2}n| < |\nu-\tfrac{1}{2}n|, \tag{2.11}$$

and (ii) *no two of*

$$x_\mu - x_{\mu'}, \quad y_\nu - y_{\nu'}, \tag{2.2}$$

where

$$\mu < \mu', \quad \nu < \nu', \quad |\mu'-\tfrac{1}{2}n| = |\mu-\tfrac{1}{2}n|, \quad |\nu'-\tfrac{1}{2}n| = |\nu-\tfrac{1}{2}n|, \tag{2.21}$$

have different signs, the common sign being positive or negative at our discretion.

Suppose, for example, that all the x's and all the y's are unequal. Then, of two x's not equidistant from the middle, that which is nearer the middle is the greater, and similarly for the y's. If two x's are equidistant from the middle, their difference may have either sign; but if, in one such pair of x's, that nearer the beginning is greater, then the same must be true of every other such pair of x's and of every such pair of y's. It is easy to see that these conditions define just two possible arrangements. Thus, when $n = 5$, x_3 and y_3 must be the greatest x and y. If $x_2 > x_4$, then

$$x_3 > x_2 > x_4 > x_1 > x_5, \quad y_3 > y_2 > y_4 > y_1 > y_5,$$

and if $x_4 > x_2$ then

$$x_3 > x_4 > x_2 > x_5 > x_1, \quad y_3 > y_4 > y_2 > y_5 > y_1.$$

These two arrangements give the same maximum value for S.

The corresponding result for T is much simpler: T is a maximum when x_r and y_s decrease steadily when r and s increase†. For if, *e.g.*,

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* Generally, the maximum arrangement will be unique (with a reservation which will appear in the sequel). We state the theorem so as to cover exceptional cases such as that in which every a, every x, or every y is zero.

† This was observed by F. Wiener, "Elementarer Beweis eines Reihensatzes von Herrn Hilbert", *Math. Annalen*, 68 (1910), 361–366.

$r' > r$ and $x_{r'} > x_r$, an exchange of x_r and $x_{r'}$ alters T by

$$(x_{r'} - x_r) \sum_s (a_{r+s} - a_{r'+s}) y_s,$$

which is positive. It is plain that we must prove Theorem 1 by arguments of the same character, but the nature of the appropriate substitutions is much less obvious.

Proof of Theorem 1.

2.2. Suppose that $1 < p < n$, and consider the pairs

$$(x_1, x_p), \quad (x_2, x_{p-1}), \quad \ldots, \quad (y_1, y_p), \quad \ldots,$$

or

$$(x_i, x_{p+1-i}), \quad (y_j, y_{p+1-j}) \quad (i \leqslant \tfrac{1}{2}p, \ j \leqslant \tfrac{1}{2}p). \tag{2.21}$$

We call these pairs of elements *conjugate pairs* (for the index p). If p is even, say equal to $2q$, all x's and y's whose indices do not exceed p are paired, the central pairs being (x_q, x_{q+1}) and (y_q, y_{q+1}). If $p = 2q+1$, the elements x_{q+1} and y_{q+1} are left unpaired. These unpaired elements, if they exist, and the unpaired elements

$$x_{p+1}, \ x_{p+2}, \ \ldots, \ x_n, \qquad y_{p+1}, \ \ldots, \ y_n$$

in any case, we call *free elements* (for the index p). If $p = n$, there is no free element when n is even, and two (the central elements) when n is odd.

We suppose first that $p < n$, and we proceed to show that, among the "maximum" arrangements, there is one in which every difference

$$x_i - x_{p+1-i}, \quad y_j - y_{p+1-j}$$

is negative or zero, and this for every value of p.

Let us suppose, in fact, that some of these differences are positive, and let us denote the values of i and j for which this is so by I and J. We shall prove that *S is unaltered or increased when the elements of each conjugate pair*

$$(x_I, x_{p+1-I}), \quad (y_J, y_{p+1-J})$$

are interchanged, and this will establish our conclusion.

We must distinguish two cases.

(*a*) *p is even.* The free elements are those whose rank is greater than p. We divide up S into partial sums as follows :—

(1) S_1: x_r and y_s are both free.

(2) S_2: x_r is free, s is j or $p+1-j$ $(j \neq J)$.

(3) S_3: y_s is free, r is i or $p+1-i$ $(i \neq I)$.

(4) S_4: x_r is free, s is J or $p+1-J$.

(5) S_5: y_s is free, r is I or $p+1-I$.

(6) S_6: r is i or $p+1-i$, s is j or $p+1-j$ $(i \neq I,\ j \neq J)$.

(7) S_7: r is i or $p+1-i$ $(i \neq I)$, s is J or $p+1-J$.

(8) S_8: r is I or $p+1-I$, s is j or $p+1-j$ $(j \neq J)$.

(9) S_9: r is I or $p+1-I$, s is J or $p+1-J$.

The sums S_2 to S_5 include those terms $a_{r-s}x_r y_s$ in which just one of x_r and y_s is free, the sums S_6 to S_9 those in which neither is free.

It is obvious that S_1, S_2, S_3, and S_6 are unaltered by the substitution. We shall prove that S_9 is also unaltered, while S_4, S_5, S_7, and S_8 are unaltered or increased.

2.3. The discussion of S_4 will apply by symmetry to S_5, and that of S_7 to S_8. It is therefore only necessary to consider S_4, S_7, and S_9.

In S_4, the coefficient of x_r is

$$\sum_J (a_{r-J}y_J + a_{r-p-1+J}y_{p+1-J}),$$

and the change in this is

$$\sum_J (a_{r-J} - a_{r-p-1+J})(y_{p+1-J} - y_J).$$

The second factor here is by hypothesis negative*, and the first is so also because $J < \frac{1}{2}(p+1)$, $r-J > r-p-1+J$. Hence S_4 is increased.

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* In what follows we use "positive" and "negative" to include "zero", and "increased" and "decreased" as meaning "not decreased" and "not increased".

In S_7 we consider the terms of the conjugate pair (x_i, x_{p+1-i}) together. The contribution of this pair to S_7 is

$$x_i \sum_J (a_{i-J} y_J + a_{i-p-1+J} y_{p+1-J}) + x_{p+1-i} \sum_J (a_{p+1-i-J} y_J + a_{J-i} y_{p+1-J}).$$

The resultant change is

$$(x_i - x_{p+1-i}) \sum_J (a_{i-J} - a_{i-p-1+J})(y_{p+1-J} - y_J).$$

The outside factor is negative, and so is the second factor inside. The first factor inside is positive if $|i-p-1+J| > |i-J|$. This condition is either $p+1-i-J > i-J$ or $p+1-i-J > J-i$, *i.e.* $p+1 > 2i$ or $p+1 > 2J$, both of which inequalities are satisfied. Hence S_7 is increased.

Finally, in S_9 we consider the contribution of the two pairs (x_I, x_{p+1-I}) and (y_J, y_{p+1-J}). This is

$$a_{I-J} x_I y_J + a_{p+1-I-J} x_{p+1-I} y_J + a_{I-p-1+J} x_I y_{p+1-J} + a_{J-I} x_{p+1-I} y_{p+1-J},$$

which is plainly unaltered.

It follows that S is increased by the substitution.

2.4. (*b*) *p is odd.* This case differs only slightly from case (*a*). If $p = 2q+1$, there are additional free elements x_{q+1}, y_{q+1}, and we have to consider the terms in S which involve these elements separately. It is plain that the only terms affected are those in $x_{q+1} y_J$, $x_{q+1} y_{p+1-J}$, and $x_I y_{q+1}$, $x_{p+1-I} y_{q+1}$. The sum of the terms of the first type is

$$x_{q+1} \sum_J (a_{q+1-J} y_J + a_{-q-1+J} y_{2q+2-J}),$$

which is unchanged. Similarly, the sum of the terms of the second type is unchanged.

2.5. We have thus proved that there is a maximum arrangement which satisfies the conditions of Theorem 1, in so far as they are concerned with pairs $x_\mu, x_{\mu'}$ or $y_\nu, y_{\nu'}$ whose second member is nearer to the middle than the first. It is plain by symmetry that we can account similarly for the conditions concerning pairs whose first member is nearer to the middle than the second. There remain the pairs disposed symmetrically about the middle, and to deal with these we must consider the case $p = n$. We shall prove that there is a maximum

arrangement in which all the differences

$$x_i - x_{n+1-i}, \quad y_j - y_{n+1-j} \tag{2.51}$$

have the same sign.

We denote now by I and J the values of i and j for which the differences (2.51) are positive, and repeat the argument of §§ 2.1–2.4. The sums $S_1, S_2, \ldots, S_5$ do not now occur. If not all of the differences are positive, S_6 and S_9 are unaltered, and S_7 and S_8 are increased. If, on the other hand, all are positive, then S_6, S_7, and S_8 do not occur, and S_9 (*i.e.* the whole sum) is unaltered.

To sum up, we have established the existence of a maximum arrangement in which (i)

$$x_i - x_{p+1-i} \leqslant 0, \quad y_j - y_{p+1-j} \leqslant 0 \quad (p < n, \ i \leqslant \tfrac{1}{2}p, \ j \leqslant \tfrac{1}{2}p),$$

(ii) the corresponding set of inequalities concerning pairs whose first member is nearer to the middle than the second is also satisfied, (iii) the differences

$$x_i - x_{n+1-i}, \quad y_j - y_{n+1-j} \quad (i \leqslant \tfrac{1}{2}n)$$

have all the same sign, which we may choose as we please. These are the conditions of Theorem 1.

The form $\Sigma\Sigma \dfrac{x_r y_s}{|r-s|^\lambda}$.

3.1. We consider now the special form

$$U(x, y) = \sum_{r=1}^{n} \sum_{s=1}^{n}{}' \frac{x_r y_s}{|r-s|^\lambda}, \tag{3.11}$$

where the dash excludes equal pairs r, s from the summation. This form is not, as it stands, a case of (1.1). But if we define a_{r-s} by (1.5), then

$$S(x, y) = \sum_{r=1}^{n} \sum_{s=1}^{n} a_{r-s} x_r y_s = U(x, y) + \sum_{\nu=1}^{n} x_\nu y_\nu \geqslant U(x, y), \tag{3.12}$$

and any inequality for S holds *a fortiori* for U.

THEOREM 2.—*If*

$$p > 1, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} > 1, \quad \lambda = 2 - \frac{1}{p} - \frac{1}{q}, \tag{3.13}$$

then

$$U = \Sigma' \frac{x_r y_s}{|r-s|^\lambda} \leqslant C_{p,q} (\Sigma x_r^p)^{1/p} (\Sigma y_s'^q)^{1/q}, \tag{3.14}$$

where $C_{p,q}$ is a function of p and q only. In particular we may take

$$(3.15)\quad C_{p,q} = 4 + \frac{4}{(1/p)+(1/q)-1}\left\{\left(\frac{p}{p-1}\right)^{p(q-1)/q} + \left(\frac{q}{q-1}\right)^{q(p-1)/p}\right\}*.$$

We suppose, to fix our ideas, that $n = 2k$ is even†; and we shall prove that S, and *a fortiori* U, satisfies (1.6). Since the right-hand side of (1.6) is unaltered by rearrangement of the x's or y's, we may suppose them arranged so that S has its greatest value; and therefore, by Theorem 1, we may suppose that

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k, \quad x_{2k} \leqslant x_{2k-1} \leqslant \ldots \leqslant x_{k+1},$$

and similarly for the y's.

We represent the terms of S diagrammatically by the lattice points of the square of Fig. 1, the south-west and north-east corners being (1, 1) and $(2k, 2k)$. We divide this square into four squares A, B, C, D by lines parallel to the axes through its centre. The arrows in each square

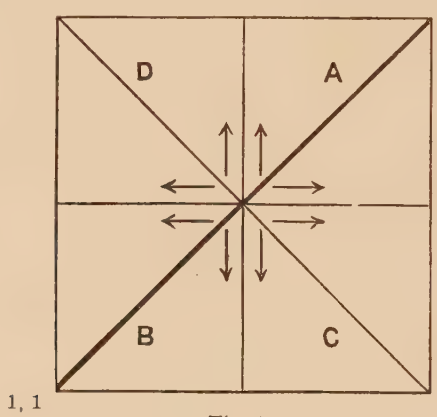

Fig. 1.

* It is possible to improve the value of $C_{p,q}$ a little by refining the argument: for example we may take

$$C_{p,q} = 1 + \frac{2^{2-(1/p)-(1/q)}}{(1/p)+(1/q)-1}\left\{\left(\frac{p}{p-1}\right)^{p(q-1)/q} + \left(\frac{q}{q-1}\right)^{q(p-1)/p}\right\}.$$

It may also be shown that we cannot take $C_{p,q}$ less than

$$2\Big/\left(\frac{1}{p}+\frac{1}{q}\right)\left(\frac{1}{p}+\frac{1}{q}-1\right).$$

It would probably be very difficult to obtain a "best possible" form for $C_{p,q}$, and we content ourselves with the form which occurs most naturally. The essential thing to notice is that $C_{p,q}$ tends to infinity if $p \to 1$, $q \to 1$, or $(1/p)+(1/q) \to 1$.

† We can plainly reduce the other case to this one by introducing an x and a y which are both zero.

indicate the directions in which the x's and y's decrease. It is plainly enough to prove that the sum of the terms in each of these four squares is less than

$$\tfrac{1}{4}C_{p,q}A^{1/p}B^{1/q}.$$

In the north-east square A we write

$$x_{k+1}=\xi_1,\ \ldots,\ x_{2k}=\xi_k,\quad y_{k+1}=\eta_1,\ \ldots,\ y_{2k}=\eta_k.$$

The corresponding sum is then

$$\sum_{r=1}^{k}\sum_{s=1}^{k}a_{r-s}\xi_r\eta_s, \tag{3.16}$$

and here ξ_r and η_s decrease when r and s increase.

In B we write

$$x_k=\xi_1,\ \ldots,\ x_1=\xi_k,\quad y_k=\eta_1,\ \ldots,\ y_1=\eta_k,$$

and then the sum corresponding to B is also reduced to the form (3.16).

In C we write

$$x_{k+1}=\xi_1,\ \ldots,\ x_{2k}=\xi_k,\quad y_k=\eta_1,\ \ldots,\ y_1=\eta_k,$$

when we obtain

$$\sum_{r=1}^{k}\sum_{s=1}^{k}a_{r+s-1}\xi_r\eta_s. \tag{3.17}$$

Since $r+s-1\geqslant|r-s|$, this sum is not greater than the sum (3.16). Finally, D may be treated similarly.

The problem is thus reduced to proving that

$$\sum_{r=1}^{k}\sum_{s=1}^{k}a_{r-s}\xi_r\eta_s\leqslant\tfrac{1}{4}C_{p,q}A^{1/p}B^{1/q}.$$

And if we change our notation again we see that, in order to prove (1.6), it is sufficient to prove the somewhat sharper inequality of the same type in which $C_{p,q}$ is replaced by $\frac{1}{4}C_{p,q}$, in the special case in which x_r and y_s are steadily decreasing functions of r and s.

3.2. We use three known inequalities as lemmas.

LEMMA α.—*If* $p>1$, $q>1$, $1/p+1/q\geqslant 1$, *and* a_ν *and* b_ν *are positive, then*

$$\sum_{1}^{n}a_\nu b_\nu\leqslant\left(\sum_{1}^{n}a_\nu^p\right)^{1/p}\left(\sum_{1}^{n}b_\nu^q\right)^{1/q}. \tag{3.21}$$

This inequality is due to Hölder; it is sometimes called the "generalized inequality of Schwarz". It is usually stated for the case when $1/p+1/q=1$. The remark that it holds also when $1/p+1/q>1$ is due to Jensen*.

LEMMA β.—*If a_ν and b_ν are positive and monotonic, and their directions of variation are opposed to one another, then*

$$n\sum_1^n a_\nu b_\nu \leqslant \sum_1^n a_\nu \sum_1^n b_\nu. \tag{3.22}$$

If the directions are the same, the sign of the inequality must be reversed. This inequality is due to Tchebicheff†.

LEMMA γ.—*If $p>1$, a_ν is positive, and*

$$A_\nu = a_1+a_2+\ldots+a_\nu,$$

then

$$\sum_1^n \left(\frac{A_\nu}{\nu}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_1^n a_\nu^p. \tag{3.23}$$

3.3. We write

$$V=\sum_{r=1}^n \sum_{s=1}^n a_{r-s}x_r y_s = \sum_1^n x_\nu y_\nu + \sum_{r>s} + \sum_{r<s} = V_1+V_2+V_3, \tag{3.31}$$

say. We have first, by Lemma α,

$$V_1 \leqslant A^{1/p}B^{1/q}. \tag{3.32}$$

Next

$$V_2 = \sum_{r>s} = \sum_{r=2}^n x_r \sum_{s=1}^{r-1} \frac{y_s}{(r-s)^\lambda}. \tag{3.33}$$

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* See O. Hölder, "Über einen Mittelwertsatz", *Göttinger Nachrichten* (1889), 38–47; J. L. W. V. Jensen, "Sur les fonctions convexes et les inégalités entre les valeurs moyennes", *Acta Math.*, 30 (1906), 175–193 (193); E. Landau, "Über einen Konvergenzsatz", *Göttinger Nachrichten* (1907), 25–27; F. Riesz, "Untersuchungen über Systeme integrierbarer Funktionen", *Math. Annalen*, 69 (1910), 449–497 (452, 455). The inequality is apparently not given in any text-book of analysis.

† See Ch. Hermite, *Cours d'analyse* (1882), 47 (the corresponding proposition for integrals).

‡ See G. H. Hardy, "Note on a theorem of Hilbert", *Math. Zeitschrift*, 6 (1920), 314–317, and "An inequality between integrals" [Notes on some points in the integral calculus, 60] *Messenger of Math.*, 54 (1925), 150–156; E. Landau, "Note on a theorem concerning series of positive terms", *Journal London Math. Soc.*, 1 (1926), 38–39; E. B. Elliott, "A simple exposition of some recently proved facts as to convergency", *ibid.*, 93–96.

Here y_s decreases, and $(r-s)^{-\lambda}$ increases, as s increases, so that we may use Lemma β, which gives

$$(3.34) \qquad \sum_{s=1}^{r-1} \frac{y_s}{(r-s)^\lambda} \leqslant \frac{Y_{r-1}}{r-1} \sum_{t=1}^{r-1} \frac{1}{t^\lambda} \leqslant \frac{Y_{r-1}}{(1-\lambda)(r-1)^\lambda},$$

where

$$(3.341) \qquad Y_r = y_1+y_2+\ldots+y_r.$$

From (3.33) and (3.34) we deduce

$$(3.35) \qquad V_2 \leqslant \frac{1}{1-\lambda} \sum_{r=2}^{n} x_r \frac{Y_{r-1}}{(r-1)^\lambda} = \frac{1}{1-\lambda} \sum_{r=1}^{n-1} x_{r+1} \frac{Y_r}{r^\lambda}.$$

We write $$q' = \frac{p}{p-1}, \quad \frac{1}{p}+\frac{1}{q'} = 1$$

(so that $q' > q$), and apply Hölder's inequality (Lemma α), for p and q', to the right-hand side of (3.35). We obtain

$$(3.36) \qquad V_2 \leqslant \frac{A^{1/p}}{1-\lambda} \left(\sum_{r=1}^{n} \frac{Y_r^{q'}}{r^{\lambda q'}} \right)^{1/q'}.$$

Now $$Y_r^q = (y_1+\ldots+y_r)^q \leqslant r^{q-1} B$$

(again by Lemma α). It follows that

$$(3.37) \qquad r^{-\lambda q'} Y_r^{q'} = r^{-\lambda q'} Y_r^q \, Y_r^{q'-q}$$

$$\leqslant r^{-\lambda q'} Y_r^q (r^{q-1} B)^{(q'-q)/q} = B^{(q'-q)/q} \left(\frac{Y_r}{r}\right)^q,$$

since $$-\lambda q' + \frac{(q'-q)(q-1)}{q} = -q.$$

From (3.36) and (3.37) we deduce

$$(3.38) \qquad V_2 \leqslant \frac{1}{1-\lambda} A^{1/p} B^{(q'-q)/qq'} \left\{ \sum_{r=1}^{n} \left(\frac{Y_r}{r}\right)^q \right\}^{1/q'}$$

$$\leqslant \frac{1}{1-\lambda} \left(\frac{q}{q-1}\right)^{q(p-1)/p} A^{1/p} B^{1/q},$$

by Lemma γ. Similarly

$$(3.39) \qquad V_3 \leqslant \frac{1}{1-\lambda} \left(\frac{p}{p-1}\right)^{p(q-1)/q} A^{1/p} B^{1/q}.$$

From (3.31), (3.32), (3.38), and (3.39) we deduce

$$V \leqslant \tfrac{1}{4} C_{p,q} A^{1/p} B^{1/q},$$

which proves the theorem.

Convergence theorems.

4.1. It is plain that, as an immediate corollary of Theorem 2, we have

THEOREM 3.—*If the summations extend from* 1 *to infinity,* p, q, *and* λ *satisfy* (3.13), *and* Σx_r^p *and* Σy_s^q *are convergent, then*

$$\Sigma\Sigma' \frac{x_r y_s}{|r-s|^\lambda} \tag{4.11}$$

is convergent.

It is important to observe that the theorem becomes false in the limiting case in which

$$\frac{1}{p}+\frac{1}{q}=1, \quad \lambda=1. \tag{4.12}$$

Suppose, in fact, that

$$p>1, \quad q>1, \quad \frac{1}{p}+\frac{1}{q}\leqslant 1, \quad \lambda=2-\frac{1}{p}-\frac{1}{q}\geqslant 1,$$

and take

$$x_r=r^{-1/p}(\log r)^{-1} \quad (r>1), \qquad y_s=s^{-1/q}(\log s)^{-1} \quad (s>1),$$

and $x_1=x_2$, $y_1=y_2$. Then Σx_r^p and Σy_s^q are convergent. But

$$\Sigma\Sigma' \frac{x_r y_s}{|r-s|^\lambda} \geqslant \sum_{s=1}^{\infty}\sum_{k=1}^{\infty} k^{-\lambda} x_{s+k} y_s \geqslant \sum_{k=1}^{\infty} k^{-\lambda} \sum_{s=k+1}^{2k} x_s y_s$$

$$\geqslant \sum_{k=1}^{\infty} k^{-\lambda} \int_{k+1}^{\infty} u^{\lambda-2}(\log u)^{-2}\,du \geqslant \sum_{k=1}^{\infty} k^{-\lambda} k^{\lambda-1} \int_{k+1}^{\infty} \frac{du}{u(\log u)^2}$$

$$\geqslant \sum_{k=2}^{\infty} \frac{1}{k\log k};$$

and this series is divergent. It may indeed easily be verified that the theorem is false for *any* pair of values of p and q which do not satisfy (3.13), except for the particular pair $p=1$, $q=1$, when it is true, but trivial.

It is instructive to compare the series (4.11) with the series

$$\Sigma\Sigma \frac{x_r y_s}{(r+s)^\lambda}, \tag{4.13}$$

It is obvious that, if (4 . 11) is convergent, then (4 . 13) is also convergent. The latter series, however, is still convergent in the limiting case (4 . 12)*. In this case the series

$$\Sigma\Sigma' \frac{x_r y_s}{r-s}$$

is convergent, as has been shown by Hilbert in the case $p = q = 2$, and by Marcel Riesz generally, but, in general, not absolutely convergent†.

4 . 2. It follows from Theorem 3 that, if Σx_r^p is convergent, and

$$X_s^{(k)} = (s-1)^{k-1} x_1 + (s-2)^{k-1} x_2 + \ldots + 1^{k-1} x_{s-1}, \tag{4 . 21}$$

then

$$\sum_s y_s X_s^{(1-\lambda)}$$

is convergent for every system of y's for which Σy_s^q converges, and therefore, by a well known theorem of Landau‡, that

$$\sum_s \{X_s^{(1-\lambda)}\}^{q/(q-1)}$$

is convergent. If $\lambda = 1-k$, so that $0 < k < 1/p$§, then

$$\frac{q}{q-1} = \frac{p}{1-kp}.$$

* See G. H. Hardy, "Note on a theorem of Hilbert concerning series of positive terms", *Proc. London Math. Soc.* (2), 23 (1925), xlv-xlvi (*Records* for 24 April, 1924); and Theorem 6 below.

The series (4 . 13) is not generally convergent when $p > 1$, $q > 1$, $1/p + 1/q < 1$. This may be shown by the example

$$x_r = r^{-1/p} (\log r)^{-\alpha} \quad (r > 1), \qquad y_s = s^{-1/q} (\log s)^{-\beta} \quad (s > 1), \qquad x_1 = x_2, \qquad y_1 = y_2.$$

where $p\alpha > 1$, $q\beta > 1$, $\alpha + \beta < 1$.

† For Hilbert's proof see H. Weyl, "Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems", *Inaugural Dissertation*, Göttingen (1908), 83. Another proof was given by I. Schur, "Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen", *Journal für Math.*, 140 (1912), 1–28. The results of Riesz were communicated to us by letter, but have not yet been published in full: see M. Riesz, "Les fonctions conjuguées et les séries de Fourier", *Comptes rendus*, 28 April, 1924; and E. C. Titchmarsh, "Reciprocal formulae involving series and integrals", *Math. Zeitschrift*, 25 (1926), 321–347.

‡ E. Landau, *loc. cit. supra*, 274, f.n. ‡

§ $k > 0$ because $1/p + 1/q > 1$ and $k < 1/p$ because $q > 1$.

We have thus proved

THEOREM 4.—*If* $p>1$, Σx_r^p *is convergent*, $0<k<1/p$, *and* $X_r^{(k)}$ *is defined by* (4.21), *then*

$$\Sigma\{X_r^{(k)}\}^{p/(1-kp)} \tag{4.22}$$

is convergent.

This theorem may be stated in a more obviously interesting form. We may define the k-th Cesaro sum, or simply the k-th sum, or sum of order k, of x_s, as the coefficient X_r^k of u^s in*

$$\frac{x_0+x_1u+x_2u^2+\dots}{(1-u)^k};$$

so that

$$X_r^k=\frac{1}{\Gamma(k)}\sum_{i=0}^{r}\frac{\Gamma(r+k-i)}{\Gamma(r+1-i)}x_i, \tag{4.23}$$

and in particular $X_r^1=x_0+x_1+\dots+x_r=X_r$.

It is plain that

$$X_r^k=x_r+\theta_r\sum_{i=1}^{r-1}(r-i)^{k-1}x_i=x_r+\theta_r X_r^{(k)},$$

where θ_r lies between fixed positive bounds, and that the convergence of $\Sigma(X_r^{(k)})^{\varpi}$, for any $\varpi\geqslant p$, involves that of $\Sigma(X_r^k)^{\varpi}$. We thus obtain

THEOREM 5.—*If* $p>1$, Σx_r^p *is convergent*, $0<k<1/p$, *and* X_r^k *is the k-th sum of* x_r, *then*

$$\Sigma(X_r^k)^{p/1-kp)}$$

is convergent.

It is easy to show, by means of the example

$$x_1=0,\quad x_r=r^{-1/p}(\log r)^{-\beta}\quad(r>1),$$

that the index $p/(1-kp)$ cannot be replaced by any smaller index, and that

$$\Sigma(X_r^{1/p})^q$$

may diverge for every q.

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* It is convenient to insert x_0, which may afterwards be taken to be zero.

4 . 3. Theorem 5 is susceptible of a further interesting generalization. In order to obtain this, we begin by proving

THEOREM 6.—*If*

$$p>1, \quad q>1, \quad \frac{1}{p}+\frac{1}{q}\geqslant 1, \quad \lambda=2-\frac{1}{p}-\frac{1}{q}, \quad 0<\alpha\leqslant\lambda,$$

and Σx_r^p, Σy_s^q *are convergent, then*

$$\Sigma\Sigma' \frac{x_r y_s}{(r+s)^\alpha |\, r-s\,|^{\lambda-\alpha}}$$

is convergent.

The case $\alpha=1$ is that given by Hardy and referred to in § 4 . 1.

If $1/p+1/q>1$, the theorem is included in Theorem 3, and we may therefore confine our attention to the case $1/p+1/q=1$, in which the series (4 . 11) is not necessarily convergent. The method which we use yields, in fact, a more general and more precise result*.

Put $$K(x, y)=(x+y)^{-\alpha}|\, x-y\,|^{\alpha-1} \quad (x\geqslant 0,\ y\geqslant 0,\ x\neq y),$$

$$K(x, x)=0 \quad (x\geqslant 0).$$

Then $K(x, y)$ has the following properties:

(1) $$K(x, y)\geqslant 0;$$

(2) $$K(\rho x, \rho y)=\rho^{-1}K(x, y),$$

so that $K(x, y)$ is homogeneous of degree -1;

(3) the integrals

$$\int_0^\infty K(u, 1)\, u^{-1/p}\, du, \quad \int_0^\infty K(1, v)\, v^{-1/q}\, dv$$

converge to a common value k;

(4) the function $x^{-\delta}K(x, y)$, where $\delta>0$, decreases from $x=1$ onwards, while the interval $(0, 1)$ can be divided into two parts $(0, \eta)$, $(\eta, 1)$, in the first of which the function decreases, and in the second of which it increases†; and the function $y^{-\delta}K(x, y)$ has similar properties.

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* It gives a generalization of the results obtained by Schur (*loc. cit. supra*, 277, f.n. †, 23–28) in the particular case $p=q=2$. Schur's method is different from ours, but may also be adapted to the same purpose.

† In general (*i.e.* when $\alpha<1$) there are infinities for $x=0$ and $x=1$. When $\alpha=1$ the second infinity and the interval $(\eta, 1)$ disappear.

It follows that, if r and s are any positive integers,

$$(4.31)\qquad \sum_{n=1}^{\infty} K\left(\frac{n}{s},\ 1\right)\left(\frac{n}{s}\right)^{-1/p}\frac{1}{s} < \int_0^{\infty} K(u,\ 1)\,u^{-1/p}\,du = k,$$

$$(4.32)\qquad \sum_{n=1}^{\infty} K\left(1,\ \frac{n}{r}\right)\left(\frac{n}{r}\right)^{-1/q}\frac{1}{r} < \int_0^{\infty} K(1,\ v)\,v^{-1/q}\,dv = k^*.$$

Now, by Lemma α,

$$(4.33)\qquad \Sigma\Sigma' \frac{x_r y_s}{(r+s)^{\alpha}|r-s|^{1-\alpha}} = \Sigma\Sigma K(r,\ s)\,x_r y_s$$

$$= \Sigma\Sigma x_r K^{1/p}\left(\frac{r}{s}\right)^{1/pq}\cdot y_s K^{1/q}\left(\frac{s}{r}\right)^{1/pq} \leqslant PQ,$$

where

$$(4.34)\qquad P^p = \sum_r x_r^p \sum_s K(r,\ s)\left(\frac{r}{s}\right)^{1/q} = \sum_r x_r^p \sum_s K\left(1,\ \frac{s}{r}\right)\left(\frac{s}{r}\right)^{-1/q}\frac{1}{r},$$

$$(4.35)\qquad Q^q = \sum_s y_s^q \sum_r K(r,\ s)\left(\frac{s}{r}\right)^{1/p} = \sum_s y_s^q \sum_r K\left(\frac{r}{s},\ 1\right)\left(\frac{r}{s}\right)^{-1/p}\frac{1}{s}.$$

From (4.31)–(4.35) it follows that

$$(4.36)\qquad \Sigma\Sigma' \frac{x_r y_s}{(r+s)^{\alpha}|r-s|^{1-\alpha}} \leqslant k(\Sigma x_r^p)^{1/p}(\Sigma y_s^q)^{1/q}.$$

This proves Theorem 6. It may be shown that k, in (4.36), cannot be replaced by any smaller constant.

4.4. From Theorem 6, with $\lambda = 1$, it follows that, if Σx_r^p is convergent, then

$$\sum_s s^{-\alpha} y_s X_s^{(\alpha)}$$

is convergent for every system of y's for which $\Sigma y_s^{p/(p-1)}$ is convergent, and therefore that

$$\Sigma\left(\frac{X_s^{(\alpha)}}{s^{\alpha}}\right)^p$$

is convergent. Arguing as in § 4.2, we deduce

* It is to be observed that as $K(1,\ 1) = 0$ the terms $n = s$ and $n = r$ in (4.31) and (4.32) vanish.

THEOREM 7.—*If* $p>1$, Σx_r^p *is convergent, and* $k>0$, *then*

$$\Sigma (r^{-k} X_r^{(k)})^p$$

is convergent.

Here k is the α of the preceding argument. The theorem is proved in the first instance for $0<k<1$, but is immediately extensible to larger values of k, since $r^{-k} X_r^{(k)}$ is a decreasing function of k. when $k=1$, Theorem 7 reduces to the theorem of Hardy, referred to in § 3 . 2, which is included in Lemma γ.

We observe finally that Theorems 5 and 7 are each special cases of a more general theorem.

THEOREM 8.—*If* $p>1$, Σx_r^p *is convergent, and*

$$0<k<\frac{1}{p}, \quad p \leqslant q \leqslant \frac{p}{1-kp},$$

then $$\Sigma r^{-(p-q+kpq)/p} (X_r^{(k)})^q$$

is convergent.

Theorems 5 and 7 are the extreme cases of this theorem in which $q=p/(1-kp)$ and $q=p$ respectively. The general theorem is immediately deducible from the two special cases by means of Hölder's inequality. In fact, we may write

$$r^{-(p-q+kpq)/p} (X_r^{(k)})^q = (X_r^{(k)})^s (r^{-k} X_r^{(k)})^t,$$

where $$s=\frac{q-p}{kp}, \quad t=\frac{p-q+kpq}{kp}.$$

It follows that

$$\Sigma r^{-(p-q+kpq)/p} (X_r^{(k)})^q \leqslant \{\Sigma (X_r^{(k)})^{ls}\}^{1/l} \left\{\Sigma \left(\frac{X_r^{(k)}}{r^k}\right)^{lt/(l-1)}\right\}^{(l-1)/l}, \tag{4.41}$$

where l is any number greater than 1. We take

$$l=\frac{kp^2}{(q-p)(1-kp)},$$

which is greater than 1, since $kpq-q+p>0$. Then

$$ls=\frac{p}{1-kp}, \quad \frac{lt}{l-1}=p,$$

and the two series on the right-hand side of (4 . 41) are convergent, by Theorems 5 and 7. This proves Theorem 8.

So far we have supposed $k < 1/p$. If, however, we write

$$r^{-(p-q+kpq)/p}(X_r^k)^q = r^{-(p-q)/p}(r^{-k}X_r^k)^q,$$

and observe that $r^{-k}X_r^{(k)}$ is a decreasing function of k, we see that Theorem 8 remains valid when $k \geqslant 1/p$, but that the second condition on q then becomes nugatory. In other words, we have

THEOREM 9.—*The result of Theorem* 8 *is valid for* $k \geqslant 1/p$, $q \geqslant p$.

4 . 5. The theorems which we have proved have many further applications. These, however, belong more to the integral calculus and the theory of functions, and form part of the further researches to which we have already referred.

THE ZEROS OF CERTAIN INTEGRAL FUNCTIONS

By E. C. TITCHMARSH.

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1.1. The functions with which we are concerned are of the form

$$F(z) = \int_a^b e^{zt} f(t)\, dt \tag{1.11}$$

where $f(t)$ is any real integrable function, or, more generally,

$$f(t) = f_1(t) + if_2(t),$$

where $f_1(t)$ and $f_2(t)$ are real integrable functions. The limits a, b, are finite. The related forms

$$\int_a^b \cos zt\, f(t)\, dt, \qquad \int_a^b \sin zt\, f(t)\, dt$$

may of course be reduced to (1.11) by simple transformations. The type includes many well known functions, such as Bessel functions.

Some striking results as to the distribution of the zeros when $f(t)$ satisfies simple special conditions have been obtained by Pólya*. Here however, the problem is to determine properties which are common to all the functions $F(z)$, without further restriction on $f(t)$ than that of integrability. The results obtained are much more precise than is the case for the general integral function of exponential type, *i.e.* such that

$$|F(z)| < Ae^{K|z|}.$$

They are more analogous to those obtained by Pólya† for functions of the form

$$P_1(z)\, e^{\alpha_1 z} + P_2(z)\, e^{\alpha_2 z} + \ldots + P_m(z)\, e^{\alpha_m z} \tag{1.12}$$

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* G. Pólya, *Math. Zeitschrift*, 2 (1918), 352–383.

† G. Pólya, *Bayer. Akademie der Wissenschaften* (1920), 285–290.

where $P_1(z)\ldots$ are polynomials. The function (1.11) may be regarded as a limiting case of (1.12) when the polynomials reduce to constants, the exponents $a_1\ldots$ are all real, and $m \to \infty$. However, the properties of (1.11) are rather different from those of (1.12), and they are in fact obtained quite independently.

1.2. It is supposed that a and b are the effective lower and upper limits of the integral; *i.e.* there is no number $a > a$ such that

$$\int_a^a |f(t)|\,dt = 0,$$

and no number $\beta < b$ such that

$$\int_\beta^b |f(t)|\,dt = 0.$$

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, ... be the zeros of $F(z)$, arranged so that r_n is a non-decreasing function of n. It is supposed throughout that $F(0)$ is not zero. We may do this without loss of generality; for if $F(0) = 0$, then

$$F(z) = -z\int_a^b e^{zt} g(t)\,dt \quad \left(g(t) = \int_a^t f(u)\,du\right),$$

so that on dividing by z we obtain a function of the same type, with a zero of smaller order. So the zero may be removed altogether.

The first theorem is one already obtained by Pólya* :—

THEOREM I.—*The function $F(z)$ has an infinity of zeros such that the series*

$$\sum_1^\infty \frac{1}{r_n} \tag{1.21}$$

is divergent.

The proof given here is simpler than the original one.

THEOREM II.—*The series*

$$\sum_1^\infty \frac{\cos\theta_n}{r_n} \tag{1.22}$$

is absolutely convergent.

It follows from these two theorems that, in general, $\cos\theta_n$ is small when n is large; and it might be supposed that $\cos\theta_n \to 0$, or that any

* G. Pólya, *Math. Zeitschrift*, 2 (1918), 380.

angular region including the imaginary axis would contain all zeros of large modulus. This would be the case if we could replace $F(z)$ by a function of the form (1.12). Actually it is not necessarily so, as is shown by

THEOREM III.—*There is a function of the given class with an infinity of real zeros.*

The principal result of the paper is

THEOREM IV.—*If $n(r)$ is the number of zeros of $F(z)$ for which $|z| \leqslant r$, then*

$$n(r) \sim \frac{b-a}{\pi} r.$$

This, of course, includes Theorem I.

THEOREM V.—*The series*

$$\sum_1^\infty \frac{\sin \theta_n}{r_n} \tag{1.23}$$

is conditionally convergent.

It follows from Theorems I and II that it is not absolutely convergent.

THEOREM VI.—*We have*

$$F(z) = F(0)\, e^{\frac{1}{2}(a+b)z} \prod_{n=1}^\infty \left(1 - \frac{z}{z_n}\right), \tag{1.24}$$

the product being conditionally convergent.

Incidentally we obtain the sums of the series (1.22) and (1.23). We have

$$\sum_1^\infty \frac{\cos \theta_n}{r_n} = \tfrac{1}{2}(a+b) - R\left\{\frac{F'(0)}{F(0)}\right\}, \tag{1.25}$$

$$\sum_1^\infty \frac{\sin \theta_n}{r_n} = I\left\{\frac{F'(0)}{F(0)}\right\}. \tag{1.26}$$

In the course of the proofs we obtain a good deal of information about the modes of variation of $|F(z)|$, which sometimes goes further than is actually required for application to the problem of the zeros. It is proved, for instance, that, if $M(r) = \text{Max } |F(z)| \, (|z| = r)$, then

$$\log M(r) \sim r \text{ Max}(|a|, |b|). \tag{1.27}$$

The main theorems may be extended so as to apply to the zeros of $F(z)-k$, where k is any constant, real or complex. The proofs only require slight modifications.

As an application of Theorem IV we have

THEOREM VII.—*If $\phi(t)$ and $\psi(t)$ are integrable functions, such that*

$$\int_0^x \phi(t)\,\psi(x-t)\,dt = 0$$

almost everywhere in the interval $0<x<\kappa$, then $\phi(t)=0$ almost everywhere in $(0,\lambda)$, and $\psi(t)=0$ almost everywhere in $(0,\mu)$, where $\lambda+\mu \geqslant \kappa$.

1.3. The proofs are based on the following known theorems :—

THEOREM OF PHRAGMÉN AND LINDELÖF*.—*If $\phi(z)$ is regular and of order ρ in an angle less than π/ρ, i.e. if for every positive ϵ*

$$|\phi(re^{i\theta})| < e^{r^{\rho+\epsilon}} \quad (\alpha \leqslant \theta \leqslant \beta,\ \beta-\alpha<\pi/\rho,\ r>r_0),$$

and $\phi(z)$ is bounded for $\theta=\alpha$, $\theta=\beta$, then it is bounded for $\alpha\leqslant\theta\leqslant\beta$, and if $\phi(z)\to 0$ as $r\to\infty$ for $\theta=\alpha$, $\theta=\beta$, then it tends uniformly to zero for $\alpha\leqslant\theta\leqslant\beta$.

THEOREM OF JENSEN.—*If $\phi(z)$ is an integral function $\big(\phi(0)\neq 0\big)$, and $n(r)$ is the number of its zeros for $|z|\leqslant r$, then*

$$\int_0^r \frac{n(x)}{x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|\phi(re^{i\theta})|\,d\theta - \log|\phi(0)|.$$

THEOREM OF CARLEMAN†.—*If $\phi(z)$ is regular for $R(z)\geqslant 0$, and $r_1e^{i\theta_1}$, $r_2e^{i\theta_2}$, ... are its zeros in the right half-plane, then*

$$\sum_{l<r_n<R}\left(\frac{1}{r_n}-\frac{r_n}{R^2}\right)\cos\theta_n = \frac{1}{\pi R}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\log|\phi(Re^{i\theta})|\cos\theta\,d\theta$$

$$+\frac{1}{2\pi}\int_l^R\left(\frac{1}{r^2}-\frac{1}{R^2}\right)\{\log|\phi(ir)|+\log|\phi(-ir)|\}\,dr+\psi(R)$$

where $\psi(R)$ is bounded as $R\to\infty$.

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* E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 381–406.

† T. Carleman, *Arkiv för Mat. Astr. o. Fys.*, 17 (1922), Memoir 9.

THEOREM OF VALIRON*.—*If $\phi(z)$ is an integral function, and $M(r) = \text{Max}\,|\phi(z)|\,(|z|=r)$, then, given $k>1$, there is a number $H=H(k)$ and a sequence of circles $|z|=R_n\,(R_n<kR_{n-1})$ on each of which*

$$|\phi(z)|>\{M(r)\}^{-H}.$$

1.4. It will be supposed throughout the proof that $a=-1$, $b=1$. The general case can be reduced to this. For

$$\int_a^b e^{zt}f(t)\,dt = \tfrac{1}{2}(b-a)\,e^{\frac{1}{2}(a+b)z}\int_{-1}^{1} e^{\xi t}g(t)\,dt$$

where $\xi = \tfrac{1}{2}(b-a)z, \quad g(t) = f\{\tfrac{1}{2}(b+a)+\tfrac{1}{2}(b-a)t\}$.

2. *Preliminary Lemmas.*

2.1. LEMMA 2.1.—*If $F(z)=0$ for all values of z, then $f(t)=0$ for almost all values of t.*

We have

$$F(iy)+F(-iy) = 2\int_0^1 \cos yt\,\{f(t)+f(-t)\}\,dt,$$

and hence

$$\int_0^\lambda \frac{\sin\xi y}{y}\,\{F(iy)+F(-iy)\}\,dy = 2\int_0^\lambda \frac{\sin\xi y}{y}\,dy\int_0^1 \cos yt\,\{f(t)+f(-t)\}\,dt$$

$$= 2\int_0^1 \{f(t)+f(-t)\}\,dt\int_0^\lambda \frac{\sin\xi y\cos yt}{y}\,dy.$$

Now if ξ and t are real

$$\left|\int_0^\lambda \frac{\sin\xi y\cos yt}{y}\,dy\right| \leqslant \int_0^\pi \frac{\sin u}{u}\,du.$$

Hence by Lebesgue's convergence theorem

$$\lim_{\lambda\to\infty}\int_0^1 \{f(t)+f(-t)\}\,dt\int_0^\lambda \frac{\sin\xi y\cos yt}{y}\,dy$$

$$= \int_0^1 \{f(t)+f(-t)\}\,dt\int_0^\infty \frac{\sin\xi y\cos yt}{y}\,dy$$

$$= \tfrac{1}{2}\pi\int_0^\xi \{f(t)+f(-t)\}\,dt,$$

* G. Valiron, *Theory of Integral Functions*, Toulouse (1923), 80. The theorem is stated by Valiron for functions of finite order, but his proof applies to any integral function, or indeed to any function which is regular in an appropriate region.

if $0<\xi<1$. Hence

$$(2.11)\qquad \int_0^{\xi}\{f(t)+f(-t)\}\,dt=\frac{1}{\pi}\int_0^{\infty}\frac{\sin\xi y}{y}\{F(iy)+F(-iy)\}\,dy.$$

Similarly

$$(2.12)\qquad \int_0^{\xi}\{f(t)-f(-t)\}\,dt=\frac{1}{\pi}\int_0^{\infty}\frac{1-\cos\xi y}{y}\{F(iy)-F(-iy)\}\,dy.$$

It follows that, if $F(z)=0$, then

$$\int_0^{\xi}f(t)\,dt=0,\qquad \int_0^{\xi}f(-t)\,dt=0\quad(0<\xi<1),$$

and the result stated follows.

2.2. LEMMA 2.2.—*As* $r\to\infty$,

$$|F(re^{i\theta})|=o(e^{r|\cos\theta|})$$

uniformly with respect to θ.

If x is real and positive

$$|F(x)|=\left|\int_{-1}^{1-\delta}e^{xt}f(t)\,dt+\int_{1-\delta}^{1}e^{xt}f(t)\,dt\right|$$

$$\leqslant e^{(1-\delta)x}\int_{-1}^{1-\delta}|f(t)|\,dt+e^{x}\int_{1-\delta}^{1}|f(t)|\,dt.$$

Now, having given ϵ, we can choose δ so small that

$$\int_{1-\delta}^{1}|f(t)|\,dt<\tfrac{1}{2}\epsilon,$$

and then x_0 so large that

$$e^{-\delta x}\int_{-1}^{1-\delta}|f(t)|\,dt<\tfrac{1}{2}\epsilon\quad(x>x_0).$$

Then $|F(x)|<\epsilon e^{x}$ $(x>x_0)$, *i.e.* $F(x)=o(e^{x})$. Again

$$F(iy)=\int_{-1}^{1}\cos yt\,f(t)\,dt+i\int_{-1}^{1}\sin yt\,f(t)\,dt,$$

which tends to zero as $y\to\infty$, by the "Riemann-Lebesgue lemma."

Hence, by the theorem of Phragmén and Lindelöf, the function

$$G(z) = e^{-z}F(z)$$

tends uniformly to zero for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. This gives the desired result in this range, and the remaining values of θ may be dealt with similarly.

2.3. LEMMA 2.3.—*When* $r \to \infty$.

$$|F(re^{i\theta})| \neq O\{e^{r(|\cos\theta|-\delta)}\} \quad (\delta > 0) \tag{2.31}$$

for any value of θ.

Suppose that, for a value α of θ $(-\frac{1}{2}\pi < \alpha < \frac{1}{2}\pi)$ and a positive value of δ,

$$|F(re^{i\alpha})| = O\{e^{r(\cos\alpha-\delta)}\}. \tag{2.32}$$

Let $$G(z) = e^{-z(1-\delta\sec\alpha)}\int_{1-\delta\sec\alpha}^{1} e^{zt}f(t)\,dt,$$

when we suppose, as obviously we may, that $\delta < 2\cos\alpha$. Then it follows from Lemma 2.2 that $G(z)$ is bounded for $R(z) \leqslant 0$. But also, for $\theta = \alpha$,

$$\begin{aligned}\int_{1-\delta\sec\alpha}^{1} e^{zt}f(t)\,dt &= F(z) - \int_{-1}^{1-\delta\sec\alpha} e^{zt}f(t)\,dt \\ &= O\{e^{r(\cos\alpha-\delta)}\} + O\{e^{r\cos\alpha(1-\delta\sec\alpha)}\} \\ &= O\{e^{r(\cos\alpha-\delta)}\},\end{aligned}$$

so that $G(z)$ is bounded for $\theta = \alpha$. Hence, by the theorem of Phragmén and Lindelöf, $G(z)$ is bounded everywhere, and so reduces to a constant, which is zero, since $G(-x) \to 0$. Hence, by Lemma 2.1, $f(t) = 0$ almost everywhere in the interval $(1-\delta\sec\alpha < t < 1)$, contrary to hypothesis.

The theorem may be proved in a similar way if $\frac{1}{2}\pi < \alpha < \frac{3}{2}\pi$. The proof fails if $\alpha = \pm\frac{1}{2}\pi$. But if

$$F(iy) = O(e^{-\delta y}), \tag{2.33}$$

then the function $$e^{-(1+i\delta)z}F(z)$$

is bounded for $z = x$, $z = iy$, and so for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. Hence

$$|F(re^{i\theta})| = O\{e^{r(\cos\theta - \delta\sin\theta)}\} \quad (0 < \theta < \tfrac{1}{2}\pi),$$

which contradicts what has been proved already.

3 . 1. *Proof of Theorem I.*—If the series (1 . 21) were convergent, we should have

$$F(z) = F(0)\, e^{(\alpha+i\beta)z} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) = e^{(\alpha+i\beta)z} G(z),$$

say, where, for every positive ϵ,

$$|G(z)| = O(e^{\epsilon|z|}).$$

But by Lemma 2 . 3

$$|G(x)| = e^{-\alpha x}|F(x)| \neq O\{e^{(1-\alpha-\delta)x}\},$$

$$|G(-x)| = e^{\alpha x}|F(-x)| \neq O\{e^{(1+\alpha-\delta)x}\}.$$

Hence $$1-\alpha-\delta \leqslant \epsilon, \quad 1+\alpha-\delta \leqslant \epsilon,$$

for all positive values of δ and ϵ; and this is impossible, whatever α is.

3 . 2. *Proof of Theorem II.*—If a function $\phi(z)$ is regular and bounded for $R(z) \geqslant 0$, and has zeros at $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2}$, ..., in the right half-plane, then

$$\Sigma \frac{\cos\theta_n}{r_n}$$

is convergent. A simple direct proof of this is given by Pólya and Szegö*. It also follows from the formula of Carleman. For we have

$$\sum_{1 < r_n < R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n < K,$$

and *a fortiori*

$$\sum_{1 < r_n < \frac{1}{2}R} \frac{\cos\theta_n}{r_n} < \tfrac{4}{3} \sum_{1 < r_n < \frac{1}{2}R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \cos\theta_n < \tfrac{4}{3}K,$$

which gives the result.

Now, by Lemma 2 . 2, $e^{-z}F(z)$ is bounded for $R(z) \geqslant 0$, and $e^{z}F(z)$ is bounded for $R(z) \leqslant 0$. So Theorem II follows.

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* G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis I*, Berlin (1924), 142, problem 298.

3.3. *Proof of Theorem III.*—Let $0<\theta<1$, $0<\delta<\theta(1-\theta)$, and

$$f(t)=(-1)^n\mu_n \quad (\theta^n-\delta^n\leqslant t\leqslant\theta^n),$$

for $n=1, 2, \ldots$, and $f(t)=0$ elsewhere. The numbers μ_n are supposed positive and such that the series

$$\Sigma\mu_n\delta^n$$

is convergent, so that $f(t)$ is integrable. Then

$$F(-x)=\sum_{\nu=1}^{\infty}(-1)^\nu\mu_\nu\int_{\theta^\nu-\delta^\nu}^{\theta^\nu}e^{-xt}\,dt.$$

Now
$$\left|\sum_{\nu=1}^{n-1}(-1)^\nu\mu_\nu\int_{\theta^\nu-\delta^\nu}^{\theta^\nu}e^{-xt}\,dt\right|<e^{-x(\theta^{n-1}-\delta^{n-1})}\sum_{\nu=1}^{n-1}\mu_\nu\delta^\nu,$$

$$\left|\sum_{\nu=n+1}^{\infty}(-1)^\nu\mu_\nu\int_{\theta^\nu-\delta^\nu}^{\theta^\nu}e^{-xt}\,dt\right|<\sum_{\nu=n+1}^{\infty}\mu_\nu\delta^\nu,$$

$$\int_{\theta^n-\delta^n}^{\theta^n}e^{-xt}\,dt>\delta^n e^{-x\theta^n}.$$

Hence
$$(-1)^nF(x)>\mu_n\delta^n e^{-x\theta^n}-e^{-x(\theta^{n-1}-\delta^{n-1})}\sum_{\nu=1}^{n-1}\mu_\nu\delta^\nu-\sum_{\nu=n+1}^{\infty}\mu_\nu\delta^\nu.$$

Taking $x=a^n$, $\mu_n=\delta^{-n}e^{-c^n}$ $(c>1)$, we have

$$(-1)^nF(-a^n)>e^{-c^n-a^n\theta^n}-e^{-a^n(\theta^{n-1}-\delta^{n-1})}\sum_{\nu=1}^{n-1}e^{-c^\nu}-\sum_{\nu=n+1}^{\infty}e^{-c^\nu}$$

$$>e^{-c^n-a^n\theta^n}-Ae^{-a^n\theta^{n-1}}-Ae^{-c^{n+1}}$$

provided that $\delta<1/a$. Let $c=a\theta$. Then

$$(-1)^nF(-a^n)>e^{-2a^n\theta^n}-Ae^{-a^n\theta^{n-1}}-Ae^{-a^{n+1}\theta^{n+1}}$$

$$=e^{-2a^n\theta^n}\{1-Ae^{a^n\theta^n(2-1/\theta)}-Ae^{a^n\theta^n(2-a\theta)}\},$$

which is positive for sufficiently large values of n if $\theta<\frac{1}{2}$, $a>2/\theta$. So with these conditions the function $F(z)$ has a real zero in every interval $(-a^{n+1}, -a^n)$ for $n>n_1$.

4. *Further Lemmas.*

4.1. LEMMA 4.1.—*Let $n(r)$ denote the number of zeros of $F(z)$ for which $|z| \leqslant r$, and let*

$$N(r) = \int_0^r \frac{n(x)}{x}\, dx.$$

Then $$N(r) < 2r/\pi \quad (r > r_0).$$

Jensen's formula is

$$N(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|F(re^{i\theta})|\, d\theta - \log|F(0)|.$$

By Lemma 2.2 $$|F(re^{i\theta})| < \delta e^{r|\cos\theta|} \quad (r > r_0).$$

Hence $$\log|F(re^{i\theta})| < \log\delta + r|\cos\theta|,$$

and
$$N(r) < \log\delta + \frac{r}{2\pi}\int_{-\pi}^{\pi} |\cos\theta|\, d\theta - \log|F(0)|$$

$$= 2r/\pi - \log\{|F(0)|/\delta\},$$

so that the result holds if $\delta < |F(0)|$.

Corollary.—$n(r) < Kr$.

For $$n(r)\log k \leqslant \int_r^{kr} \frac{n(x)}{x}\, dx \leqslant N(kr) < Kr.$$

4.2. LEMMA 4.2.—*Let Π_δ denote a product taken over terms involving zeros for which $|\cos\theta_n| < \delta$. Then, given $\delta > 0$, $\epsilon > 0$, we have*

$$\Pi_\delta \left|1 - \frac{x}{r_n}\, e^{-i\theta_n}\right|^2 > e^{-(A\delta^2+\epsilon)x}\,\Pi_\delta\left(1+\frac{x^2}{r_n^2}\right)$$

for $x > x_0(\delta, \epsilon)$, where A is an absolute constant.*

We have

$$1 - 2\frac{x}{r_n}\cos\theta_n + \frac{x^2}{r_n^2} \geqslant \left(1 - 2\frac{x}{r_n}|\cos\theta_n|\right)\left(1+\frac{x^2}{r_n^2}\right)$$

$$> e^{-4(x/r_n)|\cos\theta_n|}\left(1+\frac{x^2}{r_n^2}\right),$$

* The following work would be much simpler if all the zeros were purely imaginary—for instance, this lemma would be trivial. We use the fact (expressed by Theorem II) that $\cos\theta_n$, if not always zero, is at any rate in general small.

if $x|\cos\theta_n|/r_n < \frac{1}{4}\log 2$, which is certainly so if $r_n > 8x\delta$. Let

$$\Pi_\delta = \prod_{r_n \leqslant 8x\delta} \cdot \prod_{r_n > 8x\delta} = \Pi_1 . \Pi_2.$$

Then

$$\Pi_2\left(1-2\frac{x}{r_n}\cos\theta_n+\frac{x^2}{r_n^2}\right) > \exp\left(-4x\sum_{r_n>8x\delta}\frac{\cos\theta_n}{r_n}\right)\Pi_2\left(1+\frac{x^2}{r_n^2}\right)$$

$$> e^{-\epsilon x}\Pi_2\left(1+\frac{x^2}{r_n^2}\right)$$

for $x > x_0(\epsilon, \delta)$; and

$$\Pi_1\left(1-2\frac{x}{r_n}\cos\theta_n+\frac{x^2}{r_n^2}\right)\Big/\Pi_1\left(1+\frac{x^2}{r_n^2}\right) > \Pi_1\left\{\frac{(x-r_n)^2}{x^2+r_n^2}\right\}$$

$$> \Pi_1\left\{\frac{(x-8x\delta)^2}{x^2+8^2x^2\delta^2}\right\}$$

$$= \left\{\frac{(1-8\delta)^2}{1+8^2\delta^2}\right\}^{n(8x\delta)}$$

$$> \left\{\frac{(1-8\delta)^2}{1+8^2\delta^2}\right\}^{Kx\delta}$$

$$= e^{Kx\delta\log[(1-8\delta)^2/(1+8^2\delta^2)]}$$

$$> e^{-Ax\delta^2}.$$

Combining these two inequalities, we have the result stated.

4.3. LEMMA 4.3.—*Let Π'_δ denote a product taken over terms involving zeros for which $|\cos\theta_n| \geqslant \delta$. Let*

$$\phi(z) = \Pi'_\delta\left(1-\frac{z}{z_n}\right), \quad G(z) = \frac{F(z)}{\phi(z)}.$$

Then as $x \to \infty$,

$$G(x) = O\{e^{(1+\epsilon)x}\}, \quad \neq O\{e^{(1-\epsilon)x}\},$$

for every positive ϵ.

The series $\Sigma'_\delta 1/|z_n|$ is convergent, since no term is greater than the corresponding term of $\Sigma'_\delta|\cos\theta_n|/\delta r_n$. Hence the product for $\phi(z)$ is absolutely convergent, and

$$|\phi(z)| = O(e^{\epsilon|z|})$$

for every positive ϵ. Hence, by the Theorem of Valiron, there is a sequence of circles $|z| = R_1, R_2, \ldots$ $(R_n < kR_{n-1})$ on each of which

$$|\phi(z)| > e^{-\epsilon|z|}.$$

It follows from this and Lemma 2.2 that

$$|G(z)| < Ke^{(1+\epsilon)R_n}$$

for $|z| = R_n$, and so, since $G(z)$ is an integral function, for $|z| \leqslant R_n$. In particular, if $R_{n-1} < x \leqslant R_n$, then

$$|G(x)| < Ke^{(1+\epsilon)R_n} < Ke^{(1+\epsilon)kx}.$$

and the first result follows, since ϵ and $k-1$ may be as small as we please.

The second part follows at once from Lemma 2.3, since

$$F(x) \neq O\{e^{(1-\epsilon)x}\}, \quad \phi(x) = O(e^{\epsilon x}).$$

Similar results, of course, hold for $G(-x)$.

4.4. LEMMA 4.4.—*If x is real*

$$|F(x)| = |F(0)| \prod_1^\infty \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right|.$$

We have

$$F(z) = F(0)\, e^{(\alpha+i\beta)z} \prod_1^\infty \left(1 - \frac{z}{z_n}\right) e^{z/z_n}, \tag{4.41}$$

so that if x is real

$$|F(x)| = |F(0)|\, e^{\alpha x} \prod_1^\infty \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right| e^{(x\cos\theta_n)/r_n}.$$

But it follows from Theorem II that the product

$$\Pi e^{(x\cos\theta_n)/r_n}$$

is convergent. Hence

$$|F(x)| = |F(0)|\, e^{\gamma x} \prod_1^\infty \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right|,$$

where

$$\gamma = \alpha + \sum_1^\infty \frac{\cos\theta_n}{r_n}. \tag{4.42}$$

We have to prove that $\gamma = 0$. Define $G(z)$, as in Lemma 4.3. Then, by Lemma 4.2,

$$|G(x)|^2 = |F(0)|^2 e^{2\gamma x} \Pi_\delta \left|1 - \frac{x}{r_n} e^{-i\theta_n}\right|^2$$

$$> K e^{(2\gamma - A\delta^3 - \epsilon)x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right). \tag{4.43}$$

Also

$$|G(-x)|^2 = |F(0)|^2 e^{-2\gamma x} \Pi_\delta \left|1 + \frac{x}{r_n} e^{-i\theta_n}\right|^2$$

$$< K e^{-2\gamma x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right) \Pi_\delta \left(1 + 2\frac{x}{r_n} |\cos\theta_n|\right)$$

$$< K e^{(\epsilon - 2\gamma)x} \Pi_\delta \left(1 + \frac{x^2}{r_n^2}\right). \tag{4.44}$$

Hence

$$\left|\frac{G(x)}{G(-x)}\right|^2 > K e^{(4\gamma - A\delta^2 - 2\epsilon)x} \quad [x > x_0(\epsilon, \delta)]. \tag{4.45}$$

Now it follows from Lemma 4.3 that there is a sequence of values of x, tending to infinity, in which

$$|G(x)|^2 < e^{\epsilon x} |G(-x)|^2. \tag{4.46}$$

Comparing (4.44), (4.45), we have

$$4\gamma - A\delta^2 - 2\epsilon \leqslant \epsilon$$

for all positive values of δ and ϵ. Hence $\gamma \leqslant 0$. Applying the same argument to $F(-z)$, we obtain $\gamma \geqslant 0$. Hence $\gamma = 0$.

This proves the lemma. The relation (1.25) also follows. For, equating the coefficients of z on the two sides of (4.41), we obtain

$$\alpha + i\beta = F'(0)/F(0), \tag{1.47}$$

and (1.25), in the case $a = -1$, $b = 1$, follows from (4.42) and (4.47).

4.5. LEMMA 4.5.—*Given $k > 1$, $\epsilon > 0$, there is a sequence of numbers $x_1, x_2, \ldots$ $(x_n < k x_{n-1})$ such that for every n*

$$|F(x_n)| > e^{(1-\epsilon)x_n}.$$

Let $-1<\xi<1$, and let

$$h(t)=1 \quad (-1<t<\xi), \qquad =f(t) \quad (\xi<t<1).$$

Let
$$H(z)=\int_{-1}^{1} e^{zt}\,h(t)\,dt.$$

Let $\zeta_1, \zeta_2, \ldots$ be the zeros of $H(z)$, and let

$$\psi(z)=\Pi_\delta'\left(1-\frac{z}{\zeta_n}\right), \quad K(z)=\frac{H(z)}{\psi(z)}.$$

Applying (4 . 45) to $K(z)$, we have, since $\gamma=0$,

$$|K(x)/K(-x)|^2>Ae^{(-A\delta^2-2\epsilon)x} \quad [x>x_0(\epsilon,\delta)],$$

or
$$|H(x)|^2>Ae^{(-A\delta^2-2\epsilon)x}|H(-x)\,\psi(x)/\psi(-x)|^2.$$

Now, as in the case of $\phi(z)$,

$$|\psi(-x)|=O(e^{\epsilon x})$$

and so there is a sequence of numbers $x_1, x_2, \ldots$ $(x_n<kx_{n-1})$ in which

$$|\psi(x)|>e^{-\epsilon x}.$$

Hence for this sequence

$$|H(x)|>Ae^{(-A\delta^2-2\epsilon)x}|H(-x)|.$$

But
$$H(x)=F(x)+O(e^{\xi x}),$$

$$H(-x)=e^x/x+O(e^{-\xi x}).$$

Hence
$$|F(x)|>Ae^{(1-A\delta^2-2\epsilon)x}/x-O(e^{\xi x})-O(e^{-\xi x})$$

for the given sequence. This proves the lemma.

4 . 6.—LEMMA 4 . 6.—*For every positive* ϵ

$$\prod_1^\infty\left(1+\frac{x^2}{r_n^2}\right)>e^{(2-\epsilon)x} \quad [x>x_0(\epsilon)].$$

Applying the previous lemma to $F(-x)$, we obtain a sequence of values x_m of x for which

$$|F(-x_m)|>e^{(1-\epsilon)x_m}.$$

Since $\phi(-x_m)=O(e^{\epsilon x_m})$, it follows that

$$|G(-x_m)|>e^{(1-2\epsilon)x_m}.$$

Hence, by (4 . 44), $$\Pi_\delta\left(1+\frac{x_m^2}{r_n^2}\right) > e^{(2-5\epsilon)x_m} \quad [x > x_0(\epsilon)].$$

Hence, if $x_m \leqslant x < x_{m+1}$,

$$\prod_1^\infty\left(1+\frac{x^2}{r_n^2}\right) \geqslant \prod_1^\infty\left(1+\frac{x_m^2}{r_n^2}\right) \geqslant \Pi_\delta\left(1+\frac{x_m^2}{r_n^2}\right)$$

$$> e^{(2-5\epsilon)x_m} > e^{(2-5\epsilon)x/k},$$

and the result follows, since $k-1$ and ϵ may be as small as we please.

4 . 7. LEMMA 4 . 7.—$N(r) \sim 2r/\pi$.

In view of Lemma 4 . 1, we have only to prove that

(4 . 71) $$N(r) > (2/\pi-\epsilon)r \quad (r > r_0).$$

By Lemma 4 . 6

(4 . 72) $$\sum_1^\infty \log\left(1+\frac{x^2}{r_n^2}\right) > (2-\epsilon)x \quad (x > x_0).$$

Now $$\sum_1^\infty \log\left(1+\frac{x^2}{r_n^2}\right) = \sum_1^\infty n\left\{\log\left(1+\frac{x^2}{r_n^2}\right)-\log\left(1+\frac{x^2}{r_{n+1}^2}\right)\right\}$$

$$= \sum_1^\infty n\int_{r_n}^{r_{n+1}} \frac{2x^2\,dr}{r(x^2+r^2)}$$

$$= 2x^2\int_0^\infty \frac{n(r)\,dr}{r(x^2+r^2)}$$

(4 . 73) $$= 4x^2\int_0^\infty \frac{rN(r)}{(x^2+r^2)^2}\,dr,$$

the last form being obtained by a partial integration. If the lemma is not true, there is a number λ less than unity such that

$$N(r) < 2\lambda r/\pi$$

for indefinitely great values of r, say $r = R_1, R_2, \ldots$. Since $N(r)$ is a non-decreasing function of r, we have

$$\int_{\lambda R_n}^{R_n} \frac{rN(r)}{(x^2+r^2)^2}\,dr < \frac{2\lambda R_n}{\pi}\int_{\lambda R_n}^{R_n} \frac{r\,dr}{(x^2+r^2)^2},$$

From this and Lemma 4 . 1 it follows that

$$\int_0^\infty \frac{rN(r)}{(x^2+r^2)^2}\,dr < \frac{2}{\pi}\int_0^\infty \frac{r^2\,dr}{(x^2+r^2)^2} - \frac{2}{\pi}\int_{\lambda R_n}^{R_n} \frac{(r-\lambda R_n)\,r\,dr}{(x^2+r^2)^2} + \frac{K}{x^4}$$

$$= \frac{1}{2x} - \frac{2}{\pi}\int_{\lambda R_n}^{R_n} \frac{(r-\lambda R_n)\,r\,dr}{(x^2+r^2)^2} + \frac{K}{x^4}.$$

Hence by (4 . 72), (4 . 73),

$$\frac{2}{\pi}\int_{\lambda R_n}^{R_n} \frac{(r-\lambda R_n)\,r\,dr}{(x^2+r^2)^2} < \frac{\epsilon}{4x} + \frac{K}{x^4} < \frac{\epsilon}{2x} \quad (x > x_1).$$

Taking $x = R_n$, this gives

$$\frac{2}{\pi R_n}\int_\lambda^1 \frac{(u-\lambda)\,u\,du}{(1+u^2)^2} < \frac{\epsilon}{2R_n},$$

which is impossible if ϵ is sufficiently small. This proves the lemma.

4 . 8. LEMMA 4 . 8*.—*If $N(r)/r$ tends to a limit, then $n(r)/r$ tends to the same limit.*

Suppose that $N(r)/r \to \kappa$. Then

$$(\kappa-\epsilon)\,r < N(r) < (\kappa+\epsilon)\,r \quad (r > r_0).$$

Hence

$$\int_r^{r(1+\delta)} \frac{n(x)}{x}\,dx = N\{r(1+\delta)\} - N(r)$$

$$< (\kappa+\epsilon)(1+\delta)\,r - (\kappa-\epsilon)\,r$$

$$= \kappa\delta r + \epsilon(2+\delta)\,r.$$

But

$$\int_r^{r(1+\delta)} \frac{n(x)}{x}\,dx \geqslant n(r)\int_r^{r(1+\delta)} \frac{dx}{x} > \frac{\delta n(r)}{1+\delta}.$$

Hence

$$n(r) < \kappa(1+\delta)\,r + \epsilon(1+\delta)(2+\delta)\,r/\delta.$$

Taking *e.g.* $\delta = \sqrt{\epsilon}$, it follows that

$$\overline{\lim}\, n(r)/r \leqslant \kappa.$$

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* This is a form of a well known theorem of Landau. See E. Landau, *Rend. di Palermo*, 26 (1908), 218.

Similarly $$n(r) > \kappa(1-\delta)\, r - \epsilon(1-\delta)(2-\delta)\, r/\delta,$$

so that $$\underline{\lim}\, n(r)/r \geqslant \kappa.$$

4 . 9. *Proof of Theorem IV and* (1 . 27).—Theorem IV follows at once from Lemmas 4 . 7 and 4 . 8. Also it is clear from Lemma 2 . 2 that

$$\log M(r) < r \quad (r > r_0). \tag{4 . 91}$$

Let $$\mu_1(r) = \operatorname*{Max}_{0 \leqslant x \leqslant r} |F(x)|, \quad \mu_2(r) = \operatorname*{Max}_{0 \leqslant x \leqslant r} |F(-x)|.$$

By Lemma 4 . 5 $$\mu_1(r) > e^{(1-\epsilon)r} \quad (r = x_1, x_2, \ldots),$$

where $x_n < k x_{n-1}$. Hence, if $x_n \leqslant r < x_{n+1}$,

$$\mu_1(r) \geqslant \mu_1(x_n) > e^{(1-\epsilon)x_n} > e^{(1-\epsilon)r/k},$$

so that

$$\log \mu_1(r) > (1-\epsilon)\, r \quad (r > r_0). \tag{4 . 92}$$

Similarly $$\log \mu_2(r) > (1-\epsilon)\, r \quad (r > r_0).$$

The general result (1 . 27) easily follows from these equations.

5. *Proof of Theorem V.*—Applying Carleman's formula to $F(iz)$, $F(-iz)$, and making $l \to 0$, we have*

$$\sum_{r_n < R} \left(\frac{1}{r_n} - \frac{r_n}{R^2}\right) \sin\theta_n = \frac{1}{\pi R} \int_{-\pi}^{\pi} \log |F(Re^{i\theta})| \sin\theta\, d\theta + I\left\{\frac{F'(0)}{F(0)}\right\}. \tag{5 . 1}$$

Now it follows from Jensen's formula and Lemma 4 . 7 that

$$\lim_{R \to \infty} \frac{1}{R} \int_{-\pi}^{\pi} \log |F(Re^{i\theta})|\, d\theta = 4,$$

or $$\lim_{R \to \infty} \int_{-\pi}^{\pi} \left\{ |\cos\theta| - \frac{1}{R} \log |F(Re^{i\theta})| \right\} d\theta = 0.$$

But the integrand is positive if R is sufficiently large, by Lemma 2 . 2.

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* See Pólya and Szegö, *loc. cit.*, 1, 310 (solution 240).

Hence we have also

$$\lim_{R\to\infty}\int_{-\pi}^{\pi}\left\{|\cos\theta|-\frac{1}{R}\log|F(Re^{i\theta})|\right\}\sin\theta\,d\theta=0,$$

or

$$\lim_{R\to\infty}\frac{1}{R}\int_{-\pi}^{\pi}\log|F(Re^{i\theta})|\sin\theta\,d\theta=0. \tag{5.2}$$

Hence*

$$\lim_{R\to\infty}\sum_{r_n<R}\left(\frac{1}{r_n}-\frac{r_n}{R^2}\right)\sin\theta_n=I\left\{\frac{F'(0)}{F(0)}\right\}. \tag{5.3}$$

Put $R=\frac{1}{2}\pi m$, where m is an integer. Then

$$\lim_{m\to\infty}\sum_{n<m}\frac{\sin\theta_n}{r_n}\left(1-\frac{4r_n^2}{\pi^2m^2}\right)=I\left(\frac{F'(0)}{F(0)}\right), \tag{5.31}$$

for the number of terms which occur in one of these sums but not in the other is

$$|m-n(\tfrac{1}{2}\pi m)|=o(m)$$

and the modulus of each of these terms is $O(1/m)$. Let

$$\sigma_m=\sum_{n<m}\frac{\sin\theta_n}{r_n}\left(1-\frac{4r_n^2}{\pi^2m^2}\right).$$

Then

$$(m+1)^2\sigma_{m+1}-m^2\sigma_m=\sum_{n<m}\frac{\sin\theta_n}{r_n}(2m+1)+\frac{\sin\theta_m}{r_m}\left\{(m+1)^2-\frac{4r_m^2}{\pi^2}\right\},$$

so that

$$\sum_{n<m}\frac{\sin\theta_n}{r_n}=\frac{(m+1)^2\sigma_{m+1}-m^2\sigma_m}{2m+1}+o(1).$$

Hence

$$\begin{aligned}\frac{1}{p}\sum_{m=1}^{p}\sum_{n<m}\frac{\sin\theta_n}{r_n}&=\frac{1}{p}\sum_{m=1}^{p}\frac{(m+1)^2\sigma_{m+1}-m^2\sigma_m}{2m+1}+o(1)\\&=\frac{1}{p}\sum_{m=1}^{p}\frac{2m^2}{4m^2-1}\sigma_m+\frac{(p+1)^2}{p(2p+1)}\sigma_{p+1}+o(1),\end{aligned}$$

which, when $p\to\infty$, tends to the same limit as σ_p. Hence the series (1.23) is summable $(C,1)$ to the sum given by (1.26); and since its

* This limit is a "Rieszian mean", which we reduce to an ordinary arithmetic mean.

n-th term is $O(1/n)$, it must actually be convergent. This proves Theorem V and (1 . 26).

The result is, of course, obvious if $f(t)$ is real, since then the zeros of $F(z)$ are conjugate complex numbers, and the terms of the series (1 . 23) cancel in pairs. It may be of interest to verify (1 . 26) in a less trivial case. Let

$$f(t) = 1 \ (-1 < t \leqslant 0), \quad i \ (0 < t < 1).$$

Then
$$F(z) = (e^z - 1)(e^{-z} + i)/z.$$

The zeros of $F(z)$ are at

$$\pm 2i\pi, \ \pm 4i\pi, \ \ldots \quad \text{and} \quad i\pi/2, \ -3i\pi/2, \ 5i\pi/2, \ -7i\pi/2, \ \ldots.$$

Hence
$$\sum_1^\infty \frac{\sin\theta_n}{r_n} = \frac{2}{\pi}\left(1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots\right) = \tfrac{1}{2}.$$

Also $F(0) = i+1$, $F'(0) = (i-1)/2$, so that $F'(0)/F(0) = \frac{1}{2}i$.

6. *Proof of Theorem VI.*—In view of Theorems II and V, we may write the Weierstrassian product (4 . 41) in the form

$$F(z) = F(0)\, e^{(\gamma + i\delta)z} \prod_1^\infty \left(1 - \frac{z}{z_n}\right),$$

where
$$\gamma = \alpha + \sum_1^\infty \frac{\cos\theta_n}{r_n}, \quad \delta = \beta - \sum_1^\infty \frac{\sin\theta_n}{r_n}.$$

We have already proved that $\gamma = 0$, and the fact that $\delta = 0$ follows from (1 . 26) and (4 . 47).

7. *Proof of Theorem VII.*—Let

$$f(t) = \phi(t) \ (0 < t < \kappa), \quad 1 \ (\kappa < t < \nu),$$

$$g(t) = \psi(t) \ (0 < t < \kappa), \quad 1 \ (\kappa < t < \nu),$$

$$F(z) = \int_0^\nu e^{zt} f(t)\, dt, \qquad G(z) = \int_0^\nu e^{zt} g(t)\, dt.$$

Then
$$F(z)\, G(z) = \int_0^\nu \int_0^\nu e^{z(s+t)} f(s)\, g(t)\, ds\, dt,$$

and putting $s = \frac{1}{2}(u+v)$, $t = \frac{1}{2}(u-v)$,

$$F(z)\,G(z) = \tfrac{1}{2}\iint e^{zu} f(\tfrac{1}{2}u+\tfrac{1}{2}v)\,g(\tfrac{1}{2}u-\tfrac{1}{2}v)\,du\,dv$$

$$= \int_0^{2\nu} e^{zu}\,h(u)\,du,$$

where

$$h(u) = \tfrac{1}{2}\int_{-u}^{u} f(\tfrac{1}{2}u+\tfrac{1}{2}v)\,g(\tfrac{1}{2}u-\tfrac{1}{2}v)\,dv \quad (0 < u < \nu)$$

$$= \tfrac{1}{2}\int_{u-2\nu}^{2\nu-u} f(\tfrac{1}{2}u+\tfrac{1}{2}v)\,g(\tfrac{1}{2}u-\tfrac{1}{2}v)\,dv \quad (\nu < u < 2\nu).$$

In particular, for $0 < u < \kappa$ (putting $v = 2t-u$),

$$h(u) = \int_0^u f(t)\,g(u-t)\,dt = \int_0^u \phi(t)\,\psi(u-t)\,dt,$$

which is zero for $0 < u < \kappa$, by hypothesis. So applying Theorem IV to $F(z)\ G(z)$, and denoting by $n(r)$ the number of its zeros for $|z| \leqslant r$, we obtain

$$n(r) \sim (2\nu-\rho)\,r/\pi \quad (\rho \geqslant \kappa).$$

Let $n_1(r)$, $n_2(r)$ refer to $F(z)$, $G(z)$, respectively, and suppose that $\phi(t) = 0$ $(0 < t < \lambda)$, $\psi(t) = 0$ $(0 < t < \mu)$. Then

$$n_1(r) \sim (\nu-\lambda)\,r/\pi, \quad n_2(r) \sim (\nu-\mu)\,r/\pi.$$

But

$$n_1(r)+n_2(r) = n(r).$$

Hence $\lambda+\mu = \rho$, and the result follows.

ON THE INVARIANT THEORY OF MIXED QUATERNARY FORMS

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A complete discussion of the projective invariant theory of a quaternary quadric involves the use of three types of variables x, p, u equivalent to point, line, and plane coordinates respectively in geometry. The introduction of line and plane coordinates suggests the need to deal with more complicated ground forms than those which merely involve point coordinates. For example, there is the form F homogeneous in each set of variables and of order l in x, m in p, and n in u. This might be called a *mixed form of orders* (l, m, n).

Such a form, when both x and u are absent $(l = n = 0)$, is a *complex of order* m. It has been customary in dealing with the invariant theory of complexes and of mixed forms when m exceeds zero to make use of a special symbolism, called the *complex symbols*, apparently due to Grassmann* (or Hamilton?) which differ essentially from the Clebsch-Aronhold or *ordinary* symbols in that their product obeys an alternating law and not the commutative law of actual numbers. Thus the number ϵ_{ik} is expressed as follows:—

(i) using complex symbols (ϵ),

$$\epsilon_{ik} = \epsilon_i \,.\, \epsilon_k = -\epsilon_k \,.\, \epsilon_i ;$$

(ii) using ordinary symbols (e),

$$\epsilon_{ik} = e_i\, e_k = e_k\, e_i .$$

* Waelsch, "Zur Invariantentheorie der Liniengeometrie", *Math. Annalen*, 37 (1890), 141–152 (141).

The purpose of the present communication is to show that the ordinary symbols may with advantage be used for both complex and mixed forms, and that the salient features of projective line geometry may readily be deduced from the ordinary symbolic theory of quaternary forms. Section I is devoted to the Fundamental Theorem, Section II to the linear complex, Section III to the quadratic complex and its bearings on differential geometry and relativity.

I. *The Fundamental Theorem for the general mixed forms.*

1. Every complex of the m-th order can be expressed in a *normal* form, as Clebsch* has shown, and thereby be further expressed symbolically as the m-th power of a symbolical special linear complex. The same argument applies to the mixed ground form.

In fact, let there be the three types of homogeneous variables, (1) the point coordinate type

$$x = \{x_1, x_2, x_3, x_4\} \tag{1}$$

with cogredient variables $y, z, x', y', z', \xi, \ldots$;

(2) the line coordinate type

$$p = \{p_{14}, p_{24}, p_{34}, p_{23}, p_{31}, p_{12}\}$$
$$= \left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix} \right\|$$

with cogredient variables $q, p', q', \ldots$, so that

$$p_{ik} = x_i y_k - x_k y_i; \tag{2}$$

(3) the plane coordinate type

$$u = \{u_1, u_2, u_3, u_4\}$$
$$= \left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right\|$$

with cogredient variables $v, w, u', v', w', \ldots$.

* *Math. Annalen*, 2 (1870), 1-8. Cf. Jessop, *Line Complex*, 321-323.

Let this last be written $u = (xyz)$,

or more specifically $$u_i = (xyz)^i \quad (i = 1, 2, 3, 4). \tag{3}$$

Then, if x, y, z, ξ are four non-coplanar points, we may denote their four planes u, v, w, t by

$$u = (xyz), \quad w = (z\xi x),$$

$$v = (x\xi y), \quad t = (y\xi z). \tag{4}$$

Correlatively we may write

$$x : y : z : \xi = (uvw) : (utv) : (wtu) : (vtw). \tag{5}$$

Furthermore, if the determinant, $\Delta, = |x_1 y_2 z_3 \xi_4| = (xyz\xi) \neq 0$ is utilized, then

$$(u_3 v_4 - u_4 v_3) = -\Delta(x_1 y_2 - x_2 y_1)$$

with five similar expressions, leading to axial coordinates

$$\pi = \{\pi_{14}, \pi_{24}, \pi_{34}, \pi_{23}, \pi_{31}, \pi_{12}\}$$

which are in the same ratio as the line coordinates and in the same cyclic order, so that (taking $\Delta = -1$ for convenience)

$$p_{ik} = \pi_{lm}$$

(i, k, l, $m = 1, 2, 3, 4$ in some order). Without ambiguity we write

$$p = (xy) = (uv) = \pi,$$

or, more specifically,

$$p_{ik} = (xy)_{ik} = (uv)_{lm} = \pi_{lm}. \tag{6}$$

These suffice for ordinary symbols for variables. The identical relation between line coordinates is then

$$\omega \equiv \tfrac{1}{2}(pp) \equiv p_{14} p_{23} + p_{24} p_{31} + p_{34} p_{12} = 0. \tag{7}$$

2. Let there be a linear complex

$$C = \Sigma C_{ik} p_{ik} = \Sigma C_{ik} \pi_{lm}. \tag{8}$$

We write the coefficients C_{ik} symbolically, following Clebsch, by the use

of two symbols c, c' cogredient with u, namely

$$C_{ik} = c_i c'_k - c_k c'_i = (cc')_{ik}, \tag{9}$$

or briefly $$C = (cc').$$

By using equivalent symbols A, B, C, ..., we denote the complex by

$$(Ap) = (Bp) = (Cp) = \ldots, \tag{10}$$

and its invariant $\Sigma C_{ik} C_{lm}$

by (AB) (11)

or $(aa'bb')$

the ordinary bracket factor of quaternary symbolic forms. It should be noticed that (AB) is commutative in its symbols, so that $(AB) = (BA)$.

3. Let ϕ_m denote a complex of order m (>1), then owing to the identical relation ω,

$$\psi_m \equiv \phi_m + \phi_{m-2} \cdot \omega$$

denotes the same complex, where ϕ_{m-2} is an arbitrary complex of order $(m-2)$. Clebsch proved that a unique complex ϕ_{m-2} exists which allows a given ϕ_m to be replaced by the *normal form* ψ_m whose symbolic form is

$$(Ap)^m = (Bp)^m = (Cp)^m = \ldots, \tag{12}$$

the m-th power of a symbolic special linear complex. In particular for the quadratic complex the normal form is

$$\psi_2 = (Ap)^2 = (Bp)^2 = (Cp)^2 = \ldots, \tag{13}$$

whose linear invariant $$(AA) \tag{14}$$

vanishes identically; and each symbol A, B, C may be replaced by two symbols aa', bb', cc', such that

$$A_{ik} = a_i a'_k - a_k a'_i,$$

or briefly $$A = (aa').$$

But these symbols a, a', b, b' are not entirely equivalent, only in pairs are they equivalent, a with b and simultaneously a' with b'.

4. More generally* it has been shown that the mixed form ϕ of orders (l, m, n) can be replaced by its normal form ψ, which in turn can be written symbolically as

$$\psi = (ax)^l (abp)^m (abcu)^n,$$

where a, b, c are ordinary symbols cogredient with u; further, that the *Fundamental Theorem* for the symbolic expression of concomitants applies. Namely

Every rational integral concomitant of the mixed form ψ whose orders are (l, m, n) can be expressed as an aggregate of terms composed entirely of symbolic factors of types

$$(d_1 d_2 d_3 d_4), \quad (d_1 d_2 d_3 u), \quad (d_1 d_2 p), \quad (d_1 x),$$

where each symbol d stands for a symbol of type a, b, c.

The number of symbols a which enter such a term is

$$l+m+n,$$

the number of symbols b is $m+n$, while the number of symbols c is n. Also, for a concomitant of degree two in the actual coefficients of the ground form ψ, a second set of symbols a', b', c' equivalent to a, b, respectively, is needed; and so on for higher degrees.

5. In this enunciation nothing is told us of the relative positions of the symbols a, b, c in the symbolic form of the concomitant. I propose to remove this difficulty by forming a *prepared system* of symbolic types with the following properties:—

(1) Every ground form of orders (l, m, n) shall be written symbolically as

$$F = (ax)^l (Ap)^m (au)^n, \tag{15}$$

where the symbols a, A, a are cogredient with u, p, x respectively, a property expressed by saying they are of *currency* one, two, three respectively.

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* Weitzenböck, *loc. cit.*, 158. E. Noether, *Crelle*, 139 (1911), 118–154 (143).

(2) Every concomitant of such ground forms shall be composed of terms each of which contains l a's, m A's, and n α's.

(3) Sets of symbols (b, B, β) equivalent to (a, A, α) shall, if necessary, be used.

This may be effected as follows. First, we replace the original symbols a, b, c of ψ by a, A, α where

$$\alpha = (abc), \quad A = (ab),$$

and next we resolve the factor (Ap) into complex symbols, here denoted for distinction by ϵ, π where

$$A_{ik} = \epsilon_i \epsilon_k = -\epsilon_k \epsilon_i, \quad \epsilon_i \epsilon_i = 0,$$

$$p_{ik} = \pi_i \pi_k = -\pi_k \pi_i, \quad \pi_i \pi_i = 0,$$

so that
$$(Ap) = \Sigma A_{ik} p_{ik} = \tfrac{1}{2}(\epsilon_1\pi_1 + \epsilon_2\pi_2 + \epsilon_3\pi_3 + \epsilon_4\pi_4)^2$$
$$= \tfrac{1}{2}(\epsilon\pi)^2, \tag{16}$$

and we replace $(Ap)^m$ by using m equivalent sets

$$\epsilon, \pi, \quad \epsilon', \pi', \quad \epsilon'', \pi'', \quad \ldots$$

as
$$(Ap)^m = 2^{-m} \prod_{i=0}^{m-1} (\epsilon^{(i)}\pi^{(i)})^2. \tag{17}$$

The (l, m, n) form ψ is now expressed by symbols

$$a, \quad \alpha, \quad \epsilon$$

as Weitzenböck has it, who proved* the fundamental theorem which now states, when adapted to quaternary forms, that *the necessary and sufficient types of symbolic factors for the expression of all concomitants of mixed ground forms are of three characteristic types*

(i) *type* $(a\beta)$ *of characteristic* (1, 3),

(ii) ,, $(abcd)$,, ,, (1, 1, 1, 1),

(iii) ,, $(a\beta\gamma\delta)$,, ,, (3, 3, 3, 3);

* *Wiener Berichte*, 122 (1913).

where the symbols involved are a, ϵ, u, α, π, x and their equivalents; and the characteristic denotes the currencies of the symbols involved, arranged in the same relative order as the symbols themselves.

The first type (i) may also be written

$$-(\beta a) \quad \text{or} \quad \alpha_\beta.$$

6. Suppose now that each complex symbol ϵ or π is eliminated in turn from the concomitant

$$P = \psi(\epsilon, \pi, a, \alpha, u, x).$$

Since such a symbol ϵ occurs twice, P may be of type either

$$\epsilon_\rho \epsilon_\sigma M,$$

or
$$(\epsilon bcd)\,\epsilon_\sigma M,$$

or
$$(\epsilon bcd)(\epsilon efg)\,M,$$

or
$$(\epsilon\epsilon bc)\,M,$$

where b, c, d, e, f, g are ordinary or complex symbols, or variables, cogredient with u, and ρ, σ are cogredient with x. But expressed in terms of $a'a''$, or A instead of $\epsilon\epsilon$, these types become

$$(A\rho\sigma)\,M \qquad i.e. \quad (a'_\rho a''_\sigma - a''_\rho a'_\sigma)\,M,$$

$$(A\,.\,bcd\,.\,\sigma)\,M \qquad i.e. \quad \left((a'bcd)\,a''_\sigma - (a''bcd)\,a'_\sigma\right)M,$$

$$(A\,.\,bcd\,.\,efg)\,M \quad i.e. \quad \left((a'bcd)(a''efg) - (a''bcd)(a'efg)\right)M,$$

and
$$2(Abc)\,M \qquad i.e. \quad 2(a'a''bc)\,M.$$

These terms on the right are all aggregates, of characteristics (1, 3), (1, 1, 1, 1), and (3, 3, 3, 3), wherein one complex symbol has been removed by introducing in its place the ordinary symbols $a'a''$. This process may be repeated until all complex symbols have been removed.

The terms on the left contain A explicitly, although some of those on the right have lost the explicit form A; but in this case they are still alternating functions of a', a''. This holds for every symbol A, or cogredient symbol, or variable.

But such a state of things has already* been dealt with for concomitants of quaternary quadrics a_x^2, b_x^2, c_x^2, ... The same argument therefore

* Turnbull, *Proc. London Math. Soc.* (2), 21 (1922), 381–388.

applies here, and shows that all symbols a, A, α of the ground forms may occur explicitly and simultaneously in the symbolic expression of a concomitant, provided that the original three fundamental types of symbolic factor are supplemented by further types (iv)–(vii), to be presently defined, whose notation is

$$\left.\begin{array}{lc} \text{(iv)} & (AB),\quad (ABCD),\quad (ABCDEF),\quad \ldots \\ \text{(v)} & (aAb),\quad (aABCb),\quad (aABCDEb),\quad \ldots \\ \text{(vi)} & (aABa),\quad (aABCDa),\quad \ldots \\ \text{(vii)} & (aA\beta),\quad (aABC\beta),\quad \ldots \end{array}\right\} \tag{18}$$

Here type (iv) is of characteristic (22) or (2 ... 2),

(v) „ „ (121) or (12 ... 21),

(vi) „ „ (1223) or (12 ... 23),

(vii) „ „ (323) or (32 ... 23), dual of (v).

The number of symbols A, B, ... in a factor is called the *grade* of the factor. The expression of these factors in terms of the fundamental types is

$$\left.\begin{array}{ll} \text{(iv)} & (ABC \ldots HK) = \Omega(a''Bc')(c''\ldots)\ldots(\ldots h')(h''Ka') \\ \text{(v)} & (aAB \ldots GHb) = \Omega(aAb')(b''Cd')\ldots(g''Hb) \\ \text{(vi)} & (aAB \ldots HKa) = \Omega(aAb')(b''\ldots)\ldots(g''Hk')(k''a) \\ \text{(vii)} & (aAB \ldots GH\beta) = \Omega(aa')(a''Bc')\ldots(f''Gh')(h''\beta) \end{array}\right\} \tag{19}$$

where $A = a'b''$, $B = b'b''$, etc., and Ω denotes the simultaneous alternations among $a'a''$, and among $c'c''$, etc., as indicated by accented letters. Thus, for example, the series for expressing $(ABCDEF)$ has eight terms.

7. To illustrate a little more fully, we have

$$(AB) = (BA), \tag{20}$$

and

$$\begin{aligned}(uABCv) &= (uAb')(b''Cv) - (uAb'')(b'Cv) \\ &= \Omega_b(uABCv).\end{aligned}$$

Also $$(xABCy) = (xa')(a''Bc')(c''y) - (xa'')(a'Bc')(c''y)$$
$$- (xa')(a''Bc'')(c'y) + (xa'')(a'Bc'')(c'y)$$
$$= \Omega_{a,c}(xa')(a''Bc')(c''y).$$

Hence $$(uABCv) = -(vCBAu).$$

In particular, if $u = v$, $x = y$, these third grade types behave exactly like determinants, or ternary bracket factors, in symbols A, B, C, so that

$$\left.\begin{aligned}(uABCu) &= -(uBACu) = (uCABu) = \text{etc.}\\ (xABCx) &= -(xBACx) = (xCABx) = \text{etc.}\end{aligned}\right\} \tag{21}$$

as is seen by use of the fundamental identities.

Applied to the general type, a fundamental identity gives

$$(\lambda PQi) + (\lambda QPi) + (PQ)(\lambda i) = 0, \tag{22}$$

where P, Q are adjacent symbols of currency two, and λ, i denote respectively all that precedes and succeeds PQ, in the type under consideration. Hence, if (PQ) and (λi) are regarded as simpler than (λPQi), both being of lower grade than it, we may permute the capital letters in any type, except the first, (AB), as we permute in an ordinary bracket factor, or as we permute columns in a determinant. In this way we get alternative types, equivalent but for reducible terms.

Once more, it is clear that

$$(ABCDEF) = (CDEFAB) = (EFABCD)$$
$$= (AFEDCB) = (EDCBAF) = (CBAFED). \tag{23}$$

8. Although at first sight this system of types (i)–(vii) appears endless, a *reduced prepared system* which is finite may be chosen by rejecting all types among (iv), (v), (vi), (vii) which are reducible. The result of this is a very workable form of the quaternary fundamental theorem, the enunciation and proof of which are as follows:

Every form of orders (l, m, n) in variables x, p, u may be written symbolically as

$$(ax)^l (Ap)^m (au)^n,$$

and every concomitant of such ground forms may be expressed symbolically by bracket factors of thirteen kinds, such that every symbol a, A, a

involved occurs explicitly; and these factors are

$$\left.\begin{array}{c} (abcd),\quad (a\beta),\quad (a\beta\gamma\delta) \\ (aAb),\quad (aA\beta) \\ (AB),\quad (aABa) \\ (aABCb),\quad (aABC\beta) \\ (aABCDa) \\ (aABCDEb),\quad (aABCDE\beta) \\ \text{and} \qquad (ABCDEF). \end{array}\right\} \tag{24}$$

Proof. This list contains all possible types of grade less than six with the exception of

$$(ABCD),$$

which is reducible*; in fact

$$\begin{aligned}(ABCD) &= \Omega_{a,c}(a''Bc')(c''Da') \\ &= \tfrac{1}{2}[(BC)(AD)+(AB)(CD)-(AC)(BD)]\end{aligned} \tag{25}$$

by use of the fundamental identities. Also all types of grade six and higher are reducible to those of lower grade with the exception of the single type

$$(ABCDEF).$$

For writing such a term P as

$$(\lambda ABCDEFi),$$

where, as before, λ and i are arbitrary and their total currency is a multiple of four, we have, by (22),

$$\{AB\}P = -(AB)(\lambda CDEFi),$$

where $\{AB\}$ denotes the two-term substitution

$$A, B \quad B, A.$$

So if $A'B'C'D'E'F'$ denote any one of the 6! permutations of $ABCDEF$

* *Proc. London Math. Soc., loc. cit.*, 387.

then, if reducible terms are dropped,

$$(\lambda A' B' C' D' E' F' i) \equiv \pm (\lambda ABCDEFi),$$

according as the substitution is even or odd. Hence, if

$$\{ABCDEF\}'$$

denote the sum of these 6! alternations, then

$$\{ABCDEF\}' P \equiv 6!\, P. \tag{26}$$

But P is a form simultaneously linear and homogeneous in six sets of senary variables $A, \ldots, F$, such as

$$A = \{A_{14}, A_{24}, A_{34}, A_{22}, A_{31}, A_{12}\};$$

hence the left side of (26) contains the determinant

$$|A_{14} B_{24} C_{34} D_{23} E_{31} F_{12}|$$

as a factor*. Dropping the suffixes for brevity, we obtain the result

$$6!\, P \equiv |ABCDEF| Q, \tag{27}$$

where Q is a rational integral function of the residual symbols excluding A, B, C, D, E, F, and therefore, since the whole product as well as the determinantal factor is invariantive, so also is Q. So we may write

$$Q = \rho(\lambda i), \tag{28}$$

where ρ is numerical. The same argument shows that

$$(ABCDEF) \equiv \sigma |ABCDEF|, \tag{29}$$

where σ is numerical.

This proves the theorem.

9. Next it may be shown that this determinant is an irreducible type.

$|ABCDEF|$ *is irreducible.*

Proof.—For if not, it must be expressible by a series

$$S = \Sigma \lambda (AB)(CD)(EF),$$

* Grace and Young, *Algebra of Invariants*, 346.

where λ is numerical, and each term contains three factors like (AB) involving all six symbols $A \ldots F$. No such identity exists when the six symbols $A \ldots F$ are arbitrary, since the result of operating on the assumed identity $|ABCDEF| = \Sigma\lambda(AB)(CD)(EF)$ with $\{ABCDEF\}'$ renders at once the right-hand side zero, but not the left.

The principle of duality.

10. In accordance with the usual theory, the symbols

$$a, \quad b, \quad c, \quad u, \quad v, \quad w$$

are formal duals of $\alpha, \quad \beta, \quad \gamma, \quad x, \quad y, \quad z,$

whereas the symbols $A, \quad B, \quad C, \quad p$

are self-dual. But more precisely A_{ik} is dual of A_{lm} and p_{ik} of p_{lm} $(= \pi_{ik})$. It follows that the types

$$(AB), \quad (BA) \tag{30}$$

which happen to be equal, are dual of each other. As for the other fundamental types of grade exceeding one we consider the following as dual types arranged in columns:

$$\left.\begin{array}{lll} (aB \ldots Gh) & \text{dual of} & (aB \ldots G\theta) \\ (aB \ldots GH\lambda) & ,, \quad ,, & (aB \ldots GHl) \\ (B \ldots GH) & ,, \quad ,, & (HB \ldots G) \end{array}\right\}. \tag{31}$$

Now in any such type the alternate capital letter symbols behave alike in the original form of the type; either all are explicit or all are distributed. For example in

$$\Omega_{a,c,e}(a''Bc')(c''De')(e''Fa') = (ABCDEF),$$

A, C, E are distributed while B, D, F are explicit. The dual is formed by reversing this feature while retaining the cyclic order $ABCDEF$. So we obtain as dual

$$\Omega_{b,d,f}(d''Ef')(f''Ab')(b''Cd') = (DEFABC). \tag{32'}$$

So also the determinants $|A_{ik}|$ and $|A_{lm}|$ are dual, thus $|ABCDEF|$ and

$|DEFABC|$ are dual. Again, since

$$(ABCDEF) \equiv \sigma |ABCDEF| \pmod{R},$$

then $$(ABCDEF) = \sigma |ABCDEF| + \Sigma\nu(AB)(CD)(EF),$$

where ν is numerical. Next, by forming the dual by writing A_{ik} for A_{lm} throughout, and likewise for B, C, ..., we have

$$(DEFABC) = -\sigma |ABCDEF| + \Sigma\nu(BA)(DC)(FE),$$

whence by subtraction

$$(ABCDEF) - (DEFABC) = 2\sigma |ABCDEF| \tag{33}$$

identically; and, by taking a numerical case, we find that

$$2\sigma = 1. \tag{34}$$

Further, it is a straightforward matter to express $(DEFABC)$ as

$$-(ABCDEF) + R,$$

where R denotes terms like $(AB)(CD)(EF)$. This is effected by a few applications of relations such as

$$(DEFABC) + (DEAFBC) + (AF)(DEBC) = 0,$$

bringing the letters by adjacent interchanges into alphabetical order. Ultimately this gives

$$2(ABCDEF) = |ABCDEF| + R, \tag{35}$$

where R signifies reducible terms*.

The prepared system for any number of quaternary quadrics.

11. The preceding results may be used to simplify the existing prepared system for quadrics. Thus, if

$$f = a_x^2 = a_x'^2 = \ldots$$

* Cf. Weitzenböck, *Invariantentheorie* (Groningen, 1923), 179–181, for the treatment by use of complex symbols.

is a quadric whose line coordinate form is

$$(aa'xy)^2 = (aa'p)^2 = (Ap)^2,$$

we may regard (Ap)

as a symbolic linear complex. Also

$$(aa'a''u)^2 = (au)^2$$

gives the plane coordinate or reciprocal form of f. Similarly, for other quadrics denoted by b, B, β; c, C, γ; etc. For these, a prepared system need contain no bracket factors of grade greater than five except the types

$$(ABCDEp) \quad \text{and} \quad (ABCDEF).$$

Thus nine of the last eleven entries in the table given in *Proc. London Math. Soc.* (2), 21 (1922), 381–388 (388), may be rejected.

II. *Complete system of one linear complex.*

12. Let a linear complex K be denoted by

$$\left.\begin{aligned} K &= \Sigma A_{ik} p_{ik} = \Sigma A_{ik} \pi_{lm} \\ &= (aa'xy) = (bb'xy) = (cc'xy) \\ &= (aa'uv) = (bb'uv) = (cc'uv) \\ &= (Ap) = (Bp) = (Cp), \end{aligned}\right\} \tag{36}$$

Here $$(aa'xy) = (ax)(a'y) - (ay)(a'x);$$

and $A\ [=(aa')]$, $B\ [=(bb']$, $C\ [=(cc')]$ are equivalent symbols. So if a and b are interchanged, a' and b' must also be interchanged.

Now, by what has preceded, any concomitant is expressible by the prepared system. Also in this case the single types in this system are actual concomitants, since K is linear in $A, B, C, \ldots$. Further, all types of degree two or more in the coefficients of K are reducible, except the one invariant

$$I = (AB) = 2\Sigma A_{ik} A_{lm}. \tag{37}$$

For if $$P = (\theta AMB\phi)$$

is such a form, whatever M may be, then

$$P \equiv -(\theta BMA\phi)$$

by the general property of such forms. Thus, on interchanging A and B,

$$P \equiv -P \quad i.e. \quad 2P \equiv 0,$$

and so reduces.

Accordingly *the complete system of one linear complex is*

$$(pp), \quad (Ap), \quad \text{and} \quad (AB),$$

the identical line coordinate relation, the complex itself and its invariant, together with forms derivable from (Ap) *and* (ux) *by polarization*

$$(Axy), \quad (Ax\,.\,pu), \quad (Auv), \quad \text{etc.}$$

Complete system for any number of linear complexes.

13. This system consists of the invariants $I = (AA')$ of the complexes themselves, together with forms linear in the coefficients of the complexes. Otherwise the preceding reduction applies.

Disregarding polar forms, we deal with variables

$$u, \quad p, \quad x$$

and symbols $A, \quad B, \quad C, \quad \ldots$

of the different complexes

$$(Ap), \quad (Bp), \quad (Cp), \quad \ldots.$$

This procedure gives *three types of invariant*

$$\left.\begin{aligned} &(1) \qquad (AA) = 2\Sigma A_{ik}A_{lm}, \\ &(2) \qquad (AB) = \Sigma A_{ik}B_{lm}, \\ &(3) \qquad (ABCDEF) - (DEFABC) = |\,A_{14}B_{24}C_{34}D_{23}E_{31}F_{12}\,| \end{aligned}\right\} \tag{38}$$

involving one, two, and six complexes, respectively. All these types are irreducible and complete.

There is *one new complex concomitant* which may be taken to be

$$(ABCDEp) - (pABCDE) = |\,A_{14}B_{24}C_{34}D_{23}E_{31}p_{12}\,|. \tag{39}$$

The remainder are *mixed concomitants*

$$(ux), \quad (uABx), \quad (uABCDx), \tag{40}$$

with the covariant $(xABCx)$, (41)

and the contravariant $(uABCu)$. (42)

All others are reducible as in the case of quadrics. Thus, for example, the type $(xABCDEx)$ is reducible, for it is at once seen to be equal to $-(xEDCBAx)$, and this reduces when next A is interchanged with E and B with D.

Geometrical illustrations of the linear complex.

14. Most of these irreducible types are well known either in their non-symbolic or their complex-symbolic form*. But a few examples of the possibilities of the ordinary symbolic form seem worth while giving.

(1) The complex (Ap) is *special* if (AA') vanish; and conversely.

(2) The complexes (Ap), (Bp) are *apolar* if (AB) vanish; and conversely.

(3) The forms (xAy), (uAv) give the *polar relations* or *nul systems* of the complex A.

Thus rays of the complex passing through a fixed point (y) lie in the plane whose coordinates are (Ay), or u, where

$$u_i = (ax)\, a_i' - (a'y)\, a_i.$$

Again, the nul point $\bar{x}$ of the plane u is (Au).

(4) If $(\bar{x}\bar{y}\bar{z}\bar{r})$ are the nul points of u, v, w, u' respectively, then

$$4(\bar{x}\bar{y}\bar{z}\bar{r}) = (uvwu')\, I^2,$$

where $I = (AA')$.

(5) One complex gives a correlation $x \rightarrow u'$; two complexes give a collineation $x \rightarrow x'$. Thus, if u' is the nul plane of x for the complex A, and x' is the nul point of u' for the complex B, then the collineation $x \rightarrow x'$ is given by

$$(xABu).$$

* Cf. Waelsch, *loc. cit.*

Or, the vanishing of this concomitant expresses the plane equation of the point x', if x is given. In fact, $x' = (xAB)$ or, more specifically,

$$x'_i = (xa)(a'B)^i - (xa')(aB)^i.$$

(6) If the collineation is reversible for A and B, the complexes are apolar. For, by (22),

$$(xABu) + (xBAu) + (AB)(xu) = 0,$$

identically, in all cases. If (AB) vanish, then

$$(xABu) \quad \text{and} \quad (xBAu)$$

denote the same collineation.

Three linear complexes.

(7) The covariant $(xABCx)$ gives the point form of the quadric surface one of whose reguli is formed by the common rays of three complexes.

Correlatively $(uABCu)$ denotes the same quadric in its tangential form.

It is a good exercise in symbolic algebra to deduce the one from the other. These results may be verified by the preceding methods. The following aspect is instructive:

Let $x = (uvw)$, then

$$\begin{aligned}(xABCx) &= \Omega(uvwa')(a''Bc')(c''uvw)\\ &= -\Omega_{uvw,c}(uvA)(wBc')(c''uvw)\\ &= \begin{vmatrix}(Auv) & (Buv) & (Cuv)\\ (Avw) & (Bvw) & (Cvw)\\ (Awu) & (Bwu) & (Cwu)\end{vmatrix}.\end{aligned}$$

If $p = (uv)$, this determinant is a linear function of (Ap), (Bp), (Cp) with w arbitrary. Hence all rays common to the linear system

$$\lambda(Ap) + \mu(Bp) + \nu(Cp)$$

lie on this surface $(xABCx)$.

Correlatively, if $u = (xyz)$.

(8) The equation of the quadric through three given lines p, q, r is

$$(xpqrx) = 0 \quad \text{or} \quad (upqru) = 0.$$

If the lines intersect at all, both these forms break into linear factors for x and u respectively.

(9) The nul points of the plane u for the complexes A, B, C are collinear if

$$(uABCu) = 0.$$

(10) The nul points of u for the six complexes A, B, ..., F lie on a conic if

$$(uABCu)(uAEFu)(uBFDu)(uCDEu)$$
$$-(uDEFu)(uDBCu)(uECAu)(uFABu) = 0.$$

This is true for all values of u if and only if each of $(AB)\ldots(EF)$ vanish, giving the Klein system of six mutually apolar complexes.

These last results follow by treating A, B, C as ternary symbols for the plane u, which is possible since the factor type $(uABCu)$ behaves like a ternary bracket factor.

(11) The complex concomitant

$$(ABCDEp)-(pABCDE)$$

is the linear complex apolar to five given complexes A, B, C, D, E.

It is interesting to verify this directly by use of the fundamental identities.

III. *The quadratic complex.*

15. The full discussion of this complex, which presents great difficulties, shall not be attempted here, but a few illustrations of the preceding methods are as follows:

Let
$$F = \Sigma B_{ik,\,lm}\, p_{ik}\, p_{lm}, \tag{43}$$

summed for the usual six values of ik and of lm, independently

$$(i = 1, 2, 3, 4, \quad k = 1, 2, 3, 4, \quad \text{etc.}, \quad i \neq k, \quad l \neq m).$$

Let this be the normal form of the complex, so that

$$\tfrac{1}{2}I_1 = \tfrac{1}{2}\Sigma B_{ik,\,i'k'} = B_{14,\,23} + B_{24,\,31} + B_{34,\,12} = 0 \tag{44}$$

identically.

Here the accented suffixes denote the algebraic complements of the same unaccented letters.

Symbolically we take

$$F = (Ap)^2 = (Bp)^2 = (Cp)^2 = \text{etc.}, \tag{45}$$

where the linear invariant I_1 is

$$(AA)$$

which vanishes identically. Further, let

$$A = (aa'), \quad B = (bb'), \quad \text{etc.}$$

Then any concomitant is given symbolically by an aggregate of terms whose factors are the types in the prepared system, wherein each symbol A (or B, ...) which occurs, is present twice. Let factors (AB) be called type η and all others of grade two or higher be called type θ. Then *if a concomitant P contains AB convolved once in η and once in θ it is reducible.*

In fact
$$P = (AB)(\lambda ABi)\phi \quad \text{say},$$
$$\equiv -(AB)(\lambda BAi)\phi \quad \text{by (22)},$$
$$\equiv -(BA)(\lambda BAi)\phi$$
$$\equiv -P \quad \text{on interchanging equivalent symbols.}$$

Next, if (A_iB) denote a chain of equivalent symbols, with i repeated links and A and B unrepeated end symbols, so that

$$(A_1B) = (AC)(CB), \quad (A_2B) = (AC)(CD)(DB), \quad \text{etc.},$$

then the invariants are given by

$$0 = I_1 = (AA),$$
$$I_2 = (A_1A) = (AB)^2,$$
$$I_3 = (A_2A) = (AB)(BC)(CA),$$
$$I_4 = (A_3A),$$
$$I_5 = (A_4A),$$
$$I_6 = (A_5A),$$
$$I_{6+k} \equiv 0 \pmod{R} \quad (k > 0).$$

For, by multiplying two six-rowed determinants together, we have

$$|ABCDEF| \times |ABCDEG|$$

$$= \begin{vmatrix} (AA) & (AB) & (AC) & (AD) & (AE) & (AF) \\ (BA) & (BB) & (BC) & (BD) & (BE) & (BF) \\ (CA) & (CB) & (CC) & (CD) & (CE) & (CF) \\ (DA) & (DB) & (DC) & (DD) & (DE) & (DF) \\ (EA) & (EB) & (EC) & (ED) & (EE) & (EF) \\ (GA) & (GB) & (GC) & (GD) & (GE) & (GF) \end{vmatrix}.$$

The product determinant on expansion gives an aggregate of reducible terms like

$$(A_i A)(F_j G) \quad (i = 0, 1, 2, 3, 4),$$

and terms like

$$(F_5 G) = (FA)(AB)(BC)(CD)(DE)(EG).$$

There are 5! of these and all have a minus sign. For, writing this one as

$$\begin{bmatrix} F & A & B & C & D & E \\ A & B & C & D & E & G \end{bmatrix},$$

F may be placed in any of five ways above A, B, C, D, E, above A say: then the upper A may be placed in any of four ways above B, C, D, E, and so on. This gives all such terms. Also each term is derivable from the arrangement

$$\begin{bmatrix} A & B & C & D & E & F \\ A & B & C & D & E & G \end{bmatrix}$$

by five interchanges, and so all have a negative sign.

Further, if A, B, C, D, E are equivalent symbols, all these terms are equal, so that

$$5!\,(F_5 G) \equiv -|ABCDEF| \times |ABCDEG| \pmod{R};$$

hence, if

$$I_{6+k} = (FA)(AB)(BC)(CD)(DE)(EG)(G_{k-1}F)$$

$$= (F_5 G)(G_{k-1}F)$$

$$\equiv -\frac{1}{5!}|ABCDEF||ABCDEG|(G_{k-1}F),$$

we have, after interchanging equivalent symbols A, B, C, D, E, F in all possible ways,

$$6!\,5!\,I_{6+k} \equiv -|ABCDEF| \times [\{ABCDEF\}'\phi],$$

where $$\phi = |ABCDEG|(G_{k-1}F).$$

As in § 8 this implies the factor $|ABCDEF|$ within the bracketed portion of the right-hand side. Hence I_{6+k} contains the factor

$$\Delta = |ABCDEF|^2,$$

while I_6 itself is equivalent to Δ.

Concomitants involving coefficients A and the variable p (but not u and x) may similarly be dealt with. They consist of six quadratic complexes (including w and F) and a sextic complex with six (Klein) linear complex factors.

The singular (Kummer quartic) surface is given by

$$(xABCx)^2 \quad \text{and} \quad (uABCu)^2,$$

either of which may be deduced symbolically from the other provided that Δ does not vanish.

16. *To find the simultaneous system of a quadric, a linear complex, and a quadratic complex, of the second degree in the coefficients of these complexes.*

This problem, which Weitzenböck* proposed and solved because of its interest in the Weyl Relativity Theory, follows very directly by ordinary symbolic methods.

Let $$g = \Sigma g_{ik}\xi_i\xi_k = g_\xi^2, \tag{47}$$

$$K = \Sigma K_{ik}p_{ik} = (Kp), \tag{48}$$

$$F = \Sigma F_{ik,lm}p_{ik}p_{lm} = (Ap)^2 = (Bp)^2 \tag{49}$$

be the quadric, linear complex, and quadratic complex respectively. We require the invariants of these three ground forms of degrees

$$(2, 0), \quad (1, 1), \quad \text{or} \quad (0, 2)$$

in the coefficients of K and F.

* *Wiener Berichte*, 129 (1920), 683–708; 130 (1921), 15–23.

Let g, g', g'', g'''; $G = (gg')$; $\gamma = (gg'g'')$; etc.

all refer to the quadric, while K, K', ... are equivalent symbols for the linear complex, and A, B, ... for the quadratic complex.

Then every invariant is ultimately composed of symbolic factors

$$(PQ), \quad (\ldots PQR \ldots), \quad \text{and} \quad (gg'g''g'''),$$

where P, Q, R are symbols of type either G or K or A.

Here (PQ) may be

$$(GK), \quad (GA), \quad (KA), \quad (AB), \quad (KK'),$$

otherwise it reduces.

Also $(\ldots PQR \ldots)$ can only be

$$(gKA\gamma) \quad \text{or} \quad (gAB\gamma). \tag{50}$$

For the type $(PQRSTU)$ reduces if two of the symbols refer to the quadric, and otherwise it is of the wrong degree in K and F. Also for all other types the end symbols must refer to the quadric, consequently the intermediate symbols must refer only to K and F. Among these the first grade case reduces, and the third grade case would imply that two of P, Q, R are equal, to make the correct degree; and such a case reduces. This leaves the given set (50).

This gives the following six possible invariants of degree two or less in K and F:—

(i)	(KK')	of degree	(0, 2, 0)	in	(g, K, F)		
(ii)	$(KG)(K'G)$	,,	(2, 2, 0)	,,	,,		
(iii)	$(AB)^2$	,,	(0, 0, 2)	,,	,,		(51)
(iv)	$(AB)(AG)(GB)$	,,	(2, 0, 2)	,,	,,		
(v)	$(AG)^2$	,,	(2, 0, 1)	,,	,,		
(vi)	$(gAB\gamma)^2$	,,	(4, 0, 2)	,,	,,		

Let these be called L_1, L_2, L_3, L_4, L_5, L_6 respectively. An alternative to L_6 is the form

$$L_6' = (AG)(GB)(BG')(G'A),$$

where G, G' are equivalent symbols, the connexion being

$$6L_6' = 4L_6 + \Delta \cdot L_3 + L_5^2 \quad (\Delta = g_\gamma^2 = 24|g_{ik}|). \tag{52}$$

17. This list, which has here been readily constructed by use of ordinary symbols, agrees with the system M obtained by Weitzenböck with complex symbols. The following is the connexion :—

Ordinary symbols.		*Complex symbols.*
Quadric	g_ξ^2	——
Linear complex	(Kp)	$= \frac{1}{2}(p\pi')^2 \quad [K_{ik} = p_i p_k]$,
Quadratic complex	$(Ap)^2$	$= \frac{1}{4}(a\pi')^2(a\rho')^2$,
Discriminant $\lvert g\rvert$ *of quadric*	$\frac{1}{24}g_\gamma^2$	——
$2L_1 = 2(KK')$		$= M_1 = (pq')^2$,
$\frac{4L_2}{\lvert g\rvert} = \frac{4(KG)(K'G)}{\lvert g\rvert}$		$= \bar{M}_2 = (pg_1')(pg_2')(qg_1')(qg_2')$,
$4L_3 = 4(AB)^2$		$= M_7 = (ab')^2(a\beta')^2$,
$\frac{2L_4}{\lvert g\rvert} = \frac{2(AB)(AG)(GB)}{\lvert g\rvert}$		$= \bar{M}_9 = (ab')^2(g_1'a)(g_1'\beta)(g_2'a)(g_2'\beta)$,
$\frac{L_5}{\lvert g\rvert} = \frac{(GA)^2}{\lvert g\rvert}$		$= \bar{M}_{11} = (g_1'a)(g_1'a)(g_2'a)(g_2'a)$,
$\frac{L_6'}{\lvert g\rvert^2} = \frac{(AG)(GB)(BG')(G'A)}{\lvert g\rvert^2}$		$= \bar{M}_{12} = (g_1'a)(g_1'b)(g_2'a)(g_2'b)(g_3'a) \times (g_3'\beta)(g_4'a)(g_4'\beta)$.

Tensor Notation.

18. If ik, $i'k'$ are algebraic complements among suffixes 1, 2, 3, 4, that is if $(ik, i'k') = (12, 34)$ or $(23, 14)$, etc. grouped as in a determinantal permutation; and if

$$\sum_{ik}$$

denote the six-term summation

$$ik = 14,\ 24,\ 34,\ 23,\ 31,\ 12;$$

and if the repeated suffix i, i in a term implies a sum of such terms for

$i = 1, 2, 3, 4$ inclusive, then the nonsymbolic equivalents of the system L, in tensor notation, are

$$(KK') = L_1 = \sum_{ik} K_{ik} K_{i'k'},$$

$$(KG)(K'G) = L_2 = |g| . K^{ik} K_{ik},$$

$$(AB)^2 = L_3 = \sum_{ik} \sum_{lm} F_{ik,lm} F_{i'k',lm},$$

$$(AB)(AG)(BG) = L_4 = |g| \Sigma F^{00\,\lambda\mu}_{ik,00} F_{i'k',\lambda\mu},$$

$$(GA)^2 = L_5 = |g| . F^{00,ik}_{ik,00},$$

$$(AG)(GB)(BG')(G'A) = L_6' = |g|^2 F^{ik,lm} F_{ik,lm}.$$

19. Such a system of invariants leads to a set of six Hamiltonian functions W, namely

$$W_1 = (KK'),$$

$$W_2 = (GK)^2/\sqrt{|g|},$$

$$W_3 = (AB)^2/\sqrt{|g|},$$

$$W_4 = (AB)(BG)(GA)/|g|,$$

$$W_5 = (GA)^2(G'B)^2/|g|^{\frac{3}{2}} = L_5^2/|g|^{\frac{3}{2}},$$

$$W_6 = \tfrac{1}{4}(AG)(GB)(BG')(G'A)/|g|^{\frac{3}{2}}.$$

As Weitzenböck* has proved, these are the only rational integral functions of the coefficients of a certain set of ground form tensors which satisfy the Hamiltonian condition

$$\delta \iiiint W dx_1 dx_2 dx_3 dx_4 = 0,$$

where the multiple integral is an invariant over the field of transformation $x \to \bar{x}$ of a general functional type

$$x_i = f(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4),$$

---

* Cf. last reference.

modified by the general conditions of relativity due to Weyl. In this, the ground forms are

(1) $g_{ik}dx^i dx^k$, the fundamental metrical tensor;

(2) K_{ik}, the electromagnetic field tensor;

(3) $F_{ik,lm}$, the Weyl curvature tensor (Richtungskrümmung).

(4) $E_{ik,m}$, involving the first covariant derivate of (2).

If W is of degrees n_2, n_3, n_4 in tensors (2), (3), (4), respectively, then the Weyl transformations, characterized by a change of gauge λ, a function of the variables x_i, are

$$\bar{g}_{ik}d\bar{x}^i d\bar{x}^k = (d\bar{s})^2 = \lambda(ds)^2 = \lambda g_{ik}dx^i dx^k,$$

$$\bar{K}_{ik} = K_{ik}, \quad \bar{F}_{ik,lm} = \lambda F_{ik,lm};$$

and these require that $2n_2 + 2n_3 + 3n_4 = 4,$

which is solved in positive integers only by excluding tensor (4) and taking

$$n_2 + n_3 = 2,$$

as in the example of § 16.

The special (Einstein) case when $\lambda = 1$ can be treated similarly, but it gives rise to a vastly more complicated projective invariant system for the ground forms, since n_2 and n_3 are now unrestricted.

ON THE MEAN VALUES OF POWER SERIES

By J. E. LITTLEWOOD.

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1. The present paper is a sequel to one recently published in these *Proceedings**. Theorems 1 and 2 are new. Theorem 3 is the principal result of the original paper; but since the proof here given is an improved one and falls into line with those of Theorems 1 and 2, I have included it for the sake of completeness.

2. Let $a_n \geqslant 0$, and suppose always that $\Sigma a_n r^n$ is convergent for $r<1$. Let $a_0 = 0$, and let $\alpha_1, \alpha_2, \ldots$ be real numbers satisfying $0 \leqslant \alpha_n \leqslant 1$. Let $z = \rho e^{i\theta}$,

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i \alpha_n} z^n, \tag{2.1}$$

$$M_\lambda(\rho) = M_\lambda(\rho, f) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\lambda \, d\theta \right\}^{1/\lambda} \quad (0 < \lambda < \infty),$$

and

$$M(\rho) = \operatorname*{Max}_{|z|=\rho} |f(z)|\dagger.$$

We write also

$$\mu(\rho) = M_2(\rho, f) = \left\{ \sum_0^\infty a_n^2 \rho^{2n} \right\}^{\frac{1}{2}},$$

a number independent of the arguments α_n. Finally we use $A(x, y, \ldots)$ to denote a positive constant depending only on the parameters shown explicitly; in particular, A denotes a positive absolute constant. These numbers are not the same at every occurrence, unless affected with suffixes.

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* "On inequalities in the theory of functions", *Proc. London Math. Soc.* (2), 23 (1925), 481.

† It is well known, and almost evident, that $M(\rho) = \lim_{\lambda \to \infty} M_\lambda(\rho)$.

If we are given the moduli a_n of the coefficients in (2 . 1) we are given the function $\mu(\rho)$. If $\lambda \geqq 2$, $M_\lambda(\rho) \geqq \mu(\rho)$; if $\lambda \leqq 2$, $M_\lambda(\rho) \leqq \mu(\rho)$ (see Lemma 2 below). The results of the paper are, roughly, that we can dispose the arguments $2\pi a_n$ so that (*a*) the $M_\lambda(\rho)$ of large λ, and indeed the maximum $M(\rho)$, do not greatly exceed $\mu(\rho)$, or, again, so that (*b*) the $M_\lambda(\rho)$ of small λ are not greatly exceeded by $\mu(\rho)$.

3. LEMMA 1.—*Suppose that the numbers $\beta(m, n)$ are bounded. Then there exists a sequence (n_r), where $n_r \to \infty$ with r, and a set of numbers (β_m), such that, for each m, $\beta(m, n_r) \to \beta_m$ as $r \to \infty$.*

This is a known result, but I give the proof for completeness. By a theorem of Weierstrass there exist a β_1 and an increasing sequence $n_{1,1}, n_{1,2}, \ldots$, which we call S_1, such that $\beta(1, n_{1,r}) \to \beta_1$. Similarly there exist a β_2 and a subsequence S_2, or $n_{2,2}, n_{2,3}, n_{2,4}, \ldots$, of S_1, giving $\beta(2, n_{2,r}) \to \beta_2$. Moreover we may suppose $n_{2,2} > n_{1,1}$: for definiteness let $n_{2,2} = n_{1,2} > n_{11}$. The process continues indefinitely. There exist a β_3 and a subsequence S_3, or $n_{3,3}, n_{3,4}, \ldots$, for which $\beta(3, n_{3,r}) \to \beta_3$ and $n_{3,3} = n_{2,3} > n_{2,2}$. And so on. Let $n_r = n_{r,r}$ and consider the (increasing) sequence (n_r). $n_m, n_{m+1}, \ldots$ is a subsequence of S_m; hence $\beta(m, n_r) \to \beta_m$.

LEMMA 2.—*Suppose $0 < \alpha \leqq \beta \leqq \gamma$, $\alpha < \gamma$, and let $\phi(z)$ be regular in $|z| < 1$. Then*

$$M_\beta(\rho, \phi) \leqq \{M_\alpha(\rho, \phi)\}^\vartheta \{M_\gamma(\rho, \phi)\}^{1-\vartheta},$$

where $\vartheta = \alpha(\gamma-\beta)/\beta(\gamma-\alpha)$. Also $M_\lambda(\rho, \phi)$ is a non-decreasing function of λ for fixed ρ.

This also is known*. We have, in fact,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\phi|^\beta d\theta = \frac{1}{2\pi}\int |\phi|^{\beta_1} . |\phi|^{\beta_2} d\theta$$

$$\leqq \left\{\frac{1}{2\pi}\int |\phi|^{\beta_1 p_1} d\theta\right\}^{1/p_1} \left\{\frac{1}{2\pi}\int |\phi|^{\beta_2 p_2} d\theta\right\}^{1/p_2},$$

where

$$\beta_1 = \frac{\alpha(\gamma-\beta)}{\gamma-\alpha}, \quad \beta_2 = \frac{\gamma(\beta-\alpha)}{\gamma-\alpha}, \quad p_1 = \frac{\gamma-\alpha}{\gamma-\beta}, \quad p_2 = \frac{\gamma-\alpha}{\beta-\alpha}, \quad \frac{1}{p_1} + \frac{1}{p_2} = 1.$$

* F. Hausdorff, *Math. Zeitschrift*, 16 (1923), 163–169 (165).

For the last part we have

$$M_\alpha^\alpha = \frac{1}{2\pi}\int |\phi|^\alpha \, d\theta$$

$$\leqslant \left(\frac{1}{2\pi}\int |\phi|^{\alpha q_1} d\theta\right)^{1/q_1} \left(\frac{1}{2\pi}\int 1^{q_2} d\theta\right)^{1/q_2} = M_\gamma^\alpha,$$

where $q_1 = \gamma/\alpha > 1$ and $1/q_1 + 1/q_2 = 1$. Thus $M_\alpha \leqslant M_\gamma$ when $\alpha < \gamma$.

4. Let
$$f_N(z) = \sum_0^N a_n e^{2\pi i \alpha_n} z^n,$$

and let
$$I_N(\phi) = \int_0^1 d\alpha_1 \int_0^1 d\alpha_2 \ldots \int_0^1 \phi \, d\alpha_N.$$

LEMMA 3.—*Let k be a positive integer. Then, for $\rho \leqslant 1$,*

$$I_N(J) \leqslant k^k \left(\sum_0^N a_n^2 \rho^{2n}\right)^k,$$

where
$$J = \frac{1}{2\pi}\int_0^{2\pi} |f_N(\rho e^{i\theta})|^{2k} \, d\theta.$$

Let
$$\{f_N(z)\}^k = \sum_0^{kN} b_n z^n,$$

so that
$$J = \sum_0^{kN} |b_n|^2 \rho^{2n}.$$

Now

$$b_n = \Sigma c(\nu_1, \nu_2, \ldots)\, a_{n_1}^{\nu_1} a_{n_2}^{\nu_2} \ldots \exp\{2\pi i(\nu_1 \alpha_{n_1} + \nu_2 \alpha_{n_2} + \ldots)\}, \tag{4.1}$$

where
$$c(\nu_1, \nu_2, \ldots) = \frac{k!}{\nu_1!\, \nu_2! \ldots} \leqslant k^k,$$

and the summation is taken over all $n_1, n_2, \ldots, \nu_1, \nu_2, \ldots$, subject to

$$\begin{cases} \nu_r \geqslant 0, \quad 0 \leqslant n_1 < n_2 < \ldots \\ \nu_1 + \nu_2 + \ldots = k, \\ n_1\nu_1 + n_2\nu_2 + \ldots = n. \end{cases} \tag{4.2}$$

Hence, for $n \leqslant kN$,

$$|b_n|^2 = \Sigma_1 c^2(\nu_1, \nu_2, \ldots)\, a_{n_1}^{2\nu_1} a_{n_2}^{2\nu_2} \ldots$$

$$+ \Sigma_2 c(\nu_1, \nu_2, \ldots)\, c(\nu_1', \nu_2', \ldots)\, a_{n_1}^{\nu_1} \ldots a_{n_1'}^{\nu_1'} \ldots \exp(2\pi i L),$$

where $$L = \nu_1 a_{n_1} + \nu_2 a_{\nu_2} + \ldots - \nu_1' a_{n_1'} - \nu_2' a_{n_2'} - \ldots,$$

Σ_1 is taken over all ν_r and n_r subject to (4.2), and Σ_2 is taken over a certain set of ν_r, n_r, ν_r', n_r' for which we do not have simultaneously $n_1 = n_1'$ and $\nu_1 = \nu_1'$, $n_2 = n_2'$ and $\nu_2 = \nu_2'$, ... For each term of Σ_2 there is an a_p such that

$$L = \nu a_p + (\text{a linear function of } a\text{'s other than } a_p)$$

and $\nu \neq 0$. Then

$$\int_0^1 \exp(2\pi i L)\, da_p = 0, \qquad I_N\{\exp(2\pi i L)\} = 0,$$

$$I_N(J) = \sum_0^{kN} I_N(|b_n|^2)\rho^{2n}$$

$$= \sum_0^{kN} \{\Sigma_1 c^2(\nu_1, \nu_2, \ldots)(a_{n_1}^2)^{\nu_1}(a_{n_2}^2)^{\nu_2}\ldots \rho^{2n}\}$$

$$\leqslant k^k \sum_0^{kN} \Sigma_1 c(\nu_1, \nu_2, \ldots)(a_{n_1}^2 \rho^{2n_1})^{\nu_1}(a_{n_2}^2 \rho^{2n_2})^{\nu_2}\ldots$$

$$= k^k \left(\sum_0^N a_n^2 \rho^{2n}\right)^k.$$

5. THEOREM 1.—*Let* $a_n \geqslant 0$, *and let* $\sum_0^\infty a_n^2 = 1$. *Then there exists a set of real* α_n *such that* $f = \Sigma a_n e^{2\pi i \alpha_n} z^n$ *satisfies*

$$M_\lambda(\rho, f) \leqslant A_1(\lambda) \tag{5.1}$$

and

$$M_\lambda(\rho, f) \geqslant A_2(\lambda)\{\mu(\rho)\}^{4\lambda^{-1}+1}, \tag{5.2}$$

where $$A_1(\lambda) = (2\lambda+4)^{\frac{1}{2}}, \quad A_2(\lambda) = \exp(-8\log 2 \,.\, \lambda^{-1}),$$

$$\mu(\rho) = \left(\sum_0^\infty a_n^2 \rho^{2n}\right)^{\frac{1}{2}}.$$

Thus, if the moduli of the coefficients are given in such a manner that $\mu(\rho) = M_2(\rho, f)$ is bounded, we can find a set of arguments such that every $M_\lambda(f)$ is bounded.

Let P denote the point whose coordinates in a space of N dimensions are $\alpha_1, \alpha_2, \ldots, \alpha_N$. We suppose always that the α's satisfy $0 \leqslant \alpha \leqslant 1$, so that P is confined to the "unit cube". With each P, or set $\alpha_1, \alpha_2, \ldots, \alpha_N$, there is associated a function f_N. Let $v(k, N)$ be the

measure of the set of P's for which

$$M_{2k}^{2k}(1, f_N) > 2k^{k+2}, \tag{5.3}$$

and let $V(N)$ be the measure of the set S of P's for which (5.3) holds for *some* value of k. Then, by Lemma 3,

$$v(k, N)\, 2k^{k+2} \leqslant I_N \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f_N(e^{i\theta})|^{2k}\, d\theta \right\} \leqslant k^k \left(\sum_0^N a_n^2 \right)^k \leqslant k^k,$$

$$v(k, N) \leqslant \tfrac{1}{2} k^{-2},$$

and so

$$V(N) \leqslant \sum_{k=1}^{\infty} v(k, N) \leqslant \sum_1^{\infty} \tfrac{1}{2} k^{-2} < 1.$$

It follows that the set complementary to S (in the unit cube) is not null; that is, a point $\{\alpha(1, N), \alpha(2, N), \ldots, \alpha(N, N)\}$ exists for which the corresponding f_N satisfies

$$M_{2k}^{2k}(\rho, f_N) \leqslant M_{2k}^{2k}(1, f_N) \leqslant 2k^{k+2} \tag{5.4}$$

for every $\rho < 1$ and every k.

If now we define $\alpha(n, N) = 0$ for $n \geqslant N$, the numbers $\alpha(n, N)$ are bounded, and by Lemma 1 there exists a sequence N_s and a set of a_n such that $\alpha(n, N_s) \to a_n$ as $s \to \infty$. We prove now

LEMMA 4.— $$f_{N_s}(z) \to f(z) = \Sigma a_n e^{2\pi i a_n} z^n$$

as $s \to \infty$, *uniformly in* $|z| \leqslant r < 1$.

In fact, there exists a $\nu = \nu(r, \epsilon)$ such that

$$\sum_{\nu+1}^{\infty} a_n r^n < \epsilon,$$

and when ν is fixed

$$\left| \sum_0^{\nu} a_n e^{2\pi i \alpha(n, N_s)} z^n - \sum_0^{\nu} a_n e^{2\pi i a_n} z^n \right| \leqslant \sum_1^{\nu} a_n \left| e^{2\pi i \alpha(n, N_s)} - e^{2\pi i a_n} \right| < \epsilon$$

for $s > s_0(\epsilon, \nu) = s_0(\epsilon, r)$. Then

$$|f_{N_s}(z) - f(z)| < 3\epsilon \quad (s > s_0),$$

and this proves the lemma.

It follows now from (5.4) that, for every $\rho < 1$ and every k,

$$M_{2k}^{2k}(\rho, f) \leqslant 2k^{k+2}. \tag{5.5}$$

The rest is easy. For $\lambda \leqslant 2$

$$M_\lambda(\rho, f) \leqslant M_2(\rho) = \mu(\rho) \leqslant 1 < A_1(\lambda).$$

For $\lambda > 2$ we have, taking $k = [\frac{1}{2}\lambda]+1 > 1$,

(5.6)
$$M_\lambda(\rho) \leqslant M_{2k}(\rho) < (2k)^{1/k}\sqrt{k} \leqslant 2\sqrt{k}$$
$$\leqslant \sqrt{(2\lambda+4)} = A_1(\lambda).$$

On the other hand, if $\lambda \geqslant 2$, then

$$M_\lambda(\rho) \geqslant M_2(\rho) = \mu(\rho) \geqslant A_2(\lambda)\mu(\rho)^{4\lambda^{-1}+1};$$

and if $\lambda < 2$, then, by Lemma 2, with $\alpha = \lambda$, $\beta = 2$, $\gamma = 4$, $\vartheta = \lambda/(4-\lambda)$, and by (5.6) with $\lambda = 4$,

$$\frac{M_\lambda}{M_2} \geqslant \left(\frac{M_2}{M_4}\right)^{\vartheta^{-1}-1} \geqslant \left(\frac{M_2}{4}\right)^{\vartheta^{-1}-1}.$$

That is

$$M_\lambda(\rho) \geqslant 4^{1-\vartheta^{-1}}\{\mu(\rho)\}^{\vartheta^{-1}} \geqslant A_2(\lambda)\{\mu(\rho)\}^{4\lambda^{-1}-1} \geqslant A_2(\lambda)\{\mu(\rho)\}^{4\lambda^{-1}+1}.$$

6. THEOREM 2.—*Let $a_n \geqslant 0$ and $\Sigma a_n r^n$ be convergent for $r < 1$. Suppose, further, that Σa_n^2 is divergent. Then there exists a set of real a_n such that every mean $M_\lambda(\rho, f)$, where $\lambda > 0$ and $f = \Sigma a_n e^{2\pi i a_n} z^n$, is unbounded as $\rho \to 1$. More precisely, we can find an f and a sequence ρ_n tending to 1 such that*

$$M_\lambda(\rho_n) \geqslant A(\lambda)\mu(\rho_n)[\log\{2+\mu(\rho_n)\}]^{2-4\lambda^{-1}}$$

for $\lambda \leqslant 2$ and all n, where

$$\mu(\rho) = \{\Sigma a_n^2 \rho^{2n}\}^{\frac{1}{2}}.$$

Since $M_\lambda(\rho) \geqslant \mu(\rho)$ for $\lambda > 2$ and $\mu(\rho) \to \infty$ as $\rho \to 1$, we need consider only the case $\lambda \leqslant 2$. Now

$$\Omega(\rho) = \log\{2+\mu(\rho)\}$$

is a positive, continuous, and increasing function of ρ, tending to ∞ as $\rho \to 1$. Hence, if we define ρ_n by $n = \Omega(\rho_n)$, we obtain an increasing sequence tending to 1. By Lemma 3,

(6.1)
$$I_N\{M_4^4(\rho, f_N)\} \leqslant AM_2^4(\rho, f_N) \leqslant \tfrac{1}{2}A_1\mu^4(\rho).$$

Let $v(n, N)$ be the measure of the set of P's for which

(6.2)
$$M_4^4(\rho_n, f_N) > A_1 n^4 \mu^4(\rho_n).$$

Then $v(n, N) \leqslant \frac{1}{2}n^{-4}$. Let now $V(N)$ be the measure of the set S for which (6 . 2) holds for some n. Then

$$V(N) \leqslant \sum_{n=1}^{\infty} v(n, N) \leqslant \Sigma \tfrac{1}{2}n^{-4} < 1,$$

the complementary set of S is not null, and there exists an f_N constructed with $a(1, N), \ldots, a(N, N)$, for which

$$M_4^4(\rho_n, f_N) \leqslant A_1 n^4 \mu^4(\rho_n) = A_1 \Omega^4(\rho_n)\mu^4(\rho_n)$$

for every n. It follows, by Lemma 4, that there exists an f such that, for every n,

$$M_4^4(\rho_n, f) \leqslant A\Omega^4(\rho_n)\mu^4(\rho_n).$$

Then, by Lemma 2, with $\alpha = \lambda$, $\beta = 2$, $\gamma = 4$,

$$M_\lambda(\rho_n) \geqslant M_2\left(\frac{M_2}{M_4}\right)^{\vartheta^{-1}-1} \geqslant \mu(\rho_n) A(\lambda)\{\Omega(\rho_n)\}^{-\vartheta^{-1}+1},$$

where $\vartheta^{-1} = 4\lambda^{-1} - 1$, and this is the desired result.

7. THEOREM 3.—*Let*

$$\beta \geqslant 0, \quad a_n \geqslant 0, \quad \mu(\rho) = \sqrt{\left(\sum_0^\infty a_n^2 \rho^{2n}\right)},$$

and suppose that $\mu(\rho) \leqslant \gamma(1-\rho)^{-\beta}$ $(\rho < 1)$. *Then there exists a set of real* a_n *such that* $f = \Sigma a_n e^{2\pi i a_n} z^n$ *satisfies*

$$M(\rho, f) \leqslant B\mu(\rho) + C = B\mu(\rho)\sqrt{\left(\log \frac{2}{1-\rho}\right)} + C$$

for every $\rho < 1$, *where* B *depends only on* β, *and* C *only on* β *and* γ.

LEMMA 5*.—*For* $r < 1$ *and any set of* a'*s*

$$\operatorname*{Max}_{|\zeta| \leqslant r} |f'(\zeta)| \leqslant B\gamma r^{-1}(1-r)^{-B_1},$$

in which we may suppose $B_1 > 2$.

For

$$|f'(\zeta)| \leqslant r^{-1}\Sigma n a_n r^n \leqslant r^{-1}\sqrt{(\Sigma a_n^2 r^n \Sigma n^2 r^n)} \leqslant Ar^{-1} M_2(r^{\frac{1}{2}})(1-r)^{-2}$$
$$\leqslant Ar^{-1}\gamma(1-r^{\frac{1}{2}})^{-\beta}(1-r)^{-2} < B\gamma r^{-1}(1-r)^{-\beta-2}.$$

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* Lemmas 5 and 6 are taken from the original paper.

LEMMA 6.—*There exists an increasing sequence* $\rho_1, \rho_2, \ldots$, *such that* $\rho_1 = \frac{1}{2}$ *and* $\rho_n \to 1$ *as* $n \to \infty$,

$$\rho_n - \rho_{n-1} = (1-\rho_n)^{B_1}, \tag{7.1}$$

$$\rho_n - \rho_{n-1} > Bn^{-B}, \tag{7.2}$$

and

$$\log(n+2) < B \log \frac{2}{1-\rho_{n-1}}, \tag{7.3}$$

where B_1 *is the number occurring in Lemma* 1.

Take $\rho_1 = \frac{1}{2}$ and write $u_n = 1-\rho_n$. The equation (7.1) becomes $u_{n-1} = u_n + u_n^{B_1}$. If $u_{n-1} > 0$ there is a single positive solution of this for u_n, and $u_n < u_{n-1}$. It follows that an increasing sequence, $\frac{1}{2}, \rho_2, \rho_3, \ldots$, exists, satisfying (7.1) and $\rho_n < 1$. (7.1) shows further that $\rho_n \to 1$. We write next $v_n = nu_n$. Then (7.1) gives

$$v_n - v_{n-1} = u_{n-1} - u_n^{B_1-1} v_n \geqslant u_{n-1} - u_n v_n.$$

It follows that either $v_n \geqslant 1$ or else $v_n \geqslant v_{n-1}$ (or both): this being true for each n, it is evident that $v_n > B$ for all n. Thus $1-\rho_n > Bn^{-1}$ and (7.2) follows from (7.1).

Finally, in proving (7.3), we may suppose $n > 3$, in which case

$$1 > \rho_n = \rho_1 + \sum_2^n (\rho_m - \rho_{m-1}) > \sum_2^{n-1} (\rho_m - \rho_{m-1})$$

$$= \sum_2^{n-1} (1-\rho_m)^{B_1} > (n-3)(1-\rho_{n-1})^{B_1},$$

$$n+2 \leqslant 6(n-3) < 6(1-\rho_{n-1})^{-B_1},$$

whence we obtain (7.3).

8. Returning now to Theorem 3 we have, from Lemma 3,

$$I_N\{M_{2k}^{2k}(\rho_n, f_N)\} \leqslant k^k \left(\sum_0^N a_n^2 \rho^{2n}\right)^k \leqslant k^k \mu^{2k}(\rho_n).$$

Let $v(k, n, N)$ be the measure of the set of P's for which

$$M_{2k}^{2k}(\rho_n, f_N) > 2n^3 k^k \mu^{2k}(\rho_n), \tag{8.1}$$

and $V(N)$ that of the set for which the inequality (8.1) holds for some n and some $k \leqslant n$. Then $v(k, n, N) \leqslant \frac{1}{2}n^{-3}$ and

$$V(N) \leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{n} v(k, n, N) \leqslant \sum_{1}^{\infty} \tfrac{1}{2}n^{-2} < 1.$$

The complementary set exists, so therefore does an f_N giving

$$M_{2k}^{2k}(\rho_n, f_N) \leqslant 2n^3 k^k \mu^{2k}(\rho_n) \quad (n > 1,\ k \leqslant n).$$

We now infer by a familiar argument the existence of an f satisfying

$$M_{2k}^{2k}(\rho_n, f) \leqslant 2n^3 k^k \mu^{2k}(\rho_n) \quad (n > 1,\ k \leqslant n). \tag{8.2}$$

I say now that this f satisfies $M(\rho, f) \leqslant B\mu(\rho)+C$ for every $\rho < 1$.

Let n be the integer for which $\rho_{n-1} \leqslant \rho < \rho_n$ and let $k \leqslant n$, so that (8.2) holds. Let ϕ satisfy $0 \leqslant \phi \leqslant \rho_n - \rho_{n-1}$, and let f attain its maximum M for $|z| = \rho$ at $z = z_0 = \rho e^{i\psi}$. Then

$$|f(\rho_{n-1} e^{i\psi+i\phi})| \geqslant M - \left| \int_{z_0}^{\rho_{n-1} e^{i\psi+i\phi}} f'(\zeta)\, d\zeta \right|,$$

where the path of integration is a straight line, and therefore lies in $|\zeta| \leqslant \rho$. By Lemma 5

$$|f(\rho_{n-1} e^{i\psi+i\phi})| \geqslant M - C\rho^{-1}(1-\rho)^{-B_1} s, \tag{8.3}$$

where $$s = |\rho_{n-1} e^{i\psi+i\phi} - z_0| \leqslant \phi\rho_{n-1} + (\rho_n - \rho_{n-1}) < 2(\rho_n - \rho_{n-1}).$$

Since $\rho_{n-1} \geqslant \frac{1}{2}$ (8.3) and Lemma 6 give

$$|f(\rho_{n-1} e^{i\psi+i\phi})| > M - C(\rho_n - \rho_{n-1})(1-\rho_n)^{-B_1} = M - C.$$

Hence for (each) $k \leqslant n$, either $M < C$, or else

$$M_{2k}^{2k}(\rho_{n-1}, f) \geqslant \frac{1}{2\pi} \int_0^{\rho_n - \rho_{n-1}} |f(\rho_{n-1} e^{i\psi+i\phi})|^{2k}\, d\phi.$$

$$\geqslant \frac{1}{2\pi} (\rho_n - \rho_{n-1})(M-C)^{2k} > Bn^{-B}(M-C)^{2k}$$

by Lemma 6, and so, by (8.2),

$$(M-C)^{2k} < Bn^B k^k \mu^{2k}(\rho_{n-1}) \leqslant Bn^B k^k \mu^{2k}(\rho),$$

$$M < Bn^{B/k} \sqrt{k}\, .\, \mu(\rho) + C. \tag{8.4}$$

The inequality (8.4) holds therefore when $\rho > \frac{1}{2}$ and $k \leqslant n$. We now take $k = [\log(n+2)] \leqslant n$, when (8.4) gives

$$M < B\sqrt{k}\,\mu(\rho)+C < B\sqrt{\{\log(n+2)\}}\,\mu(\rho)+C$$

$$< B\mu(\rho)\sqrt{\left(\log\frac{2}{1-\rho_{n-1}}\right)} < B\mu(\rho)\sqrt{\left(\log\frac{2}{1-\rho}\right)}+C.$$

The desired inequality is therefore proved when $\rho > \frac{1}{2}$. But when $\rho \leqslant \frac{1}{2}$,

$$M(\rho) \leqslant \Sigma a_n(\tfrac{1}{2})^n \leqslant \sqrt{\{\Sigma a_n^2(\tfrac{1}{2})^n \,.\, \Sigma(\tfrac{1}{2})^n\}} \leqslant A\mu(\tfrac{1}{2}) < C < B\mu(\rho)+C.$$

[*Note added* 3 *May*, 1926. If $\Sigma c_n z^n = \Sigma(a_n - ib_n)z^n$ has an unbounded $M_\lambda(\rho)$, where $\lambda \leqslant 1$, then the two series $\Sigma(a_n\cos n\theta + b_n \sin n\theta)$, $\Sigma(b_n\cos n\theta - a_n\sin n\theta)$ cannot both be Fourier series. For, if they were Fourier series, say of f and g, we should have, writing

$$h(\theta) = f(\theta) - ig(\theta), \qquad P(t) = \frac{1-\rho^2}{1-2\rho\cos t+\rho^2},$$

$$\phi(z) = \Sigma c_n\rho^n(\cos n\theta + i\sin n\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\phi)\,P(\phi-\theta)\,d\phi,$$

and therefore

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|\,\phi(z)\,|\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\,\frac{1}{2\pi}\int_{-\pi}^{\pi}|\,h(\phi)\,|\,P(\phi-\theta)\,d\phi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\,|\,h(\phi)\,|\,\frac{1}{2\pi}\int_{-\pi}^{\pi} P(\phi-\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\,h(\phi)\,|\,d\phi.$$

It follows now from Theorem 2 that no condition on the *moduli* of a set of coefficients $c_n = a_n - ib_n$ which permits $\Sigma(a_n^2+b_n^2)$ to diverge can possibly be a sufficient condition for the a_n and b_n to be a set of Fourier coefficients. In other words, the Riesz-Fischer theorem is a best possible one.

I take this opportunity of correcting a mistake in my paper "On inequalities in the theory of functions" in *Proc. London Math. Soc.* (2), 23 (1925), 518. The first foot-note states a falsehood and should be deleted.]

THE FOUNDATIONS OF MATHEMATICS

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Preface.

The object of this paper is to give a satisfactory account of the Foundations of Mathematics, in accordance with the general method of Frege, Whitehead, and Russell. Following these authorities, I hold that mathematics is part of logic, and so belong to what may be called the logical school as opposed to the formalist and intuitionist schools. I have therefore taken *Principia Mathematica* as a basis for discussion and amendment, and believe myself to have discovered how, by using the work of Mr. Ludwig Wittgenstein, it can be rendered free from the serious objections which have caused its rejection by the majority of German authorities who have deserted altogether its line of approach.

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I. *Introduction.*

In this chapter we shall be concerned with the general nature of pure mathematics*, and how it is distinguished from other sciences.

* In future by "mathematics" will always be meant "pure mathematics."

Here there are really two distinct categories of things of which an account must be given, the ideas or concepts of mathematics, and the propositions of mathematics. This distinction is neither artificial nor unnecessary, for the great majority of writers on the subject have concentrated their attention on the explanation of one or other of these categories, and erroneously supposed that a satisfactory explanation of the other would immediately follow.

Thus the formalist school, of whom the most eminent representative is now Hilbert, have concentrated on the propositions of mathematics, such as "2+2 = 4", which they have pronounced to be meaningless formulae to be manipulated according to certain arbitrary rules, and mathematical knowledge they hold to consist in knowing what formulae can be derived from what others consistently with the rules. Such being the propositions of mathematics, the account of its concepts, for example the number 2, immediately follows. "2" is a meaningless mark occurring in these meaningless formulae. But, whatever may be thought of this as an account of mathematical propositions, it is obviously hopeless as a theory of mathematical concepts; for these occur not only in mathematical propositions, but also in those of everyday life. Thus "2" occurs not merely in "2+2 = 4", but also in "It is 2 miles to the station", which is not a meaningless formula, but a significant proposition, in which "2" cannot conceivably be a meaningless mark. Nor can there be any doubt that "2" is used in the same sense in the two cases, for we can use "2+2 = 4" to infer from "It is two miles to the station and two miles on to the Gogs", that "it is four miles to the Gogs via the station", so that these ordinary meanings of two and four are clearly involved in "2+2 = 4". So the hopelessly inadequate formalist theory is, to some extent, the result of considering only the propositions of mathematics, and neglecting the analysis of its concepts, on which additional light can be thrown by their occurrence outside mathematics in the propositions of everyday life.

Apart from formalism, there are two main general attitudes to the foundation of mathematics: that of the intuitionists or finitists like Brouwer and Weyl in his recent papers, and that of the logicians, Frege, Whitehead, and Russell. The theories of the intuitionists admittedly involve giving up many of the most fruitful methods of modern analysis, for no reason, as it seems to me, except that the methods fail to conform to their private prejudices. They do not, therefore, profess to give any foundation for mathematics as we know it, but only for a narrower body of truth which has not yet been clearly

defined. There remain the logicians whose work culminated in *Principia Mathematica*. The theories there put forward are generally rejected for reasons of detail, especially the apparently insuperable difficulties connected with the axiom of reducibility. But these defects in detail seem to me to be results of an important defect in principle, first pointed out by Mr. Wittgenstein.

The logical school has concentrated on the analysis of mathematical concepts, which it has shown to be definable in terms of a very small number of fundamental logical concepts, and, having given this account of the concepts of mathematics, they have immediately deduced an account of mathematical propositions, namely that they were those true propositions in which only mathematical or logical concepts occurred. Thus Russell, in the *Principles of Mathematics*, defines pure mathematics as "the class of all propositions of the form p implies q, where p and q are propositions containing one or more variables the same in the two propositions, and neither p nor q contains any constants except logical constants"*. This reduction of mathematics to symbolic logic was rightly described by Mr. Russell as one of the greatest discoveries of our age†; but it was not the end of the matter, as he seemed to suppose, because he was still far from an adequate conception of the nature of symbolic logic, to which mathematics had been reduced. I am not referring to his naive theory that logical constants were names for real objects (which he has since abandoned), but to his belief that any proposition which could be stated by using logical terms‡ alone must be a proposition of logic or mathematics§. I think the question is made clearer by describing the class of propositions in question as the completely general propositions, emphasizing the fact that they are not about particular things or relations, but some or all things and relations. It is really obvious that not all such propositions are propositions of mathematics or symbolic logic. Take, for example, "any two things differ in at least thirty ways"; this is a completely general proposition, it could be expressed as an implication involving only logical constants and variables, and it may well be true. But as a mathematical or logical truth no one could regard it; it is utterly different from such a proposition as "any two things together with any other two things make four things", which is a logical and not merely an empirical

* Russell, *Principles of Mathematics*, 1 (1903), 3.

† *Loc. cit.*, 5.

‡ *I.e.* variables and logical constants.

§ I neglect here, as elsewhere, the arbitrary and trivial proviso that the proposition must be of the form "p implies q".

truth. According to our philosophy we may differ in calling the one a contingent, the other a necessary proposition, or the one a genuine proposition, the other a mere tautology; but we must all agree that there is some essential difference between the two, and that a definition of mathematical propositions must include not merely their complete generality but some further property as well. This is pointed out, with a reference to Wittgenstein, in Russell's *Introduction to Mathematical Philosophy**; but there is no trace of it in *Principia Mathematica*, nor does Mr. Russell seem to have understood its tremendous importance, for example, in the consideration of primitive propositions. In the passage referred to in the *Introduction to Mathematical Philosophy*, Mr. Russell distinguishes between propositions which can be enunciated in logical terms from those which logic can assert to be true, and gives as the additional characteristic of the latter that they are "tautological" in a sense which he cannot define. It is obvious that a definition of this characteristic is essential for a clear foundation of our subject, since the idea to be defined is one of the essential sides of mathematical propositions, their content, and their form. Their content must be completely generalized, and their form tautological.

The formalists neglected the content altogether and made mathematics meaningless, the logicians neglected the form and made mathematics consist of any true generalizations; only by taking account of both sides and regarding it as composed of tautologous generalizations can we obtain an adequate theory.

We have now to explain a definition of tautology which has been given by Mr. Wittgenstein in his *Tractatus Logico-Philosophicus*, and forms one of the most important of his contributions to the subject. In doing this we cannot avoid some explanation of his theory of propositions in general.

We must begin with the notion of an *atomic proposition*†; this is one which could not be analysed in terms of other propositions, and could consist of names alone without logical constants. For instance, by joining to "ϕ", the name of a quality, "a", the name of an individual, and writing "ϕa", we have an atomic proposition, asserting that the individual has the quality. Thus, if we neglect the fact that "Socrates" and "wise" are incomplete symbols and regard them as names, "Socrates is wise" is an atomic proposition; but "All men are wise", "Socrates is not wise", are not atomic.

* Page 205.

† Wittgenstein calls these "elementary propositions"; I have called them "atomic" in order to follow Mr. Russell in using "elementary" with a different meaning.

Suppose now we have, say, n atomic propositions p, q, r, ... With regard to their truth or falsity there are 2^n mutually exclusive ultimate possibilities, which we could arrange in a table like this (T signifies truth, and F falsity, and we have taken $n = 2$ for brevity).

p	q
T	T
T	F
F	T
F	F

These 2^n possibilities we will call the truth-possibilities of the n atomic propositions. We may wish to pick out any sub-set of them, and assert that it is a possibility out of this sub-set which is, in fact, realized—that is, to express our agreement with some of the possibilities and our disagreement with the remainder. We can do this by setting marks T and F against the possibilities with which we agree and disagree respectively. In this way we obtain a proposition.

Thus

"

p	q	
T	T	F
T	F	T
F	T	T
F	F	T

"

is the proposition "not both p and q are true", or "p is incompatible with q", for we have allowed all the possibilities except the first, which we have disallowed.

Similarly "

p	q	
T	T	T
F	T	T
T	F	F
F	F	T

"

is the proposition "if p, then q".

A proposition which expresses agreement and disagreement with the truth-possibilities of p, q, ... (which need not be atomic) is called a truth-function of the arguments p, q, ... Or, more accurately, P is said to be the same truth-function of p, q, ... as R is of r, s, ... if P expresses agreement with the truth-possibilities of p, q, ... corresponding by the substitution of p for r, q for s, ... to the truth-possibilities of r, s, ... with which R expresses agreement. Thus "p and q" is the same truth-function of p, q as "r and s" is of r, s, in each case the only possibility allowed being that both the arguments are true. Mr. Wittgenstein has perceived that, if we accept this account of truth-functions as expressing agreement and disagreement with truth-possibilities, there is no reason why the arguments to a truth-function should not be infinite in number*. As no previous writer has considered truth-functions as capable of more than a finite number of arguments, this is a most important innovation. Of course, if the arguments are infinite in number they cannot all be enumerated and written down separately; but there is no need for us to enumerate them if we can determine them in any other way, as we can by using propositional functions.

A propositional function is an expression of the form "$f\hat{x}$", which is such that it expresses a proposition, when any symbol (of a certain appropriate logical type depending on f) is substituted for "$\hat{x}$". Thus "$\hat{x}$ is a man" is a propositional function. We can use propositional functions to collect together the range of propositions, which are all the values of the function for all possible values of x. Thus "$\hat{x}$ is a man" collects together all the propositions "a is a man", "b is a man",

* Thus the logical sum of a set of propositions is the proposition that one at least of the set is true, and it is immaterial whether the set is finite or infinite. On the other hand, an infinite algebraic sum is not really a sum at all but a *limit*, and so cannot be treated as a sum except subject to certain restrictions.

etc. Having now by means of a propositional function defined a set of propositions, we can, by using an appropriate notation, assert the logical sum or product of this set. Thus, by writing "$(x) . fx$", we assert the logical product of all propositions of the form "fx"; by writing "$(\exists x) . fx$" we assert their logical sum. Thus "$(x) . x$ is a man" would mean "everything is a man", "$(\exists x) . x$ is a man", "there is something which is a man". In the first case we allow only the possibility that all the propositions of the form "x is a man" are true; in the second, we exclude only the possibility that all the propositions of the form "x is a man" are false.

Thus general propositions containing "all" and "some" are found to be truth-functions, for which the arguments are not enumerated but given in another way. But we must guard here against a possible mistake. Take such a proposition as "all men are mortal"; this is not as might at first sight be supposed the logical product of the propositions "x is mortal" for such values of x as are men. Such an interpretation can easily be shown to be erroneous (see, for example, *Principia Mathematica*, 1st ed., I, 47). "All men are mortal" must be interpreted as meaning "(x) . if x is a man, x is mortal", *i.e.* it is the logical product of all the values of the function "if x is a man, x is mortal".

Mr. Wittgenstein maintains that all propositions are, in the sense defined, truth-functions of elementary propositions. This is hard to prove, but is on its own merits extremely plausible; it says that, when we assert anything, we are saying that it is one out of a certain group of ultimate possibilities which is realized, not one out of the remaining possibilities. Also it applies to all the propositions which could be expressed in the symbolism of *Principia Mathematica*, since these are built up from atomic propositions by using, first, conjunctions like "if", "and", "or"; and, secondly, various kinds of generality (apparent variables). And both these methods of construction have been shown to create truth-functions*.

From this account we see when two propositional symbols are to be regarded as instances of the same proposition, namely, when they express agreement and disagreement with the same sets of truth-possibilities of atomic propositions.

Thus in the symbolism of *Principia Mathematica*

$$"p \supset q : \sim p . \supset . q", \quad "q \vee : p . \sim p"$$

are both more complicated ways of writing "q".

* The form "A believes p" will perhaps be suggested as doubtful. This is clearly not a truth-function of "p", but may nevertheless be one of other atomic propositions.

Given any set of n atomic propositions as arguments, there are 2^n corresponding truth-possibilities, and therefore 2^{2^n} sub-classes of their truth-possibilities, and so 2^{2^n} truth-functions of n arguments, one expressing agreement with each sub-class and disagreement with the remainder. But among these 2^{2^n} there are two extreme cases of great importance: one in which we express agreement with all the truth-possibilities, the other in which we express agreement with none of them. A proposition of the first kind is called a *tautology*, of the second a *contradiction*. Tautologies and contradictions are not real propositions, but degenerate cases. We may, perhaps, make this clear most easily by taking the simplest case, when there is only one argument.

The tautology is "

p	
T	T
F	T

", *i.e.* "p or not p".

This really asserts nothing whatever; it leaves you no wiser than it found you. You know nothing about the weather, if you know that it is either raining or not raining*.

The contradiction is "

p	
T	F
F	F

", *i.e.* "p is neither true nor false".

This is clearly self-contradictory and does not represent a possible state of affairs whose existence could be asserted.

Tautologies and contradictions can be of all degrees of complexity; to give other examples "$(x) . \phi x : \supset : \phi a$" is a tautology. "$\sim . (\exists x) . \phi x : \phi a$" a contradiction. Clearly by negating a contradiction we get a tautology, and by negating a tautology a contradiction. It is important to see that tautologies are not simply true propositions, though for many purposes they can be treated as true propositions. A genuine proposition asserts something about reality, and it is true if reality is as it is asserted to be. But a tautology is a symbol constructed

* Wittgenstein, *Tractatus Logico-Philosophicus*, 4·461.

so as to say nothing whatever about reality, but to express total ignorance by agreeing with every possibility. The assimilation of tautologies and contradictions with true and false propositions respectively results from the fact that tautologies and contradictions can be taken as arguments to truth-functions just like ordinary propositions, and for determining the truth or falsity of the truth-function, tautologies and contradictions among its arguments must be counted as true and false respectively. Thus, if "t" be a tautology, "c" a contradiction, "t and p", "if t, then p", "c or p" are the same as "p", and "t or p", "if c, then p" are tautologies.

We have here, thanks to Mr. Wittgenstein, to whom the whole of this analysis is due, a clearly defined sense of tautology; but is this, it may be asked, the sense in which we found tautology to be an essential characteristic of the propositions of mathematics and symbolic logic? The question must be decided by comparison. Are the propositions of symbolic logic and mathematics tautologies in Mr. Wittgenstein's sense?

Let us begin by considering not the propositions of mathematics but those of *Principia Mathematica**. These are obtained by the process of deduction from certain primitive propositions, which fall into two groups; those expressed in symbols and those expressed in words. Those expressed in words are nearly all nonsense by the theory of types, and should be replaced by symbolic conventions. The real primitive propositions, those expressed in symbols, are, with one exception, tautologies in Wittgenstein's sense. So, as the process of deduction is such that from tautologies only tautologies follow, were it not for one blemish the whole structure would consist of tautologies. The blemish is, of course, the Axiom of Reducibility, which is, as will be shown below†, a genuine proposition, whose truth or falsity is a matter of brute fact, not of logic. It is, therefore, not a tautology in any sense, and its introduction into mathematics is inexcusable. But suppose it could be dispensed with, and *Principia Mathematica* were modified accordingly, it would consist entirely of tautologies in Wittgenstein's sense. And therefore if *Principia Mathematica* is on the right lines as a foundation and interpretation of mathematics, it is Wittgenstein's sense of tautology in which mathematics is tautologous.

But the adequacy of *Principia Mathematica* is a matter of detail, and, since we have seen it contains a very serious flaw, we can no longer be

* This is only because *Principia Mathematica may* be a wrong interpretation of mathematics; in the main I think it is a right one.

† See Chapter V.

sure that mathematics is the kind of thing Whitehead and Russell suppose it to be, or therefore that it consists of tautologies in Wittgenstein's sense. One thing is, however, clear: that mathematics does not consist of genuine propositions or assertions of fact which could be based on inductive evidence, as it was proposed to base the axiom of reducibility, but is in some sense necessary or tautologous. In actual life, as Wittgenstein says, "it is never a mathematical proposition which we really need, but we use mathematical propositions *only* in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics*". Thus we use "$2 \times 2 = 4$" to infer from "I have two pennies in each of my two pockets" to "I have four pennies altogether in my pockets". "$2 \times 2 = 4$" is not itself a genuine proposition in favour of which inductive evidence can be required, but a tautology which can be seen to be tautologous by anyone who can fully grasp its meaning. When we proceed further in mathematics the propositions become so complicated that we cannot see immediately that they are tautologous, and have to assure ourselves of this by deducing them from more obvious tautologies. The primitive propositions on which we fall back in the end must be such that no evidence could be required for them, since they are patent tautologies like "if p then p". But the tautologies of which mathematics consist may perhaps turn out not to be of Wittgenstein's kind, but of some other. Their essential use is to facilitate logical inference; this is achieved in the most obvious way by constructing tautologies in Wittgenstein's sense, for if "if p, then q" is a tautology, we can logically infer "q" from "p", and, conversely, if "q" follows logically from "p", "if p, then q" is a tautology†. But it is possible that there are other kinds of formulae which could be used to facilitate inference; for instance, what we may call identities such as "$a = b$", signifying that "a", "b" may be substituted for one another in any proposition without altering it. I do not mean without altering its truth or falsity, but without altering what proposition it is. "$2+2 = 4$" might well be an identity in this sense, since "I have $2+2$ hats" and "I have 4 hats" are the same proposition, as they agree and disagree with the same sets of ultimate truth-possibilities.

Our next problem is to decide whether mathematics consists of tautologies (in the precise sense defined by Wittgenstein, to which we

* Wittgenstein, *op. cit.*, 6·211.

† This may perhaps be made clearer by remarking that if "q" follows logically from "p", "$p \,.\, \sim q$" must be self-contradictory, therefore "$\sim (p \,.\, \sim q)$" tautologous or "$p \supset q$" tautologous.

shall in future confine the word "tautology") or of formulae of some other sort. It is fairly clear that geometry, in which we regard such terms as "point", "line" as meaning any things satisfying certain axioms, so that the only constant terms are truth-functions like "or", "some", consists of tautologies. And the same would be true of analysis, if we regarded numbers as any things satisfying Peano's axioms. Such a view would, however, be certainly inadequate, because since the numbers from 100 on satisfy Peano's axioms, it would give us no means of distinguishing "This equation has three roots" from "This equation has a hundred and three roots." So numbers must be defined not as variables but as constants, and the nature of the propositions of analysis becomes doubtful.

I believe that they are tautologies, but the proof of this depends on giving a detailed analysis of them, and the disproof of any other theory would depend on finding an insuperable difficulty in the details of its construction. In this chapter I propose to discuss the question in a general way, which must inevitably be rather vague and unsatisfactory. I shall first try to explain the great difficulties which a theory of mathematics as tautologies must overcome, and then I shall try to explain why the alternative sort of theory suggested by these difficulties seems hopelessly impracticable. Then in the following chapters I shall return to the theory that mathematics consists of tautologies, discuss and partially reject the method for overcoming the difficulties given in *Principia Mathematica*, and construct an alternative and, to my mind, satisfactory solution.

Our first business is, then, the difficulties of the tautology theory. They spring from a fundamental characteristic of modern analysis which we have now to emphasize. This characteristic may be called *extensionality*, and the difficulties may be explained as those which confront us if we try to reduce a calculus of extensions to a calculus of truth-functions. Here, of course, we are using "extension" in its logical sense, in which the extension of a predicate is a class, that of a relation a class of ordered couples; so that in calling mathematics extensional we mean that it deals not with predicates but with classes, not with relations in the ordinary sense but with possible correlations, or "relations in extension," as Mr. Russell calls them. Let us take as examples of this point three fundamental mathematical concepts, the idea of a real number, the idea of a function (of a real variable), and the idea of similarity of classes (in Cantor's sense).

Real numbers are defined as segments of rationals; any segment of rationals is a real number, and there are $2^{\aleph_0}$ of them. It is not

necessary that the segment should be defined by any property or predicate of its members, in any ordinary sense of predicate. A real number is therefore an extension, and it may even be an extension with no corresponding intension. In the same way a function of a real variable is a relation in extension, which need not be given by any real relation or formula.

The point is perhaps most striking in the Cantor's definition of similarity. Two classes are said to be similar (*i.e.* have the same cardinal number) when there is a one-one relation whose domain is the one class and converse domain the other. Here it is essential that the one-one relation need only be a relation in extension; it is obvious that two classes could be similar—*i.e.* capable of being correlated without there being any relation actually correlating them.

There is a verbal point which requires mention here; I do not use the word "class" to imply a principle of classification, as the word naturally suggests, but by a "class" I mean any set of things of the same logical type. Such a set, it seems to me, may or may not be definable either by enumeration or as the extension of a predicate. If it is not so definable we cannot mention it by itself, but only deal with it by implication in propositions about all classes, or some classes. The same is true of relations in extension, by which I do not merely mean the extensions of actual relations, but any sets of ordered couples. That this is the notion occurring in mathematics seems to me absolutely clear from the last of the above examples, Cantor's definition of similarity, where obviously there is no need for the one-one relation in extension to be either finite or the extension of an actual relation.

Mathematics is therefore essentially extensional, and may be called a calculus of extensions, since its propositions assert relations between extensions. This, as we have said, is hard to reduce to a calculus of truth-functions, to which it must be reduced if mathematics is to consist of tautologies, for tautologies are truth-functions of a certain special sort, namely those agreeing with all the truth-possibilities of their arguments. We can perhaps most easily explain the difficulty by an example.

Let us take an extensional assertion of the simplest possible sort: the assertion that one class includes another. So long as the classes are defined as the classes of things having certain predicates ϕ and ψ, there is no difficulty. That the class of ψ's includes the class of ϕ's means simply that everything which is a ϕ is a ψ, which, as we saw in Chapter I, is a truth-function. But we have seen that mathematics has (at least apparently) to deal also with classes which are not given

by defining predicates. (Such classes occur not merely when mentioned separately, but also in any statement about "all classes", "all real numbers".) Let us take two such classes as simple as possible, the class (a, b, c) and the class (a, b). Then that the class (a, b, c) includes the class (a, b) is, in a broad sense, tautological and apart from its triviality would be a mathematical proposition; but it does not seem to be a tautology in Wittgenstein's sense, that is a certain sort of truth-function of elementary propositions. The obvious way of trying to make it a truth-function is to introduce identity and write "(a, b) is contained in (a, b, c)" as "$(x) : . x = a . \vee . x = b : \supset : x = a . \vee . x = b . \vee . x = c$". This certainly looks like a tautological truth-function, whose ultimate arguments are values of "$x = a$", "$x = b$", "$x = c$", that is propositions like "$a = a$", "$b = a$", "$d = a$". But these are not real propositions at all; in "$a = b$" either "a", "b" are names of the same thing, in which case the proposition says nothing, or of different things, in which case it is absurd. In neither case is it the assertion of a fact; it only appears to be a real assertion by confusion with the case when "a" or "b" is not a name, but a description*. When "a", "b" are both names, the only significance which can be placed on "$a = b$" is that it indicates that we use "a", "b" as names of the same thing or, more generally, as equivalent symbols.

The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called "equations", for which I should prefer to substitute "identities". That is, formulae of the form "$a = b$" where "a", "b" are equivalent symbols. There is a certain plausibility in such an account of, for instance, "$2+2 = 4$". Since "I have 2+2 hats", "I have 4 hats" are the same proposition†, "2+2" and "4" are equivalent symbols. As it stands this is obviously a ridiculously narrow view of mathematics, which confines it to simple arithmetic, but it is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found that it was faced with what seemed to me insuperable difficulties. It would be out of place here to give a detailed survey of this blind alley, but I shall try to indicate in a general way the obstructions which block its end.

First of all we have to consider of what kind mathematical pro-

* For a fuller discussion of identity see the next chapter.

† In the sense explained in Chapter I. They clearly are not the same sentence, but they are the same truth-function of atomic propositions and so assert the same fact.

positions will on such a theory be. We suppose the most primitive type to be the identity "$a = b$", which only becomes a real proposition if it is taken to be about not the things meant by "a", "b", but these symbols themselves; mathematics then consists of propositions built up out of identities by a process analogous to that by which ordinary propositions are constructed out of atomic ones; that is to say, mathematical propositions are (on this theory), in some sense, truth-functions of identities. Perhaps this is an overstatement, and the theory might not assert all mathematical propositions to be of this form, but it is clearly one of the important forms that would be supposed to occur. Thus

$$"x^2-3x+2 = 0 : \supset_x : x = 2 \,.\, \mathrm{v} \,.\, x = 1"$$

would be said to be of this form, and would correspond to a verbal proposition which was a truth-function of the verbal propositions corresponding to the arguments "$x = 2$", etc. Thus the above proposition would amount to "If 'x^2-3x+2' means 0, 'x' means 2 or 1". Mathematics would then be, in part at least, the activity of constructing formulae which corresponded in this way to verbal propositions. Such a theory would be difficult and perhaps impossible to develop in detail, but there are, I think, other and simpler reasons for dismissing it. These arise as soon as we cease to treat mathematics as an isolated structure, and consider the mathematical elements in non-mathematical propositions. For simplicity let us confine ourselves to cardinal numbers, and suppose ourselves to know the analysis of the proposition that the class of ϕ's is n in number $\left(\hat{x}(\phi x) \epsilon n\right)$. Here ϕ may be any ordinary predicate defining a class, *e.g.* the class of ϕ's may be the class of Englishmen. Now take such a proposition as "The square of the number of ϕ's is greater by two than the cube of the number of ψ's". This proposition we cannot, I think, help analysing in this sort of way.

$$(\exists m, n) \,.\, \hat{x}(\phi x) \epsilon m \,.\, \hat{x}(\psi x) \epsilon n \,.\, m^2 = n^3+2.$$

It is an empirical not a mathematical proposition, and is about the ϕ's and ψ's, not about symbols; yet there occurs in it the mathematical pseudo-proposition $m^2 = n^3+2$, of which, according to the theory under discussion, we can only make sense by taking it to be about symbols, thereby making the whole proposition partly about symbols. Moreover, being an empirical proposition, it is a truth-function of elementary propositions expressing agreement with those possibilities which give numbers of ϕ's and ψ's satisfying $m^2 = n^3+2$. Thus "$m^2 = n^3+2$" is not, as it seems to be, one of the truth-arguments in the proposition

above, but rather part of the truth-function like "$\sim$" or "$\vee$" or "$\exists$, m, n", which determine which truth-function of elementary propositions it is that we are asserting. Such a use of $m^2 = n^3+2$ the identity theory of mathematics is quite inadequate to explain.

On the other hand, the tautology theory would do everything which is required; according to it $m^2 = n^3+2$ would be a tautology for the values of m and n which satisfy it, and a contradiction for all others. So

$$\hat{x}(\phi x) \in m \,.\, \hat{x}(\psi x) \in n \,.\, m^2 = n^3+2$$

would for the first set of values of m, n be equivalent to

$$\hat{x}(\phi x) \in m \,.\, \hat{x}(\psi x) \in n$$

simply, "$m^2 = n^3+2$" being tautologous, and therefore superfluous; and for all other values it would be self-contradictory. So that

$$"(\exists m, n) : \hat{x}(\phi x) \in m \,.\, \hat{x}(\psi x) \in n \,.\, m^2 = n^3+2"$$

would be the logical sum of the propositions

$$"\hat{x}(\phi x) \in m \,.\, \hat{x}(\psi x) \in n"$$

for all m, n satisfying $m^2 = n^3+2$, and of contradictions for all other m, n, and is therefore the proposition we require, since in a logical sum the contradictions are superfluous. So this difficulty, which seems fatal to the identity theory, is escaped altogether by the tautology theory, which we are therefore encouraged to pursue, and to see if we cannot find a way of overcoming the difficulties which we found would confront us in attempting to reduce an extensional calculus to a calculus of truth-functions. Such a solution is attempted in *Principia Mathematica*, and will be discussed in the next chapter; but before we proceed to this we must say something about the well known contradictions of the theory of aggregates which our theory will also have to escape.

It is not sufficiently remarked, and the fact is entirely neglected in *Principia Mathematica*, that these contradictions fall into two fundamentally distinct groups, which we will call A and B. The best known ones are divided as follows :—

A. (1) The class of all classes which are not members of themselves.

(2) The relation between two relations when one does not have itself to the other.

(3) Burali Forti's contradiction of the greatest ordinal.

B. (4) "I am lying".

(5) The least integer not nameable in fewer than nineteen syllables.

(6) The least indefinable ordinal.

(7) Richard's Contradiction.

(8) Weyl's Contradiction about "heterologisch"*.

The principle according to which I have divided them is of fundamental importance. Group A consists of contradictions, which, were no provision made against them, would occur in a logical or mathematical system itself. They involve only logical or mathematical terms such as class and number, and show that there must be something wrong with our logic or mathematics. But the contradictions of Group B are not purely logical, and cannot be stated in logical terms alone, for they all contain some reference to thought, language, or symbolism, which are not formal but empirical terms. So they may be due not to faulty logic or mathematics, but to faulty ideas concerning thought and language. If so, they would not be relevant to mathematics or to logic, if by "logic" we mean a symbolic system, though, of course, they would be relevant to logic in the sense of the analysis of thought†.

This view of the second group of contradictions is not original. For instance, Peano decided that "Exemplo de Richard non pertine ad Mathematica, sed ad linguistica"‡, and therefore dismissed it. But such an attitude is not completely satisfactory. We have contradictions involving both mathematical and linguistic ideas; the mathematician dismisses them by saying that the fault must lie in the linguistic elements, but the linguistician may equally well dismiss it for the opposite reason, and the contradictions will never be solved. The only solution which has ever been given§, that in *Principia Mathematica*, definitely attributed the contradictions to bad logic, and it is up to opponents of this view to show clearly the fault in what Peano called linguistics, but

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* For the first seven of these see *Principia Mathematica*, 1 (1910), 63. For the eighth see Weyl, *Das Kontinuum*, 2.

† These two meanings of "logic" are frequently confused. It really should be clear that those who say mathematics is logic are not meaning by "logic" at all the same thing as those who define logic as the analysis and criticism of thought.

‡ *Rivista di Mat.*, 8 (1906), 157.

§ Other so-called solutions are merely inadequate excuses for not giving a solution.

what I should prefer to call epistemology, to which these contradictions are due.

II. PRINCIPIA MATHEMATICA.

In the last chapter I tried to explain the difficulties which faced the theory that the propositions of mathematics are tautologies; in this we have to discuss the attempted solution of these difficulties given in *Principia Mathematica*. I shall try to show that this solution has three important defects, and the remainder of this essay will be devoted to expounding a modified theory from which these defects have been removed.

The theory of *Principia Mathematica* is that every class or aggregate (I use the words as synonyms) is defined by a propositional function—that is, consists of the values of x, for which "ϕx" is true, where "ϕx" is a symbol which expresses a proposition if any symbol of appropriate type be substituted for "x". This amounts to saying that every class has a defining property. Let us take the class consisting of a and b; why, it may be asked, must there be a function $\phi\hat{x}$ such that "ϕa", "ϕb" are true, but all other "ϕx" 's false? This is answered by giving as such a function "$x = a \,.\, \vee \,.\, x = b$". Let us, for the present, neglect the difficulties connected with identity, and accept this answer; it shows us that any finite class is defined by a propositional function constructed by means of identity, but as regards infinite classes it leaves us exactly where we were before, that is, without any reason to suppose that they are all defined by propositional functions, for it is impossible to write down an infinite series of identities. To this it will be answered that a class can only be given to us either by enumeration of its members, in which case it must be finite, or by giving a propositional function which defines it. So that we cannot be in any way concerned with infinite classes or aggregates, if such there be, which are not defined by propositional functions.* But this argument contains a common mistake, for it supposes that, because we cannot consider a thing individually, we can have no concern with it at all. Thus, although an infinite indefinable class cannot be mentioned by itself, it is, nevertheless, involved in any statement beginning "All classes" or "There is a class such that", and if indefinable classes are excluded the meaning of all such statements will be fundamentally altered.

Whether there are indefinable classes or not is an empirical ques-

* For short, I shall call such classes "indefinable classes".

tion; both possibilities are perfectly conceivable. But even if, in fact, all classes are definable, we cannot in our logic identify classes with definable classes without destroying the apriority and necessity which is the essence of logic. But in case any one still thinks that by classes we mean definable classes, and by "there is a class", "there is a definable class," let him consider the following illustration. This illustration does not concern exactly this problem, but the corresponding problem for two variables, the existence of relations in extension not definable by propositional functions of two variables. But this question is clearly so analogous to the other that the answers to both must be the same.

Consider the proposition "$\hat{x}(\phi x)\,\mathrm{sm}\,\hat{x}(\psi x)$" (*i.e.* the class defined by $\phi\hat{x}$ has the same cardinal as that defined by $\psi\hat{x}$); this is defined to mean that there is a one-one relation in extension whose domain is $\hat{x}(\phi x)$ and whose converse domain is $\hat{x}(\psi x)$. Now if by relation in extension we mean definable relation in extension, this means that two classes have the same cardinal only when there is a real relation or function $f(x, y)$ correlating them term by term. Whereas clearly what was meant by Cantor, who first gave this definition, was merely that the two classes were such that they could be correlated, not that there must be a propositional function which actually correlated them* Thus the classes of male and female angels may be infinite and equal in number, so that it would be possible to pair off completely the male with the female, without there being any real relation such as marriage correlating them. The possibility of indefinable classes and relations in extension is an essential part of the extensional attitude of modern mathematics which we emphasized in Chapter I, and that it is neglected in *Principia Mathematica* is the first of the three great defects in that work. The mistake is made not by having a primitive proposition, asserting that all classes are definable, but by giving a definition of class which applies only to definable classes, so that all mathematical propositions about some or all classes are misinterpreted. This misinterpretation is not merely objectionable on its own account in a general way, but is especially pernicious in connexion with the multiplicative axiom, which is a tautology when properly interpreted; but when misinterpreted after the fashion of *Principia Mathematica* becomes a significant empirical proposition, which there is no reason to suppose true. This will be shown in Chapter V.

The second defect in *Principia Mathematica* represents a failure to

* Cf. Johnson, *Logic*, 2 (1922), 159.

overcome, not, like the first, the difficulties raised by the extensionality of mathematics, but those raised by the contradictions discussed at the end of Chapter I. These contradictions it was proposed to remove by what is called the Theory of Types, which consists really of two distinct parts directed respectively against the two groups of contradictions. These two parts were unified by being both deduced in a rather sloppy way from the "vicious-circle principle", but it seems to me essential to consider them separately.

The contradictions of Group A are removed by pointing out that a propositional function cannot significantly take itself as argument, and by dividing functions and classes into a hierarchy of types according to their possible arguments. Thus the assertion that a class is a member of itself is neither true nor false, but meaningless. This part of the theory of types seems to me unquestionably correct, and I shall not discuss it further.

The first part of the theory, then, distinguishes types of propositional functions by their arguments; thus there are functions of individuals, functions of functions of individuals, functions of functions of functions of individuals, and so on. The second part designed to meet the second group of contradictions requires further distinctions between the different functions which take the same arguments, for instance between the different functions of individuals. The following explanation of these distinctions is based on the introduction to the second edition of *Principia Mathematica*.

We start with atomic propositions, which have been explained in Chapter I. Out of these by means of the stroke (p/q = not both p and q are true) we can construct any truth-function of a finite number of atomic propositions as arguments. The assemblage of propositions so obtained are called elementary propositions. By substituting a variable for the name of an individual in one or more of its occurrences in an elementary proposition we obtain an elementary function of individuals. An *elementary function* of individuals, "$\phi\hat{x}$" is therefore one whose values are elementary propositions, that is, truth-functions of a finite number of atomic propositions. Such functions were called, in the first edition of *Principia Mathematica*, *predicative functions*. We shall speak of them by their new name, and in the next chapter use "predicative function" in a new and original sense, for which it seems more appropriate. In general, an elementary function or "matrix" of one or more variables, whether these are individuals or not, is one whose values are elementary propositions. Matrices are denoted by a mark of exclamation after the functional symbol. Thus "$F!(\hat{\phi}!\hat{z}, \hat{\psi}!\hat{z}, \hat{x}, \hat{y})$" is a matrix

having two individuals and two elementary functions of individuals as arguments.

From an elementary function "$\phi!\,\hat{x}$" we obtain, as in Chapter I, the propositions "$(x).\phi!\,x$" and "$(\exists x).\phi!\,x$" which respectively assert the truth of all, and at least one of the values of "$\phi!\,x$". Similarly from an elementary function of two individuals $\phi!(\hat{x}, \hat{y})$ we obtain functions of one individual such as $(y).\phi!(\hat{x}, y)$, $(\exists y).\phi!(\hat{x}, y)$. The values of these functions are propositions such as $(y).\phi!(a, y)$ which are not elementary propositions; hence the functions themselves are not elementary functions. Such functions, whose values result from generalizing a matrix all whose values are individuals, are called first-order functions, and written $\phi_1\hat{x}$.

Suppose a is a constant. Then "$\phi!\,a$" will denote for the various values of ϕ, all the various elementary propositions of which a is a constituent. We can thus form the propositions $(\phi).\phi!\,a$, $(\exists\phi).\phi!\,a$ respectively asserting the truth of all, and at least one of the above assemblage of propositions. More generally we can assert by writing $(\phi).F!(\phi!\,\hat{z})$, $(\exists\phi).F!(\phi!\,\hat{z})$ the truth of all and at least one of the values of $F!(\phi!\,\hat{z})$. Such propositions are clearly not elementary, so that such a function as $(\phi).F!(\phi!\,\hat{z}, x)$ is not an elementary function of x. Such a function involving the totality of elementary functions is said to be of the second order and written $\phi_2 x$. By adopting the new variable ϕ_2 "we shall obtain other new functions

$$(\phi_2).f!(\phi_2\hat{z}, x), \quad (\exists\phi_2).f!(\phi_2\hat{z}, x),$$

which are again not among values for $\hat{\phi}_2 x$ (where ϕ_2 is the argument), because the totality of values of ϕ_2, which is now involved, is different from the totality of values of $\phi!$, which was formerly involved. However much we may enlarge the meaning of ϕ, a function of x in which ϕ occurs as apparent variable has a correspondingly enlarged meaning so that, however ϕ may be defined, $(\phi).f!(\phi\hat{z}, x)$ and $(\exists\phi).f!(\phi\hat{z}, x)$ can never be values for $\hat{\phi}x$. To attempt to make them so is like attempting to catch one's own shadow. It is impossible to obtain one variable which embraces among its values all possible functions of individuals"*.

For the way in which this distinction of functions into orders of which no totality is possible is used to escape the contradictions of Group B, which are shown to result from the ambiguities of language which disregard this distinction, reference may be made to *Principia*

* *Principia Mathematica*, 1 (1925), 2 ed., xxxiv.

*Mathematica**. Here it may be sufficient to apply the method to a contradiction not given in that work, which is peculiarly free from irrelevant elements; I mean, Weyl's contradiction concerning "heterologisch"†, which must now be explained. Some adjectives have meanings which are predicates of the adjective word itself; thus the word "short" is short, but the word "long" is not long. Let us call adjectives whose meanings are predicates of them, like "short", autological; others heterological. Now is "heterological" heterological? If it is, its meaning is not a predicate of it, that is, it is not heterological. But if it is not heterological, its meaning is a predicate of it, and therefore it is heterological. So we have a complete contradiction.

According to the principles of *Principia Mathematica*, this contradiction would be solved in the following way. An adjective word is the symbol for a propositional function, *e.g.* "ϕ" for $\phi\hat{x}$. Let R be the relation of meaning between "ϕ" and $\phi\hat{x}$. Then "w is heterological" is "$(\exists\phi).wR(\phi\hat{x}).\sim\phi w$". In this, as we have seen, the apparent variable ϕ must have a definite range of values (*e.g.* the range of elementary functions), of which $Fx =: .(\exists\phi): xR(\phi\hat{x}).\sim\phi x$ cannot itself be a member. So that "heterological" or "F" is not itself an adjective in the sense in which "ϕ" is. We do not have $(\exists\phi).$"F"$R(\phi\hat{x})$ because the meaning of "F" is not a function included in the range of "ϕ". So that when heterological and autological are unambiguously defined, "heterological" is not an adjective in the sense in question, and is neither heterological nor autological, and there is no contradiction.

Thus this theory of a hierarchy of orders of functions of individuals escapes the contradictions, but it lands us in an almost equally serious difficulty, for it invalidates many important mathematical arguments which appear to contain exactly the same fallacy as the contradictions. In the first edition of *Principia Mathematica* it was proposed to justify these arguments by a special axiom, the axiom of reducibility, which asserted that to every non-elementary function there is an equivalent elementary function‡. This axiom there is no reason to suppose true, and if it were true, this would be a happy accident and not a logical necessity, for it is not a tautology. This will be shown positively in Chapter V; but for the present it should be sufficient that it does not seem to be a tautology and that there is no reason to suppose that it is one. Such an axiom

* *Principia Mathematica*, 1 ed., 1 (1910), 117.

† Weyl, *Das Kontinuum*, 2.

‡ Two functions are called equivalent when the same arguments render them both true or both false. (German *umfangsgleich*.)

has no place in mathematics, and anything which cannot be proved without using it cannot be regarded as proved at all.

It is perhaps worth while, parenthetically, to notice a point which is sometimes missed. Why, it may be asked, does not the axiom of reducibility reproduce the contradictions, which the distinction between elementary and other functions avoided? For it asserts that to any non-elementary there is an equivalent elementary function, and so may appear to lose again whatever was gained by making the distinction. This is not, however, the case, owing to the peculiar nature of the contradictions in question; for, as pointed out above, this second set of contradictions are not purely mathematical, but all involve the ideas of thought or meaning, in connexion with which equivalent functions (in the sense of equivalent explained above) are not interchangeable; for instance, one can be meant by a certain word or symbol, but not the other; and one can be definable, and not the other*. On the other hand, any purely mathematical contradiction which arose from confusing elementary and non-elementary functions would be reinstated by the axiom of reducibility, owing to the extensional nature of mathematics, in which equivalent functions are interchangeable. But no such contradiction has been shown to arise, so that the axiom of reducibility does not seem to be self-contradictory. These considerations bring out clearly the peculiarity of this second group of contradictions, and make it even more probable that they have a psychological or epistemological and not a purely logical or mathematical solution; so that there is something wrong with the account of the matter given in *Principia*.

The principal mathematical methods which appear to require the axiom of reducibility are mathematical induction and Dedekindian section, the essential foundations of arithmetic and analysis respectively. Mr. Russell has succeeded in dispensing with the axiom in the first case†, but holds out no hope of a similar success in the second. Dedekindian section is thus left as an essentially unsound method, as has often been

* Dr. L. Chwistek appears to have overlooked this point that, if a function is definable, the equivalent elementary function need not also be definable in terms of given symbols. In his paper "Über die Antinomien der Prinzipien der Mathematik" in *Math. Zeitschrift*, 14 (1922), 236–243, he denotes by S a many-one relation between the natural numbers and the classes defined by functions definable in terms of certain symbols. $\phi\hat{z}$ being a non-elementary function of this kind, he concludes that there must be an n, such that $nS\hat{z}(\phi z)$. This is, however, a fallacy since $nS\hat{z}(\phi z)$ means by definition

$$(\exists\psi):\psi!\,x\equiv_x\phi x\,.\,nS(\psi!\,\hat{z})$$

and since $\psi!\,\hat{z}$ is not necessarily definable in terms of the given symbols there is no reason for there being any such n.

† See *Principia Mathematica*, 1 (1925), 2 ed., Appendix B.

emphasized by Weyl*, and ordinary analysis crumbles into dust. That these are its consequences is the second defect in the theory of *Principia Mathematica*, and, to my mind, an absolutely conclusive proof that there is something wrong. For as I can neither accept the axiom of reducibility nor reject ordinary analysis, I cannot believe in a theory which presents me with no third possibility.

The third serious defect in *Principia Mathematica* is the treatment of identity. It should be explained that what is meant is numerical identity, identity in the sense of counting as one, not as two. Of this the following definition is given :

$$"x = y \,.=: (\phi) : \phi!\, x \,.\supset.\, \phi!\, y : \quad \text{Df}" \dagger$$

that is, two things are identical if they have all their elementary properties in common.

In *Principia* this definition is asserted to depend on the axiom of reducibility, because, apart from this axiom, two things might have all their elementary properties in common, but still disagree in respect of functions of higher order, in which case they could not be regarded as numerically identical‡. Although, as we shall see, the definition is to be rejected on other grounds, I do not think it depends in this way on the axiom of reducibility. For though rejecting the axiom of reducibility destroys the obvious general proof that two things agreeing in respect of all elementary functions agree also in respect of all other functions, I think that this would still follow and could probably be proved in any particular case. For example, take a typical function of the second order

$$(\phi) : f!(\phi!\, \hat{z},\, x), \quad (\exists \phi) \,.\, f!(\phi!\, \hat{z},\, x).$$

Then, if we have $\qquad (\phi) : \phi!\, x \,.\equiv.\, \phi!\, y \quad (x = y),$

it follows that $\qquad (\phi) : f!(\phi!\, \hat{z},\, x) \,.\equiv.\, f!\,(\phi!\, \hat{z},\, y),$

because $f!(\phi!\, \hat{z},\, x)$ is an elementary function of x. Whence

$$(\phi) \,.\, f!(\phi!\, \hat{z},\, x) : \equiv : (\phi) \,.\, f!(\phi!\, \hat{z},\, y)$$

and $\qquad (\exists \phi) \,.\, f!\,(\phi!\, \hat{z},\, x) : \equiv : (\exists \phi) \,.\, f!(\phi!\, \hat{z},\, y).$

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* See H. Weyl, *Das Kontinuum*, and "Ueber die neue Grundlagenkrise der Mathematik", *Math. Zeitschrift*, 10 (1921), 39–79.

† 13 . 01.

‡ *Principia Mathematica*, 1 ed., 1 (1910), 177.

Hence rejecting the axiom of reducibility does not immediately lead to rejecting the definition of identity.

The real objection to this definition of identity is the same as that urged above against defining classes to be definable classes, that it is a misinterpretation, that it does not define the meaning with which the symbol for identity is actually used. This can be easily seen in the following way: the definition makes it self-contradictory for two things to have all their elementary properties in common. Yet this is really perfectly possible, even if, in fact, it never happens. Take two things, a and b. Then there is nothing self-contradictory in a having any self-consistent set of elementary properties, nor in b having this set, nor therefore, obviously, in both a and b having them, nor therefore in a and b having all their elementary properties in common. Hence, since this is logically possible, it is essential to have a symbolism which allows us to consider this possibility and does not exclude it by definition.

It is futile to raise the objection that it is not possible to distinguish two things which have all their properties in common, since to give them different names would imply that they had the different properties of having those names. For although this is perfectly true, that is to say, I cannot, for the reason given, know of any two particular indistinguishable things, yet I can perfectly well consider the possibility, or even know that there are two indistinguishable things, without knowing which they are. To take an analogous situation: since there are more people on the earth than hairs on any one person's head, I know that there must be at least two people with the same number of hairs, but I do not know which two people they are.

These arguments are reinforced by Wittgenstein's discovery that the sign of identity is not a necessary constituent of logical notation, but can be replaced by the convention that different signs must have different meanings. This will be found in *Tractatus Logico-Philosophicus*, p. 139; the convention is slightly ambiguous, but it can be made definite, and is then workable, although generally inconvenient. But even if of no other value, it provides an effective proof that identity can be replaced by a symbolic convention, and is therefore no genuine propositional function, but merely a logical device.

We conclude, therefore, that the treatment of identity in *Principia Mathematica* is a misinterpretation of mathematics, and just as the mistaken definition of classes is particularly unfortunate in connexion with the Multiplicative Axiom, so the mistaken definition of identity is especially misleading with regard to the Axiom of Infinity; for the two propositions, "There are an infinite number of things" and "There are

an infinite number of things differing from one another with regard to elementary functions", are, as we shall see in Chapter V, extremely different.

III. PREDICATIVE FUNCTIONS.

In this chapter we shall consider the second of the three objections which we made in the last chapter to the theory of the foundations of mathematics given in *Principia Mathematica*. This objection, which is perhaps the most serious of the three, was directed against the "Theory of Types," which seemed to involve either the acceptance of the illegitimate Axiom of Reducibility, or the rejection of such a fundamental type of mathematical argument as Dedekind section. We saw that this difficulty came from the second of the two parts into which the theory was divided, namely that part which concerned the different ranges of functions of given arguments, *e.g.* individuals, and we have to consider whether this part of the theory of types cannot be amended so as to get out of the difficulty. We shall see that this can be done in a simple and straightforward way, which is a natural consequence of the logical theories of Mr. Wittgenstein.

We shall start afresh from part of his theory of propositions, of which something was said in the first chapter. We saw there that he explains propositions in general by reference to atomic propositions, every proposition expressing agreement and disagreement with truth-possibilities of atomic propositions. We saw also that we could construct many different symbols all expressing agreement and disagreement with the same sets of possibilities. For instance,

$$"p \supset q", \quad "\sim p \,.\vee.\, q", \quad "\sim : p \,.\sim q", \quad "\sim q \,.\supset.\sim p"$$

are such a set, all agreeing with the three possibilities

$$"p \,.\, q", \quad "\sim p \,.\, q", \quad "\sim p \,.\sim q",$$

but disagreeing with "$p \,.\sim q$.". Two symbols of this kind, which express agreement and disagreement with the same sets of possibilities, are said to be instances of the same proposition. They are instances of it just as all the "the" 's on a page are instances of the word "the". But whereas the "the" 's are instances of the same word on account of their physical similarity, different symbols are instances of the same proposition because they have the same sense, that is, express agreement with the same sets of possibilities. When we speak of propositions

we shall generally mean the types of which individual symbols are instances, and we shall include types of which there may be no instances. This is inevitable, since it cannot be any concern of ours whether anyone has actually symbolized or asserted a proposition, and we have to consider all propositions in the sense of all possible assertions whether or not they have been asserted.

Any proposition expresses agreement and disagreement with complementary sets of truth-possibilities of atomic propositions; conversely given any set of these truth-possibilities, it would be logically possible to assert agreement with them and disagreement with all others, and the set of truth-possibilities therefore determines a proposition. This proposition may in practice be extremely difficult to express through the poverty of our language, for we lack both names for many objects and methods of making assertions involving an infinite number of atomic propositions, except in relatively simple cases, such as "$(x) . \phi x$", which involves the (probably) infinite set of (in certain cases) atomic propositions, "ϕa", "ϕb", &c. Nevertheless, we have to consider propositions which our language is inadequate to express. In "$(x) . \phi x$" we assert the truth of all possible propositions which would be of the form "ϕx" whether or not we have names for all the values of x. General propositions must obviously be understood as applying to everything, not merely to everything for which we have names.

We come now to a most important point in connection with the Theory of Types. We explained in the last chapter what was meant by an elementary proposition, namely one constructed explicitly as a truth-function of atomic propositions. We have now to see that, on the theory of Wittgenstein, elementary is not an adjective of the proposition type at all, but only of its instances. For an elementary and a non-elementary propositional symbol could be instances of the same proposition. Thus suppose a list was made of all individuals as "a", "b", ..., "z". Then, if $\phi\hat{x}$ were an elementary function, "$\phi a . \phi b \ldots \phi z$" would be an elementary proposition, but "$(x) . \phi x$" non-elementary; but these would express agreement and disagreement with the same possibilities and therefore be the same proposition. Or to take an example which could really occur, "ϕa" and "$\phi a : (\exists x) . \phi x$", which are the same proposition, since $(\exists x) . \phi x$ adds nothing to ϕa. But the first is elementary, the second non-elementary.

Hence some instances of a proposition can be elementary, and others non-elementary; so that elementary is not really a characteristic of the proposition, but of its mode of expression. "Elementary proposition" is like "spoken word"; just as the same word can be both spoken and

written, so the same proposition can be both elementarily and non-elementarily expressed.

After these preliminary explanations we proceed to a theory of propositional functions. By a propositional function of individuals we mean a symbol of the form "$f(\hat{x}, \hat{y}, \hat{z}, \ldots)$" which is such that, were the names of any individuals substituted for "$\hat{x}$", "$\hat{y}$", "$\hat{z}$", ... in it, the result would always be a proposition. This definition needs to be completed by the explanation that two such symbols are regarded as the same function when the substitution of the same set of names in the one and in the other always gives the same proposition. Thus if "$f(a, b, c)$" "$g(a, b, c)$" are the same proposition for any set of a, b, c, $f(\hat{x}, \hat{y}, \hat{z})$ and $g(\hat{x}, \hat{y}, \hat{z})$ are the same function, even if they are quite different to look at.

A function* "$\phi\hat{x}$" gives us for each individual a proposition, in the sense of a proposition type (which may not have any instances, for we may not have given the individual a name). So the function collects together a set of propositions, whose logical sum and product we assert by writing respectively "$(\exists x) . \phi x$", "$(x) . \phi x$". This procedure can be extended to the case of several variables. Consider "$\phi(\hat{x}, \hat{y})$"; give y any constant value η, and "$\phi(\hat{x}, \eta)$" gives a proposition when any individual name is substituted for $\hat{x}$, and is therefore a function of one variable, from which we can form the propositions

$$\text{"}(\exists x) . \phi(x, \eta)\text{"}, \quad \text{"}(x) . \phi(x, \eta)\text{"}.$$

Consider next "$(\exists x) . \phi(x, \hat{y})$"; this, as we have seen, gives a proposition when any name (*e.g.* "η") is substituted for "y", and is therefore a function of one variable from which we can form the propositions

$$(\exists y) : (\exists x) . \phi(x, y) \quad \text{and} \quad (y) : (\exists x) . \phi(x, y).$$

As so far there has been no difficulty, we shall attempt to treat functions of functions in exactly the same way as we have treated functions of individuals. Let us take, for simplicity, a function of one variable which is a function of individuals. This would be a symbol of the form "$f(\hat{\phi}\hat{x})$", which becomes a proposition on the substitution for "$\hat{\phi}\hat{x}$" of any function of an individual. "$f(\phi\hat{x})$" then collects together

* By "function" we shall in future always mean propositional function unless the contrary is stated.

a set of propositions, one for each function of an individual, of which we assert the logical sum and product by writing respectively "$(\exists\phi).f(\phi\hat{x})$", "$(\phi).f(\phi\hat{x})$".

But this account suffers from an unfortunate vagueness as to the range of functions $\phi\hat{x}$ giving the values of $f(\phi\hat{x})$ of which we assert the logical sum or product. In this respect there is an important difference between functions of functions and functions of individuals which is worth examining closely. It appears clearly in the fact that the expressions "function of functions", "function of individuals" are not strictly analogous; for, whereas functions are symbols, individuals are objects, so that to get an expression analogous to "function of functions" we should have to say "function of names of individuals". On the other hand, there does not seem any simple way of altering "function of functions" so as to make it analogous to "function of individuals", and it is just this which causes the trouble. For the range of values of a function of individuals is definitely fixed by the range of individuals, an objective totality which there is no getting away from. But the range of arguments to a function of functions is a range of symbols, all symbols which become propositions by inserting in them the name of an individual. And this range of symbols, actual or possible, is not objectively fixed, but depends on our methods of constructing them and requires more precise definition.

This definition can be given in two ways, which may be distinguished as the subjective and objective methods. The subjective* method is that adopted in *Principia Mathematica*; it consists in defining the range of functions as all those which could be constructed in a certain way, in the first instance by sole use of the "/" sign. We have seen how it leads to the impasse of the Axiom of Reducibility. I, on the other hand, shall adopt the entirely original objective method which will lead us to a satisfactory theory in which no such axiom is required. This method is to treat functions of functions as far as possible in the same way as functions of individuals. The signs which can be substituted as arguments in "$\phi\hat{x}$", a function of individuals, are determined by their meanings; they must be names of individuals. I propose similarly to determine the symbols which can be substituted as arguments in "$f(\hat{\phi}\hat{x})$" not by the manner of their construction, but by their meanings. This is more difficult, because functions do not mean single objects as names do, but have meaning in a more complicated way derived from the meanings

* I do not wish to press this term; I merely use it because I can find no better.

of the propositions which are their values. The problem is ultimately to fix as values of $f(\phi\hat{x})$ some definite set of propositions so that we can assert their logical product and sum. In *Principia Mathematica* they are determined as all propositions which can be constructed in a certain way. My method, on the other hand, is to disregard how we could construct them, and determine them by a description of their senses or imports; and in so doing we may be able to include in the set propositions which we have no way of constructing, just as we include in the range of values of ϕx propositions which we cannot express from lack of names for the individuals concerned.

We must begin the description of the new method with the definition of an atomic function of individuals, as the result of replacing by variables any of the names of individuals in an atomic proposition expressed by using names alone; where if a name occur more than once in the proposition it may be replaced by the same or different variables, or left alone in its different occurrences. The values of an atomic function of individuals are thus atomic propositions.

We next extend to propositional functions the idea of a truth-function of propositions. (At first, of course, the functions to which we extend it are only atomic, but the extension works also in general, and so I shall state it in general.) Suppose we have functions $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc., then by saying that a function $\psi(\hat{x}, \hat{y})$ is a certain truth-function (*e.g.* the logical sum) of the functions $\phi_1(\hat{x}, \hat{y})$, $\phi_2(\hat{x}, \hat{y})$, etc., and the propositions p, q, &c., we mean that any value of $\psi(x, y)$, say $\psi(a, b)$, is that truth-function of the corresponding values of $\phi_1(x, y)$, $\phi_2(x, y)$, etc., *i.e.* $\phi_1(a, b)$, $\phi_2(a, b)$, etc., and the propositions p, q, etc. This definition enables us to include functions among the arguments of any truth-function, for it always give us a unique function which is that truth-function of those arguments, *e.g.* the logical sum of $\phi_1(\hat{x})$, $\phi_2(\hat{x})$, ... is determined as $\psi(x)$, where $\psi(a)$ is the logical sum of $\phi_1 a$, $\phi_2 a$, ..., a definite proposition for each a, so that $\psi(x)$ is a definite function. It is unique because, if there were two, namely $\psi_1(x)$ and $\psi_2(x)$, $\psi_1(a)$ and $\psi_2(a)$ would for each a be the same proposition, and hence the two functions would be identical.

We can now give the most important definition in this theory, that of a predicative function. I do not use this term in the sense of *Principia Mathematica*, 1st ed., for which I follow Mr. Russell's later work in using "elementary". The notion of a predicative function, in my sense, is one which does not occur in *Principia*, and marks the essential divergence of the two methods of procedure. A *predicative function* of

individuals is one which is any truth-function of arguments which, whether finite or infinite in number, are all either atomic functions of individuals or propositions*. This defines a definite range of functions of individuals which is wider than any range occurring in *Principia*. It is essentially dependent on the notion of a truth-function of an infinite number of arguments; if there could only be a finite number of arguments our predicative functions would be simply the elementary functions of *Principia*. Admitting an infinite number involves that we do not define the range of functions as those which could be constructed in a certain way, but determine them by a description of their meanings. They are to be truth-functions, not explicitly in their appearance, but in their significance, of atomic functions and propositions. In this way we shall include many functions which we have no way of constructing, and many which we construct in quite different ways. Thus, supposing $\phi(\hat{x}, \hat{y})$ is an atomic function, p a proposition,

$$\phi(\hat{x}, \hat{y}), \quad \phi(\hat{x}, \hat{y}) . \vee . p, \quad (y) . \phi(\hat{x}, y)$$

are all predicative functions. [The last is predicative because it is the logical product of the atomic functions $\phi(\hat{x}, y)$ for different values of y.]

For functions of functions there are more or less analogous definitions. First, an atomic function of (predicative†) functions of individuals and of individuals can only have one functional argument, say ϕ, but may have many individual arguments, x, y, &c., and must be of the form $\phi(x, y, \ldots, a, b, \ldots)$ where "a", "b", ... are names of individuals. In particular, an atomic function $f(\phi\hat{z})$ is of the form ϕa. A predicative function of (predicative) functions of individuals and of individuals is one which is a truth-function whose arguments are all either propositions or atomic functions of functions of individuals and of individuals

e.g. $\hat{\phi}a . \supset . \hat{\psi}b : \vee : p$ (a function of ϕ, ψ),

$(x) . \hat{\phi}x$, the logical product of the atomic functions $\hat{\phi}a$, $\hat{\phi}b$, etc.

It is clear that a function only occurs in a predicative function through

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* Before "propositions" we could insert "atomic" without narrowing the sense of the definition. For any proposition is a truth-function of atomic propositions, and a truth-function of a truth-function is again a truth-function.

† I put "predicative" in brackets because the definitions apply equally to the non-predicative functions dealt with in the next chapter.

its values. In this way we can proceed to define predicative functions of functions of functions and so on to any order.

Now consider such a proposition as $(\phi) . f(\phi\hat{x})$ where $f(\hat{\phi}\hat{x})$ is a predicative function of functions. We understand the range of values of ϕ to be all predicative functions; *i.e.* $(\phi) . f(\phi\hat{x})$ is the logical product of the propositions $f(\phi\hat{x})$ for each predicative function, and as this is a definite set of propositions, we have attached to $(\phi) . f(\phi\hat{x})$ a definite significance.

Now consider the function of x, $(\phi) . f(\phi\hat{z}, x)$. Is this a predicative function? It is the logical product of the propositional functions of x, $f(\phi\hat{z}, x)$ for the different ϕ's which, since f is predicative, are truth-functions of ϕx and propositions possibly variable in ϕ but constant in x (*e.g.* ϕa). The ϕx's, since the ϕ's are predicative, are truth-functions of atomic functions of x. Hence the propositional functions of x, $f(\phi\hat{z}, x)$ are truth-functions of atomic functions of x and propositions. Hence they are predicative functions, and therefore their logical product $(\phi) . f(\phi\hat{z}, x)$ is predicative. More generally it is clear that by generalization, whatever the type of the apparent variable, we can never create non-predicative functions; for the generalization is a truth-function of its instances, and, if these are predicative, so is it.

Thus all the functions of individuals which occur in *Principia* are in our sense predicative and included in our variable ϕ, so that all need for an axiom of reducibility disappears.

But, it will be objected, surely in this there is a vicious circle; you cannot include $F\hat{x} = (\phi) . f(\phi\hat{z}, \hat{x})$ among the ϕ's, for it presupposes the totality of the ϕ's. This is not, however, really a vicious circle. The proposition Fa is certainly the logical product of the propositions $f(\phi\hat{z}, a)$, but to express it like this (which is the only way we can) is merely to describe it in a certain way, by reference to a totality of which it may be itself a member, just as we may refer to a man as the tallest in a group, thus identifying him by means of a totality of which he is himself a member without there being any vicious circle. The proposition Fa in its significance, that is the fact it asserts to be the case does not involve the totality of functions; it is merely our symbol which involves it. To take a particularly simple case, $(\phi) . \phi a$ is the logical product of the propositions ϕa, of which it is itself one; but this is no more remarkable and no more vicious than is the fact that $p . q$ is the logical product of the set p, q, $p . q$, of which it is itself a member. The only difference is that, owing to our inability to write propositions

of infinite length, which is logically a mere accident, $(\phi) . \phi a$ cannot, like $p . q$, be elementarily expressed, but must be expressed as the logical product of a set of which it is also a member. If we had infinite resources and could express all atomic functions as $\psi_1 x$, $\psi_2 x$, then we could form all the propositions ϕa, that is all the truth-functions of $\psi_1 a$, $\psi_2 a$, etc., and among them would be one which was the logical product of them all, including itself, just as $p . q$ is the product of p, q, $p \vee q$, $p . q$. This proposition, which we cannot express directly, that is elementarily, we express indirectly as the logical product of them all by writing "$(\phi) . \phi a$". This is certainly a circuitous process, but there is clearly nothing vicious about it.

In this lies the great advantage of my method over that of *Principia Mathematica*. In *Principia* the range of ϕ is that of functions which can be elementarily expressed, and since $(\phi) . f(\phi ! \hat{z}, x)$ cannot be so expressed it cannot be a value of $\phi !$; but I define the values of ϕ not by how they can be expressed, but by what sort of senses their values have, or, rather, by how the facts their values assert are related to their arguments. I thus include functions which could not even be expressed by us at all, let alone elementarily, but only by a being with an infinite symbolic system. And any function formed by generalization being actually predicative, there is no longer any need for an axiom of reducibility.

It remains to show that my notion of predicative functions does not involve us in any contradictions. The relevant contradictions, as I have remarked before, all contain some word like "means", and I shall show that they are due to an essential ambiguity of such words and not to any weakness in the notion of a predicative function.

Let us take first Weyl's contradiction about "heterological", which we discussed in the last chapter. It is clear that the solution given there is no longer available to us. For, as before, if R is the relation of meaning between "ϕ" and $\phi\hat{x}$, "x is heterological" is equivalent to "$(\exists \phi) . xR(\phi\hat{z}) . \sim \phi x$", the range of ϕ being here understood to be that of predicative functions. Then

$$(\exists \phi) : xR(\phi\hat{z}) . \sim \phi x,$$

which I will call Fx, is itself a predicative function.

So $$"F" R(F\hat{x})$$

and $$(\exists \phi) : "F" R(\phi\hat{x}),$$

and therefore $F(\text{“}F\text{”}) . \equiv . \sim F(\text{“}F\text{”})$,

which is a contradiction.

It will be seen that the contradiction essentially depends on deducing $(\exists \phi) : \text{“}F\text{”} R(\phi\hat{x})$ from $\text{“}F\text{”} R(F\hat{x})$. According to *Principia Mathematica* this deduction is illegitimate because $F\hat{x}$ is not a possible value of $\phi\hat{x}$. But if the range of $\phi\hat{x}$ is that of predicative functions, this solution fails, since $F\hat{x}$ is certainly a predicative function. But there is obviously another possible solution—to deny $\text{“}F\text{”} R(F\hat{x})$ the premiss of the deduction. $\text{“}F\text{”} R(F\hat{x})$ says that “F” means $F\hat{x}$. Now this is certainly true for some meaning of “means”, so to uphold our denial of it we must show some ambiguity in the meaning of meaning, and say that the sense in which “F” means $F\hat{x}$, *i.e.* in which “heterological” means heterological, is not the sense denoted by “R”, *i.e.* the sense which occurs in the definition of heterological. We can easily show that this is really the case, so that the contradiction is simply due to an ambiguity in the word “meaning” and has no relevance to mathematics whatever.

First of all to speak of “F” as meaning $F\hat{x}$ at all must appear very odd in view of our definition of a propositional function as itself a symbol. But the expression is merely elliptical. The fact which we try to describe in those terms is that we have arbitrarily chosen the letter “F” for a certain purpose, so that “Fx” shall have a certain meaning (depending on x). As a result of this choice “F”, previously non-significant, becomes significant; it has meaning. But it is clearly an impossible simplification to suppose that there is a single object F, which it means. Its meaning is more complicated than that, and must be further investigated.

Let us take the simplest case, an atomic proposition fully written out “aSb”, where “a”, “b” are names of individuals and “S” the name of a relation. Then “a”, “b”, “S” mean in the simplest way the separate objects a, b, and S. Now suppose we define

$$\phi x . = . aSx \quad \text{Df.}$$

Then “ϕ” is substituted for “aS” and does not mean a single object, but has meaning in a more complicated way, in virtue of a three-termed relation to both a and S. Then we can say “ϕ” means $aS\hat{x}$, meaning by this that “ϕ” has this relation to a and S. We can extend this

account to deal with any elementary function, that is, to say that "$\phi!$" means $\phi!\hat{x}$ means that "$\phi!$" is related in a certain way to the objects a, b, etc., involved in $\phi!\hat{x}$.

But suppose now we take a non-elementary functional symbol, for example,

$$\phi_1 x : = : (y) . yRx \quad \text{Df.}$$

Here the objects involved in $\phi_1\hat{x}$ include all individuals as values of y. And it is clear that "ϕ_1" is not related to them in at all the same way, as "$\phi!$" is to the objects in its meaning. For "$\phi!$" is related to a, b, etc., by being short for an expression containing names of a, b, etc. But "ϕ_1" is short for an expression not containing "a", "b", ..., but only an apparent variable, of which these can be values. Clearly "ϕ_1" means what it means in quite a different and more complicated way from that in which "$\phi!$" means. Of course, just as elementary is not really a characteristic of the proposition, it is not really a characteristic of the function; that is to say $\phi_1\hat{x}$ and $\phi!\hat{x}$ may be the same function, because $\phi_1 x$ is always the same proposition as $\phi!x$. Then "ϕ_1","$\phi!$" will have the same meaning, but will mean it, as we saw above, in quite different senses of meaning. Similarly a "ϕ_2" which involves functional apparent variable will mean in a different and more complicated way still*.

Hence in the contradiction which we were discussing, if "R", the symbol of the relation of meaning between "ϕ" and $\phi\hat{x}$, is to have any definite meaning, "ϕ" can only be a symbol of a certain type meaning in a certain way; suppose we limit "ϕ" to be an elementary function, by taking R to be the relation between "$\phi!$" and $\phi!\hat{x}$.

Then "Fx" or "$(\exists\phi) : xR(\phi z) . \sim \phi x$" is not elementary, but is a "ϕ_2".

Hence "F" means not in the sense of meaning denoted by "R" appropriate to "$\phi!$"s, but in that appropriate to a "ϕ_2", so that we have $\sim :$"F"$R(F\hat{x})$, which, as we explained above, solves the contradiction for this case.

The essential point to understand is that the reason why

$$(\exists\phi) : \text{"}F\text{"}\, R(\phi\hat{x})$$

* Here the range of the apparent variable in "ϕ_2" is the set of predicative functions not as in *Principia Mathematica* the set of elementary functions.

can only be true if "F" is an elementary function, is not that the range of ϕ is that of elementary functions, but that a symbol cannot have R to a function unless it (the symbol) is elementary. The limitation comes not from "$\exists\phi$", but from "R". The distinctions of "$\phi!$"s, "ϕ_1"s, and "ϕ_2"s apply to the symbols and to how they mean but not to what they mean. Therefore I always (in this section) enclosed "$\phi!$", "ϕ_1" and "ϕ_2" in inverted commas.

But it may be objected that this is an incomplete solution; for suppose we take for R the sum of the relations appropriate to "$\phi!$"s, "ϕ_1"s, and "ϕ_2"s. Then "F", since it still only contains $\exists\phi^*$, is still a "ϕ_2", and we must have in this case "F" $R(F\hat{x})$; which destroys our solution.

But this is not so because the extra complexity involved in the new R makes "F" not a "ϕ_2", but a more complicated symbol still. For with this new R, for which "ϕ_2" $R(\phi_2\hat{x})$, since "$\phi_2 x$" is of some such form as $(\exists\phi).f(\phi\hat{z}, x)$, in $(\exists\phi).$"F" $R(\phi\hat{x})$ is involved at least a variable function $f(\phi\hat{z}, x)$ of functions of individuals, for this is involved in the notion of a variable "ϕ_2", which is involved in the variable ϕ taken in conjunction with R. For if anything has R to the predicative function $\phi\hat{x}$, $\phi\hat{x}$ must be expressible by either a "$\phi!$" or a "ϕ_1" or a "ϕ_2".

Hence $(\exists\phi).$"F" $R(\phi\hat{x})$ involves not merely the variable ϕ (predicative function of an individual) but also a hidden variable f (function of a function of an individual and an individual). Hence "Fx" or "$(\exists\phi).xR(\phi\hat{x}).\sim\phi x$" is not a "$\phi_2$", but what we may call a "ϕ_3", *i.e.* a function of individuals involving a variable function of functions of individuals. (This is, of course, not the same thing as a "ϕ_3" in the sense of *Principia Mathematica*, 2nd edition.) Hence "F" means in a more complicated way still not included in R and we do not have "F" $R(F\hat{x})$ so that the contradiction again disappears.

What appears clearly from the contradictions is that we cannot obtain an all-inclusive relation of meaning for propositional functions. Whatever one we take there is still a way of constructing a symbol to mean in a way not included in our relation. The meanings of meaning form an illegitimate totality.

By the process begun above we obtain a hierarchy of propositions and a hierarchy of functions of individuals. Both are based on the fundamental hierarchy of individuals, functions of individuals, functions of functions of individuals, etc. A function of individuals we will call

* The range of ϕ in $\exists\phi$ is that of predicative functions including all "ϕ_1"s, "ϕ_2"s, etc., so it is not altered by changing R.

a function of type 1; a function of functions of individuals, a function of type 2, and so on.

We now construct the hierarchy of propositions as follows :—

Propositions of order 0 (elementary), containing no apparent variable.

,, ,, ,, 1, containing an individual apparent variable.

,, ,, ,, 2, containing an apparent variable whose values are functions of type 1.

,, ,, ,, n, containing an apparent variable whose values are functions of type $n-1$.

From this hierarchy we deduce another hierarchy of functions, irrespective of their types, according to the order of their values.

Thus functions of order 0 (matrices) contain no apparent variable;

,, ,, ,, ,, 1 contain an individual apparent variable;

and so on; *i.e.* the values of a function of order n being propositions of order n. For this classification the types of the functions are immaterial.

We must emphasize the essential distinction between order and type. The type of a function is a real characteristic of it depending on the arguments it can take; but the order of a proposition or function is not a real characteristic, but what Peano called a pseudo-function. The order of a proposition is like the numerator of a fraction. Just as from "$x = y$" we cannot deduce that the numerator of x is equal to the numerator of y, from the fact that "p" and "q" are instances of the same proposition we cannot deduce that the order of "p" is equal to that of "q". This was shown above (p. 363) for the particular case of elementary and non-elementary propositions (Orders 0 and > 0), and obviously holds in general. Order is only a characteristic of a particular symbol which is an instance of the proposition or function.

We shall now show briefly how this theory solves the remaining contradictions of group B*.

* It may be as well to repeat that for the contradictions of group A my theory preserves the solutions given in *Principia Mathematica*.

(*a*) "I am lying".

This we should analyse as "($\exists$ 'p', p); I am saying 'p' . 'p' means $p . \sim p$". Here to get a definite meaning for *means** it is necessary to limit in some way the order of "p". Suppose "p" is to be of the nth or lesser order. Then, symbolizing by ϕ_n, a function of *type* n, "p" may be $(\exists \phi_n) . \phi_{n+1}(\phi_n)$.

Hence $\exists$ "p" involves $\exists \phi_{n+1}$ and "I am lying" in the sense of "I am asserting a false proposition of order n", is at least of order $n+1$ and does not contradict itself.

(*b*) (1) The least integer not nameable in fewer than nineteen syllables.

(2) The least indefinable ordinal.

(3) Richard's Paradox.

All these result from the obvious ambiguity of "naming" and "defining". The name or definition is in each case a functional symbol which is only a name or definition by meaning something. The sense in which it means must be made precise by fixing its order; the name or definition involving all such names or definitions will be of a higher order, and this removes the contradiction. My solutions of these contradictions are obviously very similar to those of Whitehead and Russell, the difference between them lying merely in our different conceptions of the order of propositions and functions. For me propositions in themselves have no orders; they are just different truth-functions of atomic propositions—a definite totality, depending only on what atomic propositions there are. Orders and illegitimate totalities only come in with the symbols we use to symbolize the facts in variously complicated ways.

To sum up: in this chapter I have defined a range of predicative functions which escapes contradiction and enables us to dispense with the axiom of reducibility. And I have given a solution of the contradictions of group B which rests on and explains the fact that they all contain some epistemic element.

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* When I say "'p' means p", I do not suppose there to be a single object p meant by "p". The meaning of "p" is that one of a certain set of possibilities is realized, and this meaning results from the meaning-relations of the separate signs in "p" to the real objects which it is about. It is these meaning-relations which vary with the order of "p". And the order of "p" is limited not because p in $(\exists p)$ is limited, but by "means" which varies in meaning with the order of "p".

IV. PROPOSITIONAL FUNCTIONS IN EXTENSION.

Before we go on, let us look round and see where we have got to. We have seen that the introduction of the notion of a predicative function has given us a range for ϕ which enables us to dispense with the axiom of reducibility. Hence it removes the second and most important defect in the theory of *Principia Mathematica*; but how do we now stand with regard to the other two difficulties, the difficulty of including all classes and relations in extension and not merely definable ones, and the difficulty connected with identity?

The difficulty about identity we can get rid of, at the cost of great inconvenience, by adopting Wittgenstein's convention, which enables us to eliminate "=" from any proposition in which it occurs. But this puts us in a hopeless position as regards classes, because, having eliminated "=" altogether, we can no longer use $x = y$ as a propositional function in defining finite classes. So that the only classes with which we are now able to deal are those defined by predicative functions.

It may be useful here to repeat the definition of a predicative function of individuals; it is any truth-function of atomic functions and atomic propositions. We call such functions "predicative" because they correspond, as nearly as a precise notion can to a vague one, to the idea that ϕa predicates the same thing of a as ϕb does of b. They include all the propositional functions which occur in *Principia Mathematica*, including identity as there defined. It is obvious, however, that we ought not to define identity in this way as agreement in respect of all predicative functions, because two things can clearly agree as regards all atomic functions and therefore as regards all predicative functions, and yet they are two things and not, as the proposed definition of identity would involve, one thing.

Hence our theory is every bit as inadequate as *Principia Mathematica* to provide an extensional logic; in fact, if we reject this false definition of identity, we are unable to include among the classes dealt with even all finite enumerated classes. Mathematics then becomes hopeless because we cannot be sure that there is any class defined by a predicative function whose number is two; for things may all fall into triads which agree in every respect, in which case there would be in our system no unit classes and no two-member classes.

If we are to preserve at all the ordinary form of mathematics, it looks as if some extension must be made in the notion of a propositional function, so as to take in other classes as well. Such an extension is desirable on other grounds, because many things which would naturally

be regarded as propositional functions can be shown not to be predicative functions.

For example

$$F(x, y) = \text{Something other than } x \text{ and } y \text{ satisfies } \hat{z}(\phi z).$$

(Here, of course, "other than" is to be taken strictly, and not in the *Principia Mathematica* sense of "distinguishable from".)

This is not a predicative function, but is made up of parts of two predicative functions:

(1) For $x \neq y$

$$F(x, y) \quad \text{is} \quad \phi x \,.\, \phi y \,.\, \supset \,.\, Nc'\hat{z}(\phi z) \geqslant 3 : .$$
$$\phi x \,.\, \sim \phi y \,.\, \vee \,.\, \phi y \,.\, \sim \phi x : \supset : Nc'\hat{z}(\phi z) \geqslant 2 : .$$
$$\sim \phi x \,.\, \sim \phi y : \supset : Nc'\hat{z}(\phi z) \geqslant 1.$$

This is a predicative function because it is a truth function of ϕx, ϕy and the constant propositions $Nc'\hat{z}(\phi z) \geqslant 1, 2, 3$, which do not involve x, y.

(2) For $x = y$

$$F(x, x) \quad \text{is} \quad \phi(x) \,.\, \supset \,.\, Nc'\hat{z}(\phi z) \geqslant 2 :$$
$$\sim \phi x \,.\, \supset \,.\, Nc'\hat{z}(\phi z) \geqslant 1,$$

which is a predicative function.

But $F(x, y)$ is not itself a predicative function; this is perhaps more difficult to see. But it is easy to see that all functions of this kind cannot be predicative, because if they were we could find a predicative function satisfied by any given individual a alone, which we clearly cannot in general do.

For suppose fa (if not, take $\sim f\hat{x}$).

Let
$$\alpha = \hat{x}(fx),$$
$$\beta = \alpha - (a).$$

Then $\phi x =$ "There is nothing which satisfies fx except x, and members of β" applies to a and a alone. So such functions cannot always be predicative.

Just as $F(x, y)$ above, so also "$x = y$" is made up of two predicative functions :

(1) For $x \neq y$

"$x = y$" may be taken to be $(\exists \phi) . \phi x . \sim \phi x : (\exists \phi) . \phi y . \sim \phi y$, *i.e.* a contradiction.

(2) For $x = y$

"$x = y$" may be taken to be $(\phi) : . \phi x . \vee . \sim \phi x : \phi y . \vee . \sim \phi y$, *i.e.* a tautology.

But "$x = y$" is not itself predicative.

It seems, therefore, that we need to introduce non-predicative propositional functions. How is this to be done? The only practicable way is to do it as radically and drastically as possible; to drop altogether the notion that ϕa says about a, what ϕb says about b; to treat propositional functions like mathematical functions, that is, extensionalize them completely. Indeed, it is clear that, mathematical functions being derived from propositional, we shall get an adequately extensional account of the former only by taking a completely extensional view of the latter.

So in addition to the previously defined concept of a predicative function, which we shall still require for certain purposes, we define, or rather explain, for in our system it must be taken as indefinable, the new concept of a propositional function in extension. Such a function of one individual results from any one-many relation in extension between propositions and individuals; that is to say a correlation, practicable or impracticable, which to every individual associates a unique proposition, the individual being the argument to the function, the proposition its value.

Thus ϕ (Socrates) may be Queen Anne is dead.

ϕ (Plato) ,, Einstein is a great man.

$\phi \hat{x}$ being simply an arbitrary association of propositions ϕx to individuals x.

A function in extension will be marked by a suffix e thus $\phi_e \hat{x}$.

Then we can talk of the totality of such functions as the range of values of an apparent variable ϕ_e.

Consider now $(\phi_e) : \phi_e x \equiv \phi_e y$.

This asserts that in any such correlation the proposition correlated with x is equivalent to that correlated with y.

If $x = y$ this is a tautology (it is the logical product of values of $p \equiv p$).

But if $x \neq y$ it is a contradiction. For in one of the correlations some p will be associated with x, and $\sim p$ with y.

Then for this correlation $f_e\hat{x}$, $f_e x$ is p, $f_e y$ is $\sim p$, so that $f_e x \equiv f_e y$ is self-contradictory and $(\phi_e) . \phi_e x \equiv \phi_e y$ is self-contradictory.

So $(\phi_e) . \phi_e x \equiv \phi_e y$ is a tautology if $x = y$, a contradiction if $x \neq y$*.

Hence it can suitably be taken as the definition of $x = y$.

$x = y$ is a function in extension of two variables. Its value is tautology when x and y have the same value, contradictions when x, y have different values.

We have now to defend this suggested range of functions for a variable ϕ_e against the charges that it is illegitimate or leads to contradictions. It is legitimate because it is an intelligible notation, giving a definite meaning to the symbols in which it is employed. Nor can it lead to contradictions, for it will escape all the suggested contradictions just as the range of predicative functions will. Any symbol containing the variable ϕ_e will mean in a different way from a symbol not containing it, and we shall have the same sort of ambiguity of "meaning" as in Chapter III, which will remove the contradictions. Nor can any of the first group of contradictions be restored by our new notation, for it will still be impossible for a class to be a member of itself, as our functions in extension are confined to definite types of arguments by definition.

We have now to take the two notions we have defined, predicative functions and functions in extension, and consider when we shall want to use one and when the other†. First let us take the case when the arguments are individuals: then there is every advantage in taking the range of functions we use in mathematics to be that of functions in extension. We have seen how this enables us to define identity satisfactorily, and it is obvious that we shall need no axiom of reducibility, for any propositional function obtained by generalization, or in any manner whatever, is a function in extension. Further it will give us a satisfactory theory of classes, for any class will be defined by a function in extension, *e.g.* by the function which is tautology for any member of the class as argument, but contradiction for any other argument, and the null-class will be defined by the self-contradictory function. So the totality of classes can be reduced to that of functions in extension,

* On the other hand $(\phi) . \phi x \equiv \phi y$ (ϕ predicative) is a tautology if $x = y$, but not a contradiction if $x \neq y$.

† Of course, predicative functions are also functions in extension; the question is which range we want for our variable function.

and therefore it will be this totality which we shall require in mathematics, not the totality of predicative functions, which corresponds not to "all classes" but to "all predicates" or "all properties"

On the other hand, when we get to functions of functions the situation is rather different. There appears to be no point in considering any except predicative functions of functions; the reasons for introducing functions in extension no longer apply. For we do not need to define identity between functions, but only identity between classes which reduces to equivalence between functions, which is easily defined. Nor do we wish to consider classes of functions, but classes of classes, of which a simpler treatment is also possible. So in the case of functions of functions we confine ourselves to such as are predicative.

Let us recall the definition of a predicative function of functions; it is a truth-function of their values and constant propositions*. All functions of functions which occur in *Principia* are of this sort, but "I believe $(x) \,.\, \phi x$" as a function of $\phi\hat{x}$ is not. Predicative functions of functions are extensional in the sense of *Principia*, that is if the range of $f(\hat{\phi}\hat{x})$ be that of predicative functions of functions,

$$\phi_e x \equiv_x \psi_e x : \supset : f(\phi_e \hat{x}) \equiv f(\psi_e \hat{x}).$$

This is because $f(\phi_e \hat{x})$ is a truth-function of the values of $\phi_e x$ which are equivalent to the corresponding values of $\psi_e x$ so that $f(\phi_e \hat{x})$ is equivalent to $f(\psi_e \hat{x})$.

If we assumed this we should have a very simple theory of classes, since there would be no need to distinguish $\hat{x}(\phi_e x)$ from $\phi_e \hat{x}$. But though it is a tautology there is clearly no way of proving it, so that we should have to take it as a primitive proposition. If we wish to avoid this we have only to keep the theory of classes given in *Principia* based on "the derived extensional function". The range of predicative functions of functions is adequate to deal with classes of classes because, although, as we have seen, there may be classes of individuals which can only be

* It is, I think, predicative functions of functions which Mr. Russell in the introduction to the second edition of *Principia* tries to describe as functions into which functions enter only through their values. But this is clearly an insufficient description, because $\phi\hat{x}$ only enters into $F(\phi\hat{x})$ = "I believe ϕa" through its value ϕa, but this is certainly not a function of the kind meant, for it is not extensional. I think the point can only be explained by introducing, as I have, the notion of a truth function. To contend, as Mr. Russell does, that all functions of functions are predicative is to embark on a futile verbal dispute, owing to the ambiguity of the vague term functions of functions which may be used to mean only such as are predicative or to include also such as $F(\phi\hat{x})$ above.

defined by functions in extension, yet any class of classes can be defined by a predicative function, namely by $f(\alpha)$ where

$$f(\phi_e\hat{x}) = \Sigma_\psi(\phi_e x \equiv_x \psi_e x),$$

i.e. the logical sum of $\phi_e x \equiv_x \psi_e x$ for all the functions $\psi_e\hat{x}$ which define the members of the class of classes. Of course, if the class of classes is infinite, this expression cannot be written down. But, nevertheless, there will be the logical sum of these functions, though we cannot express it*.

So to obtain a complete theory of classes we must take the range of functions of individuals to be that of functions in extension; but the range of functions of functions to be that of predicative functions. By using these variables we obtain the system of *Principia Mathematica*, simplified by the omission of the axiom of reducibility, and a few corresponding alterations. Formally it is almost unaltered; but its meaning has been considerably changed. And in thus preserving the form while modifying the interpretation I am following the great school of mathematical logicians, who, in virtue of a series of startling definitions, have saved mathematics from the sceptics, and provided a rigid demonstration of its propositions. Only so can we preserve it from the Bolshevik menace of Brouwer and Weyl.

V. The axioms.

I have shown in the last two chapters how to remedy the three principal defects in *Principia Mathematica* as a foundation for mathematics. Now we have to consider the two important difficulties which remain, which concern the Axiom of Infinity and the Multiplicative Axiom. The introduction of these two axioms is not so grave as that of the Axiom of Reducibility, because they are not in themselves such objectionable assumptions, and because mathematics is largely independent of the Multiplicative Axiom, and might reasonably be supposed to require an axiom of infinity. Nevertheless, we must try to determine the logical status of these axioms, whether they are tautologies or empirical propositions or even contradictions. In this inquiry I shall

* A logical sum is not like an algebraic sum; only a finite number of terms can have an algebraic sum, for an "infinite sum" is really a limit. But the logical sum of a set of propositions is the proposition that these are not all false, and exists whether the set be finite or infinite.

include, from curiosity, the Axiom of Reducibility, although, since we have dispensed with it, it no longer really concerns us.

Let us begin with the Axiom of Reducibility, which asserts that all functions of individuals obtained by the generalization of matrices are equivalent to elementary functions. In discussing it several cases arise, of which I shall consider only the most interesting, that, namely, in which the numbers of individuals and of atomic functions of individuals are both infinite. In this case the axiom is an empirical proposition, that is to say, neither a tautology nor a contradiction, and can therefore be neither asserted nor denied by logic or mathematics. This is shown as follows :—

(*a*) The axiom is not a contradiction, but may be true.

For it is clearly possible that there should be an atomic function defining every class of individuals. In which case every function would be equivalent not merely to an elementary but to an atomic function.

(*b*) The axiom is not a tautology, but may be false.

For it is clearly possible that there should be an infinity of atomic functions, and an individual a such that whichever atomic function we take there is another individual agreeing with a in respect of all the other functions, but not in respect of the function taken. Then $(\phi) . \phi! x \equiv \phi! a$ could not be equivalent to any elementary function of x.

Having thus shown that the Axiom of Reducibility is neither a tautology nor a contradiction, let us proceed to the Multiplicative Axiom. This asserts that, given any existent class K of existent classes, there is a class having exactly one member in common with each member of K. If by "class" we mean, as I do, any set of things homogeneous in type not necessarily definable by a function which is not merely a function in extension, the Multiplicative Axiom seems to me the most evident tautology. I cannot see how this can be the subject of reasonable doubt, and I think it never would have been doubted unless it had been misinterpreted. For with the meaning it has in *Principia*, where the class whose existence it asserts must be one definable by a propositional function of the sort which occurs in *Principia*, it becomes really doubtful and, like the Axiom of Reducibility, neither a tautology nor a contradiction. We prove this by showing

(*a*) It is not a contradiction.

For it is clearly possible that every class (in my sense) should be defined by an atomic function, so that, since there is bound to be a

class in my sense having one member in common with each member of K, this would be also a class in the sense of *Principia*.

(*b*) It is not a tautology.

To show this we take not the Multiplicative Axiom itself but the equivalent theorem that any two classes are commensurable.

Consider then the following case: let there be no atomic functions of two or more variables, and only the following atomic functions of one variable :—

Associated with each individual a an atomic function $\phi_a\hat{x}$ such that

$$\phi_a x \,.\equiv_x x = a.$$

One other atomic function $f\hat{x}$ such that $\hat{x}(fx)$, $\hat{x}(\sim fx)$ are both infinite classes.

Then there is no one-one relation, in the sense of *Principia*, having either $\hat{x}(fx)$ or $\hat{x}(\sim fx)$ for domain, and therefore these two classes are incommensurable.

Hence the multiplicative axiom, interpreted as it is in *Principia*, is not a tautology but logically doubtful. But, as I interpret it, it is an obvious tautology, and this can be claimed as an additional advantage in my theory. It will probably be objected that, if it is a tautology, it ought to be able to be proved, *i.e.* deduced from the simpler primitive propositions which suffice for the deduction of the rest of mathematics. But it does not seem to me in the least unlikely that there should be a tautology, which could be stated in finite terms, whose proof was, nevertheless, infinitely complicated and therefore impossible for us. Moreover, we cannot expect to prove the multiplicative axiom in my system, because my system is formally the same as that of *Principia*, and the multiplicative axiom obviously cannot be proved in the system of *Principia*, in which it is not a tautology.

We come now to the Axiom of Infinity, of which again my system and that of *Principia* give different interpretations. In *Principia*, owing to the definition of identity there used, the axiom means that there are an infinity of distinguishable individuals, which is an empirical proposition; since, even supposing there to be an infinity of individuals, logic cannot determine whether there are an infinity of them no two of which have all their properties in common; but on my system, which admits functions in extension, the axiom of infinity asserts merely that there is an infinite number of individuals. This appears equally to be a mere question of fact; but the profound analysis of Wittgenstein has shown

that this is an illusion, and that, if it means anything, it must be either a tautology or contradiction. This will be much easier to explain if we begin not with infinity but with some smaller number.

Let us start with "There is an individual", or writing it as simply as possible in logical notation,

$$"(\exists x) . x = x"$$

Now what is this proposition? It is the logical sum of the tautologies $x = x$ for all values of x, and is therefore a tautology. But suppose there were no individuals, and therefore no values of x, then the above formula is absolute nonsense. So, if it means anything, it must be a tautology.

Next let us take "There are at least two individuals, or

$$"(\exists x, y) . x \neq y".$$

This is the logical sum of the propositions $x \neq y$, which are tautologies if x and y have different values, contradictions if they have the same value. Hence it is the logical sum of a set of tautologies and contradictions; and therefore a tautology, if any one of the set is a tautology, but otherwise a contradiction. That is, it is a tautology if x and y can take different values (*i.e.* if there are two individuals), but otherwise a contradiction.

A little reflection will make it clear that this will hold not merely of 2, but of any other number, finite or infinite. That is, "There are at least n individuals" is always either a tautology or contradiction, never a genuine proposition. We cannot, therefore, say anything about the number of individuals, since, when we attempt to do so, we never succeed in constructing a genuine proposition, but only a formula which is either tautological or self-contradictory. The number of individuals can, in Wittgenstein's phrase, only be shown, and it will be shown by whether the above formulae are tautological or contradictory.

The sequence "There is an individual".

"There are at least 2 individuals".

"There are at least n individuals".

"There are at least $\aleph_0$ individuals".

" " " $\aleph_1$ "

begins by being tautologous; but somewhere it begins to be contradictory,

and the position of the last tautologous term shows the number of individuals.

It may be wondered how, if we can say nothing about it, we can envisage as distinct possibilities that the number of individuals in the world is so-and-so. We do this by imagining different universes of discourse, to which we may be confined, so that by "all" we mean all in the universe of discourse and then that such and such a universe contains so and so many individuals is a real possibility, and can be asserted in a genuine proposition. It is only when we take, not a limited universe of discourse, but the whole world, that nothing can be said about the number of individuals in it.

We can do logic not only for the whole world but also for such limited universes of discourse; if we take one containing n individuals,

$$Nc'\hat{x}(x = x) \geqslant n \text{ will be a tautology,}$$

$$Nc'\hat{x}(x = x) \geqslant n+1 \text{ a contradiction.}$$

Hence $Nc'\hat{x}(x = x) \geqslant n+1$ cannot be deduced from the primitive propositions common to all universes, and therefore for a universe containing $n+1$ individuals must be taken as a primitive proposition.

Similarly the axiom of infinity in the logic of the whole world, if it is a tautology, cannot be proved, but must be taken as a primitive proposition. And this is the course which we must adopt, unless we prefer the view that all analysis is self-contradictory and meaningless. We do not have to assume that any particular set of things, *e.g.* atoms, is infinite, but merely that there is some infinite type which we can take to be the type of individuals.

ON PLANE QUARTIC CURVES WITH A TRIPLE POINT

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1. Plane quartic curves with a triple point have not attracted much attention. Some of their properties were given, apparently for the first time in the main, on pages 290 to 296 of Hilton's *Plane Algebraic Curves* (Clarendon Press, 1920) (referred to subsequently under the initials "H.P.A.C."). We propose here to investigate such curves more thoroughly. To save space we shall leave the details of analysis to the reader, and shall not repeat results already given in H.P.A.C., ch. 17, § 8.

In §§ 2 to 7 we take the case in which the tangents t_1, t_2, t_3 at the triple point O are all real. In §§ 2 to 6 we exclude the case in which three bitangents are concurrent, and we take that case in § 7. In § 8 we point out some analogies between the theory of the trinodal quartic and of the quartic with a triple point. In § 9 we consider the trinodal quartic with three concurrent bitangents. In § 10 we take the case in which only one tangent at O is real. Finally in § 11 we give drawings of the distinct types previously discussed.

2. Take the case in which the tangents at the triple point O are all real. Taking two of them as $x=0$, $y=0$, and then replacing z by $z+lx+my$, where l and m are suitably chosen, we can make the coefficients of x^3y and xy^3 zero in the equation of the quartic and bring it into the form

$$zxy(x+y) = (p+r)x^4-2rx^2y^2+(q+r)y^4. \qquad \text{(i)}$$

If we write

$$-x-y \text{ for } x, \qquad x \text{ for } y, \qquad z+4px+4(p+r)y \text{ for } z,$$

which is equivalent to a change of triangle of reference, (i) becomes

$$zxy(x+y) = (q+p)x^4 - 2px^2y^2 + (r+p)y^4, \qquad \text{(ii)}$$

and similarly the equation (i) may be transformed into

$$zxy(x+y) = (r+q)x^4 - 2qx^2y^2 + (p+q)y^4. \qquad \text{(iii)}$$

In these three equations each diagonal of the quadrilateral formed by the bitangents has been taken as $z = 0$ in turn. The diagonals are all real.

It is readily seen from (i), (ii), (iii) that we may suppose $p+r$, r, $q+r$ all positive in (i) without loss of generality, and that the following essentially distinct cases can occur :—

(1) $$qr+rp+pq > 0.$$

The bitangents are real, one of them being "ideal" (having unreal points of contact). No diagonal of the bitangents-quadrilateral meets the curve in a real point. The inflexions are all unreal and the conic through the inflexions does not meet the curve again in real points.

(2) $$qr+rp+pq < 0, \quad p+q > 0.$$

The bitangents are real. If

$$p+q+r-(p+q)^{\frac{1}{2}}(p+r)^{\frac{1}{2}}-(q+r)^{\frac{1}{2}}(q+p)^{\frac{1}{2}}-(r+p)^{\frac{1}{2}}(r+q)^{\frac{1}{2}} < 0,$$

one bitangent is ideal and no inflexion is real; otherwise all bitangents have real points of contact and two inflexions are real. The inflexions-conic meets the quartic again in two real points. Two of the diagonals of the bitangents-quadrilateral meet the curve in four real points, and the third meets it in none.

(3) $$qr+rp+pq < 0, \quad p+q < 0.$$

The bitangents are all unreal. Two inflexions are real and the inflexions-conic meets the curve again in real points. The diagonals of the bitangents-quadrilateral are real, one of them meets the curve in four real points and the other two meet it in two real points each.

(4) $$qr+rp+pq = 0.$$

The bitangents are real. One of them is ideal, and two of the other bitangents intersect at a point on it. The inflexions are unreal, the inflexions-conic touches the quartic.

3. In order to throw our equations into the most concise and symmetrical form, we shall consider a triangle ABC with area Δ such that

$$q+r=a^2, \quad r+p=b^2, \quad p+q=c^2. \tag{i}$$

The triangle will be real only for case (1) of § 2. For cases (2) and (3) the equations will, however, still hold good, if we consider $\cos A$ defined as $(b^2+c^2-a^2)/2bc$, $\sin B \sin C$ as $\cos A+\cos B \cos C$, etc.

In case (3) c^2 is negative, but the equations obtained will still represent real loci in general.

In § 2 (i) substitute

$$a(ax-b\cos Cy-c\cos Bz) \text{ for } x, \quad b(-a\cos Cx+by-c\cos Az) \text{ for } y,$$

$$16\Delta^2 z \text{ for } z. \tag{ii}$$

We then obtain the equation of the quartic referred to the triangle formed by the diagonals d_1, d_2, d_3 of the bitangents-quadrilateral, the triple point O becoming (1, 1, 1), the tangents t_1, t_2, t_3 at O becoming

$$ax-b\cos Cy-c\cos Bz=0, \quad -a\cos Cx+by-c\cos Az=0,$$

$$-c\cos Bx-a\cos By+cz=0, \tag{iii}$$

and the bitangents becoming

$$ax \pm by \pm cz = 0. \tag{iv}$$

Writing § 2 (i) in the form

$$(z+4rx+4ry)\,xy(x+y)=px^4+qy^4+r(x+y)^4, \tag{v}$$

we see that the equation of the quartic becomes

$$a^3\cos A(ax-b\cos Cy-c\cos Bz)^4+\ldots+\ldots$$

$$+4abc\cos A\cos B\cos C(ax-b\cos Cy-c\cos Bz)(\ldots)(\ldots)$$

$$\times(a\sec Ax+b\sec By+c\sec Cz)=0. \tag{vi}$$

The line $$\tan Ax+\tan By+\tan Cz=0 \tag{vii}$$

is that called by G. Fubini in *Atti reale Accad. Lincei*, 32 (1923, 2nd sem.), 273, the *principal line* of the quartic. In general, $z+fx+gy=0$ is the "principal line' of a curve of degree n with a multiple point of

order $n-1$, when its equation is put in the form

$$(y-m_1x)\ldots(y-m_{n-1}x)(z+fx+gy)$$
$$+a_1(y-m_1x)^n+\ldots+a_{n-1}(y-m_{n-1}x)^n=0.$$

If $n=3$, the principal line passes through the inflexions.

We shall say that the tangent t_1 at O,

$$ax-b\cos Cy-c\cos Bz=0,$$

is "associated" with the diagonal d_1, $x=0$, of the bitangents-quadrilateral; and so for t_2 and d_2, t_3 and d_3. We have the theorem:

The intersections of t_1 and d_1, t_2 and d_2, t_3 and d_3 lie on the principal line.

The corresponding theorem for the trinodal quartic is given in H.P.A.C., 271, Ex. 9.

4. The following equations may be obtained from those given in "H.P.A.C.", 292–294, replacing a, b, h by b^2, a^2, $-ab\cos C$, and x, y, z by the expressions (ii) of § 3*. Alternative forms of the equations are sometimes given.

Conic (C_b) through the points of contact of the bitangents.

$$\left.\begin{aligned}\sin A(\operatorname{cosec}B\operatorname{cosec}C-\cos A)x^2+\ldots+\ldots-2\cot Ayz-\ldots-\ldots&=0\\ \sin B\cos B\sin C\cos C\lambda^2+\ldots+\ldots+2\sin^2A\mu\nu+\ldots+\ldots&=0\end{aligned}\right\}. \quad \text{(i)}$$

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* Note the following misprints in "H.P.A.C.," Ch. 17:—

Page 274, line 13. *Read* "$-u^2$" *for* "u^2."

,, 274, line 5 from bottom. *Read* "4" *for* "2."

,, 274, line 4 from bottom. *Read* "Ax^2" *for* "Cx^2."

,, 277, foot-note. *Read* "changing Δ into 2Δ" *for* "omitting the Δ."

,, 279, Ex. 3, solution. *Read* "$A(gh+af)x+\ldots+\ldots=0$" *for* "$AFx+BGy+CHz=0$."

,, 280, line 3. *Read* "$2S_3-S_2$" *for* "S_3-2S_2."

,, 284, Ex. 4. *Read* "eccentricity $\sqrt{(2/3)}$" *for* "its asymptotes, etc."

,, 285, line 6. *Read* "$bx^2+2hxy+ay^2$" *for* "$cx^2+2hxy+by^2$."

,, 286, Ex. 11. *Read* "$2^{-\frac{1}{2}}$" *for* "$2^{\frac{1}{2}}$."

,, 292, Ex. 4. *Read* "Show that OQ and OR, and also OP and OS, are harmonically conjugate with respect to, etc."

,, 294, line 23. *Read* "$6A_2^2a_0a_2$" *for* "$6A_2^2a_1a_2$."

,, 294, line 25. *Read* "$C'_{31,1}$" *for* "C'_{31}."

,, 294, line 27. *Read* "$2b(a+3h)y^2$" *for* "$2h(a+3h)y^2$."

,, 297, Ex. 4, solution. *Read* "$27yz+19x^2$" *for* "$8yz+35x^2$."

Conic (C_i) through the inflexions.

$$\left.\begin{aligned}
&\sin A(\operatorname{cosec} B \operatorname{cosec} C - \cos A)x^2 + \ldots + \ldots \\
&\qquad - 2(\cot A + \sin A \sin B \sin C)yz - \ldots - \ldots = 0 \\
&(1 + \cos A \cos B \cos C) \\
&\qquad \times (\sin^2 A x^2 + \ldots + \ldots - 2\cos A \sin B \sin C yz - \ldots - \ldots) \\
&\quad = (\sin A \cos A x + \ldots + \ldots)\{\sin A \cos(B-C)x + \ldots + \ldots\} \\
&\cot A(\cos A \sin B \sin C - 3)\lambda^2 + \ldots + \ldots \\
&\qquad + 2\sin A(\cos B \cos C + 3 \operatorname{cosec} B \operatorname{cosec} C)\mu\nu - \ldots - \ldots = 0
\end{aligned}\right\}. \quad \text{(ii)}$$

Conic (C_t) touching the inflexional tangents and enveloping lines divided equianharmonically by the quartic.

$$\left.\begin{aligned}
&\sin A(\operatorname{cosec} B \operatorname{cosec} C - \cos A)x^2 + \ldots + \ldots \\
&\qquad - 2(\cot A - \tfrac{1}{2}\sin A \sin B \sin C)yz - \ldots - \ldots = 0 \\
&(3 + 4\cot^2 A)\lambda^2 + \ldots + \ldots + 2(3 - 2\cot B \cot C)\mu\nu + \ldots + \ldots = 0
\end{aligned}\right\}. \quad \text{(iii)}$$

Lines joining the triple point to the remaining intersections of C_i with the quartic.

$$\sin^2 A x^2 + \ldots + \ldots - 2\cos A \sin B \sin C yz - \ldots - \ldots = 0. \quad \text{(iv)}$$

Hessian of the tangents at the triple point.

$$(1 + \cos A \cos B \cos C)(\sin^2 A x^2 + \ldots + \ldots - 2\cos A \sin B \sin C yz - \ldots - \ldots)$$
$$= \sin^2 A \sin^2 B \sin^2 C(x^2 + y^2 + z^2 - yz - zx - xy). \quad \text{(v)}$$

Lines joining the inflexions to the triple point O.

$$\operatorname{cosec} A(-a\cos Cx + by - c\cos Az)^3(-a\cos Bx - b\cos Ay + cz)^3 + \ldots$$
$$+ \ldots = 0. \quad \text{(vi)}$$

Conic touching the bitangents and a tangent at O.

$$\sin(B-C)\operatorname{cosec} A\lambda^2 - \mu^2 + \nu^2 = 0, \quad \text{etc.} \quad \text{(vii)}$$

Conic touching the diagonals of the bitangents-quadrilateral and two of the tangents at O.

$$-\cos B \cos C \sec A \sin^2 A\mu\nu + \sin^2 B\nu\lambda + \sin^2 C\lambda\mu = 0, \quad \text{etc.} \quad \text{(viii)}$$

Conic touching two tangents at O and having the harmonic triangle of the bitangents-quadrilateral as a self-conjugate triangle.

$$2\cot A(\cot A\lambda^2+\cot B\mu^2+\cot C\nu^2) = (-\lambda^2+\mu^2+\nu^2), \quad \text{etc.} \qquad \text{(ix)}$$

Conic through O and two pairs of vertices of bitangents-quadrilateral.

$$\sin^2 Ax^2-\sin^2 By^2-\sin^2 Cz^2+2\cos A\sin B\sin Cyz = 0, \quad \text{etc.} \qquad \text{(x)}$$

The coordinates of any point on the quartic may be obtained symmetrically and rationally in terms of a parameter t by solving for $x : y : z$ from § 3 (vi) and $u = tv$, where

$$u \equiv (\beta\gamma\sin A-\gamma\alpha\sin B\cos C-\alpha\beta\cos B\sin C)\,ax+\ldots+\ldots,$$

$$v \equiv \{(\beta+\gamma-\alpha)-\gamma\sin B\cos C-\beta\cos B\sin C\}\,ax+\ldots+\ldots,$$

so that

$$ax-b\cos Cy-c\cos Bz = (u-\alpha v)\operatorname{cosec} A/(\alpha-\beta)(\alpha-\gamma), \quad \text{etc.}$$

Here α, β, γ are any convenient symmetrical constants, *e.g.* a, b, c or a^2, b^2, c^2.

5. From § 4 we get the following results:

(1) "The conics C_b, C_i, C_t all pass through the same four points. Through these points passes a conic circumscribing the harmonic triangle of the bitangents-quadrilateral. The tangent to this conic at each vertex of the triangle meets the opposite side on the polar of the triple point O."

The conic is $\qquad yz+zx+xy = 0.$

(2) "The tangents from O to the conic C_t are the Hessian of the tangents t_1, t_2, t_3 to the quartic at O, and the principal line is their chord of contact."

(3) "The conic C_i meets the quartic at the inflexions and at two more points H and K, the line joining which meets the quartic again on the Hessian of t_1, t_2, t_3."

The line is $\qquad \sin A\cos(B-C)x+\ldots+\ldots = 0.$

(4) "The conics through O touching the bitangents touch at O one or other of the lines OH and OK.

"The same is true of the conics through O touching the principal line and the diagonals of the bitangents-quadrilateral."

The lines are § 4 (iv). They are unreal or real for cases (1) and (2) of § 2 respectively.

(5) "A conic touches d_1, d_2, d_3, while the polar of O with respect to this conic joins the second intersections of C_i with OH and OK. Then its point of contact with d_1 is the same as the point of contact with d_1 of the conic touching d_1, d_2, d_3, t_2, t_3."

The conic is $\sin^2 A\mu\nu+\sin^2 B\nu\lambda+\sin^2 C\lambda\mu = 0,$

and the polar of O is

$$\sin 2Ax+\sin 2By+\sin 2Cz = 0.$$

(6) "The points of contact with the tangents t_1, t_2, t_3 at O of the three conics touching one of these tangents and the bitangents are collinear. If from the intersection of t_1 with the associated diagonal d_1 of the bitangents-quadrilateral the second tangent is drawn to the conic touching t_1 and the bitangents, and so for t_2 and d_2, t_3 and d_3, then the three points of contact are also collinear."

The lines of collinearity are

$$\sin A(\cos A-\cos B\cos C)x+\ldots+\ldots = 0, \quad x+y+z = 0.$$

(7) "A conic touches t_1, has the triangle formed by d_1, d_2, d_3 as self-conjugate triangle, and touches at its intersections with the polar of the intersection of d_2 and d_3 the conic which has the triangle formed by d_1, d_2, d_3 as self-conjugate triangle and has O and the principal line as pole and polar."

The latter conic is

$$\cot A\lambda^2+\cot B\mu^2+\cot C\nu^2 = 0.$$

We get the result by adding two of the equations (ix) of § 4. A similar theorem comes by subtraction.

(8) "A conic touches t_1 at O and passes through two pairs of vertices of the bitangents-quadrilateral. It meets d_1 on the Hessian of the tangents at O.

"The fourth intersection of this conic and the conic derived similarly from t_2 and d_2 lies on t_3".

(9) "The polars of any point with respect to the conics C_b, C_i, C_t, and the conic through their intersections and O form a pencil with cross-ratio $-1/3$".

The last conic is

$$\sin A(\operatorname{cosec} B \operatorname{cosec} C - \cos A)x^2 + \ldots + \ldots$$
$$-2(\cot A - \tfrac{1}{3}\sin A \sin B \sin C)yz - \ldots - \ldots = 0.$$

6. There are two cases in which the principal line touches the inflexions-conic C_i.

The first is that in which A, B, or C (say C) is $\frac{1}{2}\pi$.

The quartic is of type (1) in § 2. The principal line is the diagonal d_3 of the bitangents-quadrilateral; while the tangents t_1 and t_2 at O go through the intersections of d_2, d_3 and d_1, d_3 respectively. The conic C_b touches d_1 and d_2, t_3 being the chord of contact.

The other case is that in which

$$1 + \cos A \cos B \cos C = 0.$$

The quartic is of type (3) of § 2, with $p+q+r=0$. The conic C_i splits up into two real lines, through whose intersection the principal line passes. The conic C_b splits up into two unreal lines intersecting on one of the lines of the line-pair C_i. Two of the inflexions lie on the Hessian of the tangents at O.

There is no other case in which the lines joining O to a pair of inflexions makes with a pair of tangents at O an involution pencil having the third tangent at O as double ray, excluding the case in which the triangle ABC is equilateral (the quartic can be projected into $r = a\cos 3\theta$) or isosceles.

Another special case worth noticing is that in which

$$\sqrt{a} \pm \sqrt{b} \pm \sqrt{c} = 0,$$

when the quartic has a point of undulation.

7. Suppose now that in § 2

$$qr + rp + pq = 0.$$

Three bitangents are concurrent, and § 2 (i) becomes

$$(p+q)zxy(x+y) = (px^2 + qy^2)^2. \qquad \text{(i)}$$

If we substitute

$$\left.\begin{array}{l} \frac{1}{3}(\gamma-\beta) \text{ for } p, \quad \frac{1}{3}(\alpha-\gamma) \text{ for } q, \quad \frac{1}{3}(\alpha-\gamma)(y-\beta x) \text{ for } x \\ \frac{1}{3}(\gamma-\beta)(y-\alpha x) \text{ for } y, \quad \frac{4}{9}(\gamma-\beta)(\alpha-\gamma)(z-\gamma x) \text{ for } z \end{array}\right\}, \tag{ii}$$

where

$$\alpha+\beta+\gamma = 0, \tag{iii}$$

the equation (i) becomes

$$4z(y-\alpha x)(y-\beta x)(y-\gamma x) = \{y^2+\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)x^2\}^2+8\alpha\beta\gamma x^3 y. \tag{iv}$$

Writing this equation in the forms

$$4(z+2y)(y-\alpha x)(y-\beta x)(y-\gamma x) = \{3y^2-\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)x^2\}^2,$$

$$4(z-\alpha x)(y-\alpha x)(y-\beta x)(y-\gamma x) = \{(y-\alpha x)^2-(\alpha-\beta)(\alpha-\gamma)\,x^2\}^2,$$

$$4(z-y)(y-\alpha x)(y-\beta x)(y-\gamma x)$$

$$= \{(\beta-\gamma)^2(y-\alpha x)^4/(\alpha-\beta)(\alpha-\gamma)\}+\ldots+\ldots,$$

we see that the three concurrent bitangents b_1, b_2, b_3 are

$$z = \alpha x, \quad z = \beta x, \quad z = \gamma x$$

meeting the tangents t_1, t_2, t_3 at the triple point O, namely

$$y = \alpha x, \quad y = \beta x, \quad y = \gamma x$$

respectively, on the principal line $y = z$, while the fourth bitangent is $2y+z = 0$.

If $\alpha > \beta > \gamma$, b_2 is "ideal," while the other bitangents have real points of contact.

The conic C_b through the points of contact of the bitangents is

$$(\beta\gamma+\gamma\alpha+\alpha\beta)\,x^2-y^2+2yz+2z^2 = 0. \tag{v}$$

The conic C_i through the inflexions is

$$3(\beta\gamma+\gamma\alpha+\alpha\beta)\,x^2-y^2+2yz+8z^2 = 0. \tag{vi}$$

The following properties of a quartic with a triple point O, three bitangents concurrent at J, and a fourth bitangent meeting the principal line at L, are worth noting.

"The inflexional tangents and all lines divided equianharmonically by the quartic pass through one or other of the intersections of the principal line with the Hessian of the tangents at O."

"The conics C_b and C_i have double contact, the principal line being the chord of contact and J its pole. The lines OJ and OL are conjugate with respect to C_b and C_i, while they and the Hessian of the tangents at O both divide the fourth bitangent harmonically."

8. The method employed for the quartic with a triple point can be also used for the trinodal quartic.

Put (see "H.P.A.C.," 272, Ex. 11; 292, Ex. 9)

$$p \equiv \sin\tfrac{1}{2}A \cos\tfrac{1}{2}B \cos\tfrac{1}{2}C\,(-\sin Ax+\sin By+\sin Cz),$$

$$q \equiv \cos\tfrac{1}{2}A \sin\tfrac{1}{2}B \cos\tfrac{1}{2}C\,(\sin Ax-\sin By+\sin Cz),$$

$$r \equiv \cos\tfrac{1}{2}A \cos\tfrac{1}{2}B \sin\tfrac{1}{2}C\,(\sin Ax+\sin By-\sin Cz),$$

$$s \equiv \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C\,(\sin Ax+\sin By+\sin Cz),$$

and the quartic $$\sqrt{p}+\sqrt{q}+\sqrt{r}+\sqrt{s} = 0,$$

i.e. $$\{2(p^2+q^2+r^2+s^2)-(p+q+r+s)^2\}^2 = 64pqrs,$$

becomes

$$\{[\sin^2\tfrac{1}{2}(A+B+C)-\cos A \sin B \sin C]\sin^2 Ax^2+\ldots+\ldots$$

$$-2\sin\tfrac{1}{2}(A+B+C)\sin\tfrac{1}{2}(-A+B+C)\sin B \sin Cyz-\ldots-\ldots\}^2$$

$$+\sin^2 A \sin^2 B \sin^2 C\,(\sin^4 Ax^4+\ldots+\ldots-2\sin^2 B \sin^2 Cy^2z^2-\ldots-\ldots)=0, \qquad \text{(i)}$$

which is the equation of a quartic with bitangents

$$\pm\sin Ax \pm \sin By+\sin Cz = 0$$

and nodes $$\{\sin(B+C)\operatorname{cosec} A,\ 1,\ 1\},\ \text{etc.},$$

referred to the harmonic triangle of the bitangents-quadrilateral.

If $$A+B+C = \pi,$$

we get the special case of the quartic with a triple point discussed in §§ 2–6.

Putting $\alpha \equiv \cos\frac{1}{2}(A-B-C)$, $\beta \equiv \cos\frac{1}{2}(-A+B-C)$,

$$\gamma = \cos\tfrac{1}{2}(-A-B+C), \quad \delta = \cos\tfrac{1}{2}(A+B+C),$$

$\alpha' \equiv \sin\frac{1}{2}(A-B-C)$, $\beta' \equiv \sin\frac{1}{2}(-A+B-C)$,

$$\gamma' = \sin\tfrac{1}{2}(-A-B+C), \quad \delta' = \sin\tfrac{1}{2}(A+B+C),$$

$$\xi \equiv \delta' \sin Ax + \gamma' \sin By + \beta' \sin Cz, \quad \text{etc.,} \tag{ii}$$

i.e. $\sin(B+C)\,\xi + \sin B\eta + \sin C\zeta$

$$\equiv 2\sin A \sin B \sin C \cos\tfrac{1}{2}(A+B+C)\,x, \quad \text{etc.,}$$

the quartic becomes

$$\delta(\eta^2\zeta^2+\zeta^2\xi^2+\xi^2\eta^2)+2\xi\eta\zeta(\alpha\xi+\beta\eta+\gamma\zeta) = 0, \tag{iii}$$

the lines joining the nodes in pairs being

$$\xi = 0, \quad \eta = 0, \quad \zeta = 0.$$

The equations of the conics connected with (iii) are well known ("H.P.A.C.", Ch. 17, § 4), and their equations for the quartic (i) are obtainable by means of (ii).

For instance, the intersections of the tangents from each node with the line joining the other two nodes lie on

$$\delta' \sin^2 Ax^2 + \ldots + \ldots + 2\alpha' \sin B \sin Cyz + \ldots + \ldots = 0.$$

If $A+B+C = \pi$, this conic becomes the tangents at the triple point to the two conics passing through the triple point and touching the bitangents.

Again, the points of contact of the bitangents lie on

$$(\delta'^2-\alpha^2+\beta'^2+\gamma'^2)\sin^2 Ax^2 + \ldots + \ldots + 4\alpha'\delta' \sin B \sin Cyz + \ldots + \ldots = 0.$$

We notice that—

"Through the intersections of the bitangents-conic and the conic through the intersections of the tangents from each node with the line

joining the other two nodes passes a conic for which the harmonic triangle of the bitangents-quadrilateral is self-conjugate".

Other theorems worth noting are:—

"If conics are drawn touching the bitangents of a trinodal quartic and passing through a node, the six tangents to these conics at the nodes touch a conic".

"The polar of each node of a trinodal quartic with respect to any conic having the harmonic triangle of the bitangents-quadrilateral as self-conjugate triangle meets the corresponding side of this harmonic triangle in collinear points".

9. In § 8 we excluded the case in which three bitangents of the trinodal quartic were concurrent (at a point J, say).

The equation of such a quartic may be put in the form

$$l(-yz+zx+xy)^2+m(yz-zx+xy)^2+n(yz+zx-xy)^2 = 0, \qquad \text{(i)}$$

the point of concurrency J being (1, 1, 1).

The concurrent bitangents are

$$-(m+n)\,x+my+nz = 0, \quad \text{etc.}, \qquad \text{(ii)}$$

and the fourth bitangent is

$$(m+n)\,x+(n+l)\,y+(l+m)\,z = 0. \qquad \text{(iii)}$$

We may evidently suppose $l \geqslant m > 0 > n$ for a real quartic.

One of the bitangents (ii) is "ideal", the other two have real points of contact. The bitangent (iii) is ideal, if

$$mn+nl+lm > 0.$$

The vertex C of the triangle of reference ABC is a crunode, each of the other two may be a crunode or an acnode.

Using the notation of "H.P.A.C.", Ch. 17, § 4, we have readily, if we call the first of the bitangents (ii) the bitangent "associated" with the line joining the nodes B and C, etc.:—

"Each of the concurrent bitangents meets the associated line joining two nodes on a line which is the polar of J for the conics I, VI, VII, $U = 0$".

The line is $lx+my+nz=0.$

It meets the conic VI in unreal points.

"The polars of J with respect to the conics I, III, V are concurrent. The polars of J with respect to III and $T=0$ coincide".

"The polars of J with respect to the conics I and II meet on the fourth bitangent".

"The conics IV and IX both degenerate into line-pairs intersecting at J".

See "H.P.A.C.", 281, Ex. 24. It should be noticed that the conic VII does *not* touch the quartic, contrary to what the case of the quartic with a triple point might have led us to expect. The condition for such contact is not $\Delta^2=4FGH$ but $\Delta^2=4FGH+\Delta abc.$

10. The case in which two tangents at the triple point O of the quartic are unreal does not give such neat results as the case in which all three tangents are real, and a brief discussion will suffice.

Putting in § 2 (i)

$$y+ix \text{ for } x, \quad y-ix \text{ for } y, \quad \tfrac{1}{2}(p+q)z \text{ for } z,$$

$$m^2\cos 2\epsilon \text{ for } -2(p+q+2r)/(p+q), \quad m^2\sin 2\epsilon \text{ for } 2i(p-q)/(p+q),$$

we get

$$zy(x^2+y^2)=(x^2+y^2)^2-4m^2xy(x\cos\epsilon+y\sin\epsilon)(x\sin\epsilon-y\cos\epsilon) \qquad \text{(i)}$$

as a canonical form of the equation of such quartics.

The curve has two real bitangents

$$z+4m(m\sin\epsilon\pm 1)(x\cos\epsilon+y\sin\epsilon)=0, \qquad \text{(ii)}$$

touching where

$$x^2+y^2\pm 2my(x\cos\epsilon+y\sin\epsilon)=0, \qquad \text{(iii)}$$

and two unreal bitangents

$$z+4m(m\cos\epsilon\pm i)(x\sin\epsilon-y\cos\epsilon)=0, \qquad \text{(iv)}$$

touching where $$x^2+y^2\pm 2miy(x\sin\epsilon-y\cos\epsilon)=0. \qquad \text{(v)}$$

The conic

$$yz = 2(x^2+y^2)-4m^2 \sin\epsilon y(x\cos\epsilon+y\sin\epsilon) \qquad \text{(vi)}$$

passes through the points of contact of the real bitangents (ii) and touches the quartic at O.

Taking $m>0$ and $\sin\epsilon>0$, the real bitangents are ideal and no inflexions real, if

$$\frac{1}{1+\sin\epsilon} > m > 0.$$

One real bitangent is ideal and two inflexions real, if

$$\frac{1}{1-\sin\epsilon} > m > \frac{1}{1+\sin\epsilon};$$

while both bitangents have real points of contact and four inflexions are real, if

$$m > \frac{1}{1-\sin\epsilon}.$$

11. If in § 2 (i) and § 10 (i) we replace y by a and z by by/a, we have the equations

$$byx(x+a) = (p+r)x^4-2ra^2x^2+(q+r)a^4, \qquad \text{(i)}$$

$$by(x^2+a^2) = (x^2+a^2)^2-4m^2ax(x\cos\epsilon+a\sin\epsilon)(x\sin\epsilon-a\cos\epsilon), \qquad \text{(ii)}$$

as Cartesian equations of curves into which any quartic with a triple point may be projected. The triple point is $(0, \infty)$ and the line at infinity is one of the tangents thereat.

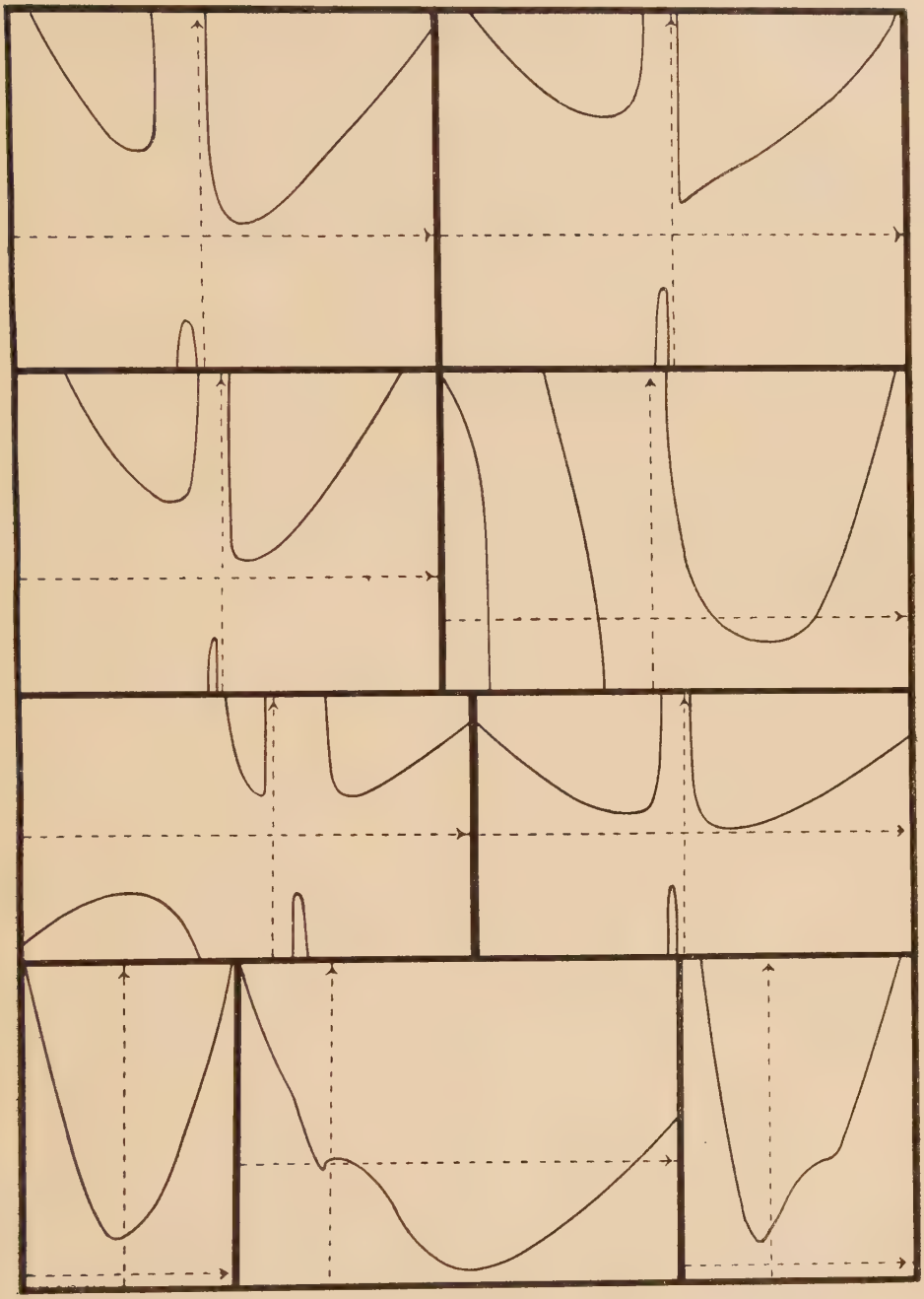
For the quartic of § 7 we may take the alternative method of replacing x, y, z by $1, x, y$ in § 7 (iv). The equation becomes

$$4y(x-\alpha)(x-\beta)(x-\gamma) = \{x^2+\tfrac{1}{2}(\alpha^2+\beta^2+\gamma^2)\}^2+8\alpha\beta\gamma x. \qquad \text{(iii)}$$

The tangents at the triple point are $x=\alpha$, $x=\beta$, $x=\gamma$ and the bitangents are $y=\alpha$, $y=\beta$, $y=\gamma$, $2x+y=0$; while $\alpha+\beta+\gamma=0$.

For the quartic of § 10 (i) we may also put 1 for z and c for $2+m^2\cos 2\epsilon$, getting the polar equation

$$2\sin\theta/r = c-m^2\cos 4(\theta-\tfrac{1}{2}\epsilon).$$



In the diagram are shown the following curves, reading bookwise from left to right and top to bottom :—

$18xy(x+1) = 4x^4+3x^2+9.$ (§ 2, type 1.)

$16xy(x+1) = x^4+59x^2+4.$ (§ 2, type 2, two real inflexions.)

$8xy(x+1) = x^4+11x^2+4.$ (§ 2, type 2, no real inflexions.)

$2xy(x+2) = x^4-6x^2+4.$ (§ 2, type 3.)

$8y(2x-1)(2x-2)(2x+3) = (4x^2+7)^2-96x.$ (§ 7.)

$25xy(x+1) = x^4+8x^2+16.$ (§ 2, type 4.)

$25y(x^2+1) = 25x^4-12x^3+57x^2+12x+25.$ (§ 10, no real inflexions.)

$300y(x^2+1) = 25x^4-432x^3+302x^2+432x+25.$
(§ 10, four real inflexions.)

$25y(x^2+1) = 25x^4-48x^3+78x^2+48x+25.$ (§ 10, two real inflexions.)

In each case the triple point O is at infinity on the axis of y, and one tangent at O is the line at infinity.

A THEORY OF RELATIVITY IN WHICH THE DYNAMICAL MANIFOLD CAN BE CONFORMALLY REPRESENTED UPON THE METRICAL MANIFOLD

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1. *Introduction.*

The object of this paper is to give an account of a theory of relativity which occupies a position intermediate between the Special and General Theories. The study of such a theory will provide a natural avenue of approach to the General Relativistic Theory of Einstein, and will afford a simple illustration of the analytical methods employed therein. The ten non-linear differential equations

$$G_{\mu\nu} = 0 \qquad (1.1)$$

of Einstein's theory are here replaced by a single linear equation, equivalent to the equation

$$G = 0 \qquad (1.2)$$

applied to a dynamical manifold of the particular type

$$dJ^2 = \psi(c^2dt^2 - dx^2 - dy^2 - dz^2).$$

The guiding principles of the theory here exposed appear to be in accordance with the views of Prof. Whitehead, as expressed in his series of three classical works on the principle of relativity.

2. *Action in classical mechanics and in relativistic theories.*

2.1. *The principle of least action.* Consider the motion of a particle which is projected from a fixed point P with a given velocity and then moves under the influence of a conservative field of force of potential U in accordance with the Newtonian laws of motion. If x, y, z are the Cartesian coordinates of the particle at any time t, its

velocity v is given by the equations

$$v = ds/dt, \tag{2.11}$$

$$ds^2 = dx^2+dy^2+dz^2. \tag{2.12}$$

The total energy of the particle remains constant throughout the motion. Hence

$$v^2 = 2(h-U), \tag{2.13}$$

where h is the constant total energy per unit mass. Let us now form the integral (actually, twice the action per unit mass).

$$A = \int_P^Q \sqrt{(h-U)}\,.\,ds \tag{2.14}$$

to be taken along any curve joining the points P and Q. Then it is a consequence of Newton's Laws of Motion that the actual path pursued by the particle is such as to render a minimum the integral A taken along this path. In fact, there is an exact equivalence between Newton's Laws of Motion and the conditions that the integral of action should be a minimum, while the total energy should be constant.

2.2. *Silberstein's theory.*—In the system of classical dynamics, as outlined above, the spatial relations of a moving particle, *i.e.* its trajectory, and its temporal relations, *i.e.* the time taken to describe its trajectory, are given by two distinct equations (2.14) and (2.13). A form of the special theory of relativity due to Silberstein* allows us to determine both sets of relations from one condition, analogous to the Newtonian principle of least action.

If c denotes the velocity of light *in vacuo*, we shall write

$$x = dx/d\sigma, \quad \dot{y} = dy/d\sigma, \quad \dot{z} = dz/d\sigma, \quad \dot{t} = dt/d\sigma, \tag{2.21}$$

where

$$d\sigma^2 = c^2dt^2-dx^2-dy^2-dz^2. \tag{2.22}$$

Equation (2.13) is now replaced by the identity

$$c^2\dot{t}^2-\dot{x}^2-\dot{y}^2-\dot{z}^2 \equiv 1. \tag{2.23}$$

Let ϕ denote the "retarded potential" at any point (x, y, z) at time t. Then we form the integral.

$$\mathbf{A} = \int_P^Q \phi\,.\,d\sigma, \tag{2.24}$$

* L. Silberstein, "General relativity without the equivalence hypothesis", *Phil. Mag.*, 36 (1918), 94-128.

to be taken along a route in "space-time" connecting the events P and Q.

We may now summarize the laws of motion in the relativistic mechanics of Silberstein by asserting that the actual route traversed by a particle is such as to render a maximum the integral **A** taken along that route.

In a paper by Dr. Jeffreys†, it has been implied that this condition is equivalent to four equations of the form

$$\ddot{x} = -\partial\phi/\partial x, \quad \ddot{y} = -\partial\phi/\partial y, \quad \ddot{z} = -\partial\phi/\partial z, \quad \ddot{t} = \partial\phi/c^2\partial t. \quad (2.25)$$

This statement, unfortunately, overlooks the fact that, in virtue of the identity (2 . 23), we have the relation

$$c^2\ddot{t} \,.\, \dot{t} \equiv \ddot{x} \,.\, \dot{x} + \ddot{y} \,.\, \dot{y} + \ddot{z} \,.\, \dot{z}$$

whence, from equation (2 . 25),

$$\frac{\partial\phi}{\partial x} \cdot \frac{\partial x}{\partial\sigma} + \frac{\partial\phi}{\partial y} \cdot \frac{\partial y}{\partial\sigma} + \frac{\partial\phi}{\partial x} \cdot \frac{\partial z}{\partial\sigma} + \frac{\partial\phi}{\partial t} \cdot \frac{\partial t}{\partial\sigma} = 0,$$

so that ϕ would be constant along the route traversed by the particle.

The simplest way to compare the consequences of making the integral **A** stationary with Newton's laws of motion is to express this condition by means of four Lagrangian equations. Let

$$L = \phi(c^2\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{\frac{1}{2}} \equiv \phi. \quad (2.26)$$

Then the conditions that the integral **A** should be stationary are

$$\frac{d}{d\sigma} \cdot \left(\frac{\partial L}{\partial \dot{u}}\right) = \frac{\partial L}{\partial u}, \quad (2.27)$$

where $u = x$, y, z, or t. Assuming ϕ to be independent of t, we see that $\partial L/\partial\dot{t}$ must be a constant,

i.e. $$\phi c^2 \dot{t} = \eta \quad (2.28)$$

(η being a constant).

The other three equations of (2 . 27) may now be written

$$\frac{d^2}{dt^2} \cdot (x, y, z) = -\frac{c^4}{2\eta^2} \cdot \frac{\partial(\phi^2)}{\partial x, y, z}, \quad (2.29)$$

i.e. we have obtained the equations of motion of Newtonian dynamics for a conservative field of force of potential $(c^4/2\eta^2) \,.\, \phi^2$.

† H. Jeffreys, "The force function in the theory of ordinary relativity", *Phil. Mag.*, 38 (1919), 113–115.

2.3. *Venkata Row's theory.*—A theory of special relativity has been suggested by Venkata Row* which, although apparently original, is intimately connected with Silberstein's theory. On writing

$$\phi = e^{\varphi} \tag{2.31}$$

and

$$L = e^{\varphi}(c^2\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2)^{\frac{1}{2}}, \tag{2.32}$$

we find that the conditions that the integral

$$\mathbf{A} = \int_P^Q L d\sigma \tag{2.24}$$

should be stationary are expressible in the form

$$\left.\begin{aligned} \ddot{x} &= -\partial\varphi/\partial x - \dot{x}\,.\,\partial\varphi/\partial\sigma \\ \ddot{y} &= -\partial\varphi/\partial y - \dot{y}\,.\,\partial\varphi/\partial\sigma \\ \ddot{z} &= -\partial\varphi/\partial z - \dot{z}\,.\,\partial\varphi/\partial\sigma \\ \ddot{t} &= \partial\varphi/c^2\partial t - \dot{t}\,.\,\partial\varphi/\partial\sigma \end{aligned}\right\}, \tag{2.33}$$

and

which are formally identical with the equations obtained by Venkata Row.

2.4. *Transformation of the electrodynamical equations.*—The researches which we have quoted above suffice to indicate the importance of the quadratic form,

$$d\mathbf{A}^2 = \phi^2(c^2dt^2 - dx^2 - dy^2 - dz^2), \tag{2.41}$$

in the special theory of relativity. Further significance may be given to this form in the light of the investigations of Bateman†, who has shown that the only biuniform, punctual transformations for which the electromagnetic equations of Maxwell are covariant, while the total change in corresponding elements of volume is invariant, are those which leave invariant the equation,

$$d\sigma^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = 0. \tag{2.42}$$

It is convenient to write

$$w = ict, \tag{2.421}$$

* C. K. Venkata Row, "Relativity referred to a flat space-time", *Nature*, 115 (1925), 261–2.

† H. Bateman, "The transformation of the electrodynamical equations", *Proc. London Math. Soc.* (2), 8 (1909), 223–264.

when the preceding equation (2 . 42) becomes

$$dw^2+dx^2+dy^2+dz^2 = 0. \qquad (2.422)$$

The differential equation of the first order which is associated with this Mongian equation is the Hamiltonian equation

$$\left(\frac{\partial u}{\partial w}\right)^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2 = 0, \qquad (2.43)$$

and the corresponding equation of the second order is the d'Alembertian equation

$$\frac{\partial^2 v}{\partial w^2}+\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2} = 0. \qquad (2.44)$$

The last equation is satisfied by any quantity v, the changes in which are propagated with the speed c. The preceding equation (2 . 43), which defines the characteristic surface of (2 . 44), indicates, as has been shown by J. Hadamard†, that a surface of separation at which v and the first differential coefficients of v are continuous, but the second differentials discontinuous, advances parallel to itself with the same velocity c. Finally equation (2 . 42) characterizes a particle which always moves with the speed c. These three equations admit the same group of transformations, which is usually named "the conformal group".

3. *The conformal group.*

3 . 1. *Comparison of the special and general theories of relativity.*— In all theories of relativity the route described by a material particle is specified in a manner analogous to that adopted in § 2 . 2, as that which would render a maximum a certain integral

$$\int_P^Q dJ, \qquad (3.11)$$

taken between two events P and Q, where the differential element of action dJ is determined by means of ten functions

$$J_{\mu\nu} \ (= J_{\nu\mu}), \ (\mu, \nu = 1, 2, 3, 4)$$

through the equation

$$dJ^2 = \Sigma(\mu, \nu) J_{\mu\nu} . dx_\mu dx_\nu, \qquad (3.12)$$

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* J. Hadamard, "Leçons sur la propagation des ondes" (1903), 292.

in which x_1, x_2, x_3, x_4 are written for x, y, z, w $(= ict)$. Different species of relativistic theories are distinguished by the group of transformations which is admitted by the differential equations determining the functions $J_{\mu\nu}$ in the various theories.

The relativistic theories, usually described as "special", are characterized by the invariance in magnitude and in form of the expression

$$d\sigma^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

$$= -(dw^2 + dx^2 + dy^2 + dz^2). \qquad (2.42)$$

The transformations of dw, dx, dy, and dz which leave unaltered this quadratic form constitute the quaternary orthogonal group L, and their explicit expressions in terms of six independent parameters may be found in Scott and Mathews' Treatise on Determinants*. These transformations correspond to rotations in the planes of (y, z), (z, x), and (x, y), and to Lorentz transformations in (x, t), (y, t), and (z, t). When we assume the invariance in magnitude (but not in form) of dJ, equation (3.12) allows us to "extend" the group of transformations L to include the group of transformations of the function $J_{\mu\nu}$ and their derivatives to any order. The differential equations which determine the functions $J_{\mu\nu}$ and thereby specify the character of the field of force which is acting are assumed to admit the operations of this extended group L'. Thus in the theories of Silberstein† and Venkata Row‡ all the functions $J_{\mu\nu}$ are respectively equal to ϕ^2 or $e^{2\varphi}$, while the functions $J_{\mu\nu}$ vanish when $\mu \neq \nu$ and ϕ and φ satisfy the invariant d'Alembertian equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial w^2} = 4\pi\rho_0, \qquad (3.13)$$

where ρ_0 denoted the "proper-density" of the matter at $(x, y, z\, w)$.

In Whitehead's first law of gravitation§, the functions $J_{\mu\nu}$, although unequal, all depend upon a potential function ψ determined by the above equation (3.13). An alternative law‖, proposed by the same author, defines the functions $J_{\mu\nu}$ in empty space by the equations

$$\Sigma(\rho)\frac{\partial}{\partial x_\rho}\left\{\frac{\partial J_{\mu\rho}}{\partial x_\nu} + \frac{\partial J_{\nu\rho}}{\partial x_\mu} - \frac{\partial J_{\mu\nu}}{\partial x_\rho}\right\} = 0 \quad (\mu, \rho = 1, 2, 3, 4). \qquad (3.14)$$

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* R. F. Scott and G. B. Mathews, "The theory of determinants" (1904), 200.

† *Loc. cit.*

‡ *Loc. cit.*

§ A. N. Whitehead, "The principle of relativity . . ." (1922), 81-82.

‖ *Loc. cit.*, 86, Equation (21).

The covariance of these equations for the group L' is established in pages 185 and 186 of the work quoted above.

In Einstein's General Theory of Relativity the coordinates x_1, x_2, x_3, x_4 are allowed to suffer any arbitrary transformation Γ, and then by means of equation (3 . 12) and the assumed invariance in magnitude of dJ we construct the extended group of transformations Γ' for the functions $J_{\mu\nu}$ and their derivatives. In an elaborate memoir*, Forsyth has calculated an exhaustive list of the invariantive functions of $J_{\mu\nu}$ and their derivatives up to the second order. The evanescence of a particular set of these invariants is equivalent to Einstein's equations

$$G_{\mu\nu} = 0. \qquad (1 . 1)$$

In the relativistic theory of this paper, we assume that the functions $J_{\mu\mu}$ are all equal to ψ; while the functions $J_{\mu\nu}$ vanish when $\mu \neq \nu$, as in the theory of Silberstein. We then allow the variables x_1, x_2, x_3, x_4 to suffer any transformation of the conformal group C, for which dJ^2 is invariant in form. Assuming the invariance in magnitude also of dJ^2 we construct the extended group C' of the transformations of ψ and its derivatives to the second order. The evanescence of a particular invariantive function of ψ and its derivatives is then taken as the characteristic equation of the gravitational field determining ψ.

3 . 2. *The transformations and invariants of the conformal group.*— The conformal group in four variables has been studied by Lie†‡, who has shown that all transformations of the group are composed of translations, rotations, magnifications, and inversions. A complete set of invariantive functions is best obtained by utilizing the infinitesimal transformations of the group, but considerably more insight into the problem is obtained by studying separately the effect of each of the four sub-groups enumerated above; especially since our object is to construct one invariant only, the vanishing of which shall be approximately equivalent to the d'Alembertian equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial w^2} = 0. \qquad (3 . 21)$$

* A. R. Forsyth, "The concomitants . . . of quadratic differential forms in four variables", *Proc. Royal Soc. Edinburgh*, 42 (1922), 147–213.

† M. S. Lie, "Ueber diejenige Theorie eines Raumes mit beliebig vielen Dimensionen, der Krummungs-Theorie des gewöhnlicher Raumes entsprich", *Göttinger Nachrichten* (1871), 191–209.

‡ M. S. Lie and Engel, "Theorie der Transformationsgruppen", 3 (1893), Kapitel 17 and 18.

Since $d\sigma$ (2 . 42) is an invariant of the rotation sub-group, so also are the functions ψ,

$$P=\left(\frac{\partial\psi}{\partial w}\right)^2+\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+\left(\frac{\partial\psi}{\partial z}\right)^2 \quad \text{and} \quad S=\frac{\partial^2\psi}{\partial\omega^2}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}.$$

None of these functions, however, is an invariant of the sub-group of magnifications, a typical transformation of which is given by the equations

$$w'=\mu w, \quad x'=\mu x, \quad y'=\mu y, \quad z'=\mu z \quad \text{(where } \mu \text{ is a constant)},$$

whence $$d\sigma'^2=\mu^2 d\sigma^2;$$

but $$\psi' d\sigma'^2=\psi d\sigma^2,$$

therefore $$\psi'=\mu^{-2}\psi, \quad P'=\mu^{-6}P \quad \text{and} \quad S'=\mu^{-4}S.$$

Hence the two functions S/ψ^2 and P/ψ^3 are invariants both of the rotation sub-group and the magnification sub-group.

The finite equations of the inversion sub-group are [in the notation of equation (3 . 12)]

$$x_i'=\frac{x_i-\tau_i\Sigma(j)(x_j^2)}{1-2\Sigma(j)(x_j\tau_j)+\Sigma(j)(x_j^2)\,\Sigma(j)(\tau_j^2)}* \quad (i=1,\,2,\,3,\,4). \quad (3\,.\,22)$$

Let I_C denote this transformation, which belongs to a continuous four-parameter group. Let T denote the translation

$$x_i'=x_i-\tau_i R^2,$$

which belongs to a group of similar type; and let I_D denote the inversion

$$x_i'=R^2 x_i/\Sigma(j)(x_j^2), \quad (3\,.\,23)$$

which belongs to a discontinuous one-parameter group. Then it is easily seen that the transformation I_C is the conjugate to T with respect to I_D, or in symbols

$$I_C=I_D\,.\,T\,.\,I_D^{-1}.$$

Hence any invariant of the groups I_D and T is also an invariant of the group I_C, and we need only study the effect of the inversion I_D on the functions S/ψ^2 and P/ψ^3.

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* M. S. Lie, "Ueber diejenige Theorie", etc., 350.

We make Green's substitution

$$\left.\begin{aligned} w &= \rho \sin\alpha \sin\beta \cos\phi \\ x &= \rho \sin\alpha \sin\beta \sin\phi \\ y &= \rho \sin\alpha \cos\beta \\ z &= \rho \cos\alpha \end{aligned}\right\}, \tag{3.24}$$

whence the transformation of equation (3.23) becomes

$$\rho' . \rho = R^2; \quad \alpha' = \alpha, \ \beta' = \beta, \ \phi' = \phi. \tag{3.25}$$

Since $$d\sigma^2 = d\rho^2 + \rho^2\{d\alpha^2 + \sin^2\alpha(d\beta^2 + \sin^2\beta . d\phi^2)\},$$

we have $$d\sigma'^2 = R^4 . \rho^{-4} . d\sigma^2$$

and $$\psi' = \rho^4 . R^{-4} . \psi.$$

Now $$P = \left(\frac{\partial\psi}{\partial\rho}\right)^2 + \frac{1}{\rho^2} \cdot \left\{\left(\frac{\partial\psi}{\partial\alpha}\right)^2 + \left(\frac{\partial\psi}{\sin\alpha\,\partial\beta}\right)^2 + \left(\frac{\partial\psi}{\sin\alpha\sin\beta\,\partial\phi}\right)^2\right\}$$

and $$S = \left(\frac{\partial^2\psi}{\partial\rho^2}\right) + \frac{3}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{2}{\rho^2}\cot\alpha\,\frac{\partial\psi}{\partial\alpha} + \frac{\partial^2\psi}{\rho^2\partial\alpha^2}$$
$$+ \frac{1}{\rho^2\sin^2\alpha\sin^2\beta} \cdot \frac{\partial}{\partial\beta}\left(\sin\beta\frac{\partial\psi}{\partial\beta}\right) + \frac{1}{\rho^2\sin^2\alpha\sin^2\beta} \cdot \frac{\partial^2\psi}{\partial\phi^2}. \tag{3.26}$$

Hence $$P' = \rho^{12} . R^{-12} . P + 8 . \rho^{11} . R^{-12} . \psi \frac{\partial\psi}{\partial\rho} + 16\rho^{10}R^{-12}\psi^2$$

and $$S = \rho^8 . R^{-8} . S + 4 . \rho^7 R^{-8} \frac{\partial\psi}{\partial\rho} + 8\rho^6 . R^{-8} . \psi.$$

Therefore $S/\psi^2 - P/2\psi^3$ is an invariant of the conformal group.

We shall take as our characteristic equation of the gravitational field

$$\frac{S}{\psi^2} - \frac{P}{2\psi^3} = \frac{8\pi\gamma}{c^2} \cdot \rho_0, \tag{3.27}$$

when ρ_0 is the "proper density" of matter, γ the constant of gravitation, and c the velocity of light. This is analogous to Einstein's 10 equations (1.1).

3.3. *The characteristic equation of the gravitational field.* Let $\psi = \phi^2$ and let $\partial^2\phi/\partial x_\mu \partial x_\nu = \phi_{\mu\nu}$, and we find that

$$\frac{S}{\psi^2} - \frac{P}{2\psi^3} = \frac{2}{\phi^3}\left(\frac{\partial^2\phi}{\partial w^2} + \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)$$
$$= 2(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44})\phi^{-3}.$$

Hence the function ϕ satisfies the equation

$$\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44} = \frac{4\pi\gamma}{c^2} \cdot \rho_0 \cdot \phi^3, \tag{3.31}$$

which is linear *in vacuo*, *i.e.* when $\rho_0 = 0$.

To compare this result with Einstein's equations, we find it convenient to use the notation of Campbell's paper*, in which the four index symbols $(rkih)$ are replaced by two index symbols α_{pq} by taking 23 in this order as 1, 31 as 2, 12 as 3, 14 as 4, 15 as 5, and 16 as 6, so that $(2331) = \alpha_{12}$, etc. Then for the differential form

$$dJ^2 = \psi(dx_1^2+dx_2^2+dx_3^2+dx_4^2), \tag{2.41}$$

we find that

$$\alpha_{11} = -\tfrac{1}{2}(\psi_{22}+\psi_{33})+\frac{1}{2\psi}(\psi_2^2+\psi_3^2-\tfrac{1}{2}\psi_1^2-\tfrac{1}{2}\psi_4^2),$$

with similar expressions for α_{22} and α_{33} obtained by the cyclic interchange of the suffixes 1, 2, and 3, and

$$\alpha_{44} = -\tfrac{1}{2}(\psi_{11}+\psi_{44})+\frac{1}{2\psi}(\psi_1^2+\psi_4^2-\tfrac{1}{2}\psi_2^2-\tfrac{1}{2}\psi_3^2)$$

with similar expressions for α_{55} and α_{66} obtained by the cyclic interchange of the suffixes 1, 2, and 3, while ϕ_{44} and ϕ_4^2 remain unaltered.

Also $$\alpha_{14} = \alpha_{25} = \alpha_{36} = 0.$$

Now
$$-G_{11} = (\alpha_{22}+\alpha_{33}+\alpha_{44})/\psi$$
$$-G_{22} = (\alpha_{33}+\alpha_{11}+\alpha_{55})/\psi$$
$$-G_{33} = (\alpha_{11}+\alpha_{22}+\alpha_{66})/\psi$$
$$-G_{44} = (\alpha_{44}+\alpha_{55}+\alpha_{66})/\psi.$$

Therefore $$G_{\mu\mu} = S/2\psi+\psi_{\mu\mu}/\psi-3\psi_\mu^2/\psi^2.$$

Hence
$$\begin{aligned} G &= (G_{11}+G_{22}+G_{33}+G_{44})/\psi \\ &= -2(\alpha_{11}+\alpha_{22}+\alpha_{33}+\alpha_{44}+\alpha_{55}+\alpha_{66}) \cdot \psi^{-2} \\ &= 3\{S/\psi^2-P/2\psi^3\}, \end{aligned} \tag{3.32}$$

* J. E. Campbell, "Einstein's theory of gravitation as an hypothesis in differential geometry", *Proc. London Math. Soc.* (2), 20 (1921), 1–14.

so that our gravitational equation is equivalent to the condition

$$G = \frac{24\pi\gamma}{c^2}\rho_0. \tag{3.33}$$

Finally, we may note that Schouten* has shown that the necessary and sufficient conditions that a manifold whose linear element is given by the equation

$$dJ^2 = \Sigma(\mu\nu) J_{\mu\nu} dx_\mu dx_\nu$$

should be conformally representable upon the manifold whose linear element is given by the equation $d\sigma^2 = \Sigma(\mu) dx_\mu^2$ are that there should exist a set of 10 functions $L_{\mu\nu}$ $(= L_{\nu\mu})$,

such that $$2(\rho\mu\nu\sigma) = J_{\rho\nu}L_{\mu\sigma} - J_{\mu\nu}L_{\rho\sigma} - J_{\rho\sigma}L_{\mu\nu} + J_{\mu\sigma}L_{\rho\nu}.$$

4. *Some special gravitational fields.*

In this section we shall consider two special problems illustrative of the general gravitational equation obtained in the previous paragraphs.

4.1. *Gravitational fields with spherical symmetry (external problem).*—Take any plane passing through the centre of the sun and let u, $\Im$ denote the reciprocal of the radius vector r, and the angular distance of any point in this plane. We assume that ϕ is a function of r only, whence, from Equation (3.31), we obtain the condition

$$\frac{d^2\phi}{dr^2} + \frac{2d\phi}{rdr} = 0, \tag{4.11}$$

holding outside the surface of the sun.

We take as the solution

$$\phi = 1 - \frac{\gamma M}{c^2 r}, \tag{4.111}$$

where M = the mass of the sun.

As in (2.2), we form the Lagrangian function for a planet

$$L = \phi(c^2\dot{t}^2 - \dot{r}^2 - r^2\dot{\Im}^2)^{\frac{1}{2}},$$

and notice that $\phi \cdot r^2 \cdot \dot{\Im}$ is equal to a constant, say h/c, where h is approximately twice the areal velocity of the planet. Hence the equation of the orbit will be

$$\frac{d^2u}{d\Im^2} + u = -\frac{c^2}{2h^2}\frac{\partial(\phi^2)}{\partial u}. \tag{4.12}$$

* J. A. Schouten, "Ueber die Konforme Abbildung . . .", *Math. Zeit.*, 11 (1921), 58–88.

Therefore $$\frac{d^2u}{d\vartheta^2}+\kappa^2 u = \frac{\gamma M}{h^2} \tag{4.13}$$

and $$l.u = 1+\epsilon\cos\kappa\vartheta, \tag{4.131}$$

where $$\kappa^2 h^2 = l\gamma M \quad \text{and} \quad \kappa^2 = 1+\frac{\gamma^2 M^2}{h^2 c^2}. \tag{4.132}$$

The equation (4.131) shows that the orbit is an ellipse whose major axis has a retrograde angular motion of $\gamma^2 M^2\pi/h^2c^2$ radians per revolution—one-sixth of the amount predicted by Einstein's and Whitehead's theories. This result, of course, agrees with that obtained by Silberstein in the paper quoted above*.

We may compare this result with that of Venkata Row's theory†. On writing in (2.31)

$$\varphi = \log\phi = -\frac{\gamma M}{c^2}u, \tag{4.14}$$

we find that the equation to the orbit is

$$\frac{d^2u}{d\vartheta^2}+u = -\frac{c^2}{2h^2}\cdot\frac{d}{du}(e^{2\varphi}) \tag{4.15}$$

$$\doteqdot \frac{\gamma M}{h^2}\cdot e^{2\varphi}.$$

Therefore $$\frac{d^2u}{d\vartheta^2}+\kappa^2 u \doteqdot \frac{\gamma M}{h^2}, \tag{4.16}$$

where $$\kappa^2 = 1+\frac{2\gamma^2 M^2}{h^2 c^2}. \tag{4.161}$$

Hence the retrograde movement of the perihelion has twice the value it has in Silberstein's theory.

4.2. *Gravitational fields with spherical symmetry (internal problem).*—In the interior of a homogeneous sphere of constant density ρ_0 the characteristic equation (3.31) becomes

$$\frac{d^2\phi}{dr^2}+\frac{2d\phi}{r.dr} = \lambda\phi^3, \tag{4.21}$$

where $$\lambda = \frac{4\pi\gamma}{c^2}\rho_0. \tag{4.22}$$

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* *Loc. cit.*

† *Loc. cit.*

On writing $z = r^2$, equation (4 . 21) becomes

$$4z\frac{d^2\phi}{dz^2}+6\frac{d\phi}{dz} = \lambda\phi^3. \tag{4.23}$$

Let the integral of this equation which is regular at $z = 0$ be

$$\phi \equiv \phi(r) = \sum_{n=0}^{\infty} a_n z^n. \tag{4.24}$$

Then $$\sum_{n=1}^{\infty}(4n+2)na_n z^{n-1} = \lambda\left\{\sum_{n=0}^{\infty} a_n z^n\right\}^3.$$

From this equation we see that a_n is expressible as a homogeneous function of a_0, a_1, ..., a_{n-1} of the third degree; and since λ is essentially positive, the coefficients of all the terms in this function are positive. Hence, by induction, we see that a_n has the same sign as a_0, and so $\phi(r)$ has the same sign as $\phi(0)$ throughout the interior of the sphere*.

To prove that this sign is positive, we observe that the differential equations (4 . 11) and (4 . 21) may be replaced by the integral equations

$$\phi(r) = \phi(0)+\lambda\int_0^r\{\phi(\xi)\}^3\left(\frac{1}{\xi}-\frac{1}{r}\right)\xi^2 d\xi \quad (r \leqslant R),$$

$$\phi(r) = \phi(0)+\lambda\int_0^R\{\phi(\xi)\}^3\xi d\xi-\frac{\lambda}{r}\int_0^R\{\phi(\xi)\}^3\xi^2 d\xi \quad (r \geqslant R),$$

R being the radius of the sphere. On comparing this last equation with (4 . 111), namely $\phi(r) = 1-\gamma M/c^2 r$, we see that

$$\tfrac{1}{3}R^3 = \int_0^R\{\phi(\xi)\}^3\xi^2 d\xi.$$

Since this integral has the same sign as $\phi(\xi)$, this sign must be positive.

Therefore, in particular, $\phi(r)$ is positive when $r = R$, so that $\gamma M < c^2 R$, and hence

$$R < c\sqrt{\left(\frac{3}{4\pi\gamma\rho_0}\right)}, \tag{4.25}$$

It is interesting to compare the upper limit to R indicated by the above analysis with the values obtained by Schwarzschild† and

* I am indebted to a referee for this proof that all the coefficients a_0, a_1, ..., a_n, ... have the same sign.

† de Donder, "La Gravifique Einsteinienne", *Gauthier-Villars* (1921), 169–176.

Einstein* on different theories :

$$R < \frac{c}{\sqrt{(3\pi\gamma\rho_0)}} \quad \text{(Schwarzschild)},$$

$$R < \frac{c}{\sqrt{(4\pi\gamma\rho_0)}} \quad \text{(Einstein)}.$$

All three values have the same order of magnitude, the smallest estimate (Einstein's) being about 300 million kilometres divided by $\sqrt{\rho_0}$ the numerical value of the square root of the density in *gms/cc.*

5. *Cosmological theory.*

The characteristic equation of § 3 . 3 may be modified by assuming that the invariant

$$(\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44}) \cdot \phi^{-3}$$

has a finite constant value $2 . R^{-2}$ in empty space. This assumption corresponds to the modified gravitation equations appropriate to de Sitter's Cosmology†, namely

$$G_{\mu\nu} = \frac{3}{R^2} \cdot g_{\mu\nu}$$

and, like them, implies that

$$G = 12 . R^{-2}. \tag{5 . 1}$$

The solution of the equation

$$\phi_{11}+\phi_{22}+\phi_{33}+\phi_{44} = 2\phi^3 . R^{-2}$$

which depends only on $x_1^2+x_2^2+x_3^2+x_4^2 = \rho^2$ is

$$\phi = (\alpha+\beta\rho^2)^{-1},$$

where the two constants α, β are related by the condition

$$4\alpha\beta . R^2 = -1.$$

On writing $\alpha = 1$ we obtain the element of dJ^2 in the form

$$\frac{dx_1^2+dx_2^2+dx_3^2+dx_4^2}{\{1-\rho^2/4R^2\}^2} = dJ^2, \tag{5 . 2}$$

given by Riemann‡ as the linear element in a manifold of uniform

* Prof. A. S. Eddington, "The mathematical theory of relativity", *Cambridge* (1925), 170.

† de Sitter, "On Einstein's theory of relativity and its astronomical consequences", III, *Monthly Notices R.A.S.*, 78 (1918), 3.

‡ B. Riemann, "Ueber die Hypothesen welche der Geometrie zu Grunde liegen", *Werke* (1876), 264.

and isotropic curvature. On writing

$$x_1 = 2R \tan \tfrac{1}{2}\theta_1 . \sin \theta_2 . \sin \theta_3 . \sin \theta_4,$$

$$x_2 = 2R \tan \tfrac{1}{2}\theta_1 . \sin \theta_2 . \sin \theta_3 . \cos \theta_4,$$

$$x_3 = 2R \tan \tfrac{1}{2}\theta_1 . \sin \theta_2 . \cos \theta_3,$$

$$x_4 = 2R \tan \tfrac{1}{2}\theta_1 . \cos \theta_2,$$

we obtain as the expression of the element of action

$$dJ^2 = R^2 \{ d\theta_1^2 + \sin^2 \theta_1 [d\theta_2^2 + \sin^2 \theta_2 (d\theta_3^2 + \sin^2 \theta_3 . d\theta_4^2)] \}, \qquad (5.3)$$

while the transformation

$$\xi_1 = R \tan \theta_1 . \sin \theta_2 . \sin \theta_3 . \sin \theta_4,$$

$$\xi_2 = R \tan \theta_1 . \sin \theta_2 . \sin \theta_3 . \cos \theta_4,$$

$$\xi_3 = R \tan \theta_1 . \sin \theta_2 . \cos \theta_3,$$

$$\xi_4 = R \tan \theta_1 . \cos \theta_2,$$

enables us to write dJ^2 in the canonical form

$$R^2 . \xi^{-2} (d\xi_1^2 + d\xi_2^2 + d\xi_3^2 + d\xi_4^2 + d\xi^2), \qquad (5.4)$$

where

$$\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi^2 = R^2, \qquad (5.41)$$

and ξ_1, ξ_2, ξ_3, ξ_4 are geodetic coordinates, *i.e.* any geodesic is given by three linear equations in these variables.

Let us consider, as in de Sitter's theory, the motion of particles in such a Space-Time, apart from gravitational action. It is evident that the motion will be uniplanar, and that the plane of motion may be taken as

$$\xi_1 = 0 \quad \text{or} \quad x_1 = 0. \qquad (5.5)$$

The two equations giving the motion of the particle must be linear in ξ_2, ξ_3, ξ_4. But $\theta_4 = 0$ and

$$\xi_2 : \xi_3 : \xi_4 : 1 = x_2 : x_3 : x_4 : \left\{ 1 - \frac{x_2^2 + x_3^2 + x_4^2}{4R^2} \right\}. \qquad (5.6)$$

Hence, unless the particle passes through the origin initially (in which

case we have $x_2 = 0$, $x_3 = 0$, $x_4 = 0$), the path described has the equations

$$a\xi_3 + b\xi_4 + 1 = 0, \quad c\xi_2 + d\xi_4 + 1 = 0,$$

or
$$a(y-y_0)^2 + \beta(z-z_0)^2 = 1$$

and
$$t = \lambda y + \mu z \qquad (5.7)$$

Just as in de Sitter's theory, the velocity of particles will vary with their distance from the observer, so that we have an explanation of the high velocities of distant stars. These velocities may be directed towards or away from the observer.

6. *Conclusion.*

We have developed a theory of relativity, which possesses the same general features as Einstein's theory, but in which the analysis is considerably simpler. This new theory has been applied to the familiar problems of the secular variation in the perihelion of Mercury, the high velocities of spiral nebulae, and the problem of the limiting size of the homogeneous sphere. It is scarcely to be expected that this theory has any value as a representation of gravitational phenomena, but it may serve the useful purpose of providing an elementary introduction to Einstein's theory.

THE FUNCTIONS INVOLVED IN THE THEORY OF A THIN ELASTIC RECTANGULAR PLATE, CLAMPED AT THE EDGES, AND CERTAIN INTEGRAL EQUATIONS SATISFIED BY SUCH FUNCTIONS

By A. C. DIXON.

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THE discussion of this subject introduces functions of a complex variable with singularities of a special kind; such singularities occur at the corners in the theory of a plate bounded by two circular arcs, but in the present case they occur at all the points of the lattice to which the rectangle belongs. The determination of Green's function, that is, the deflection produced at any point of the plate by unit load at any other point, is found to depend on the solution of certain integral equations in which the nuclei are not bounded, so that special methods have to be applied. Incidentally, the solution of these integral equations leads to the determination of Green's function for a clamped plate in the form of an infinite cross (§ 17).

The general discussion is illustrated (§ 3) by two special cases in which the theory is already known. A summary of §§ 1, 2, 4–15 is given in § 16.

1. In rectangular Cartesian coordinates the image of the point (x, y) by reflexion at the straight line $x\cos\alpha+y\sin\alpha-p=0$ has for its coordinates

$$x-2\cos\alpha(x\cos\alpha+y\sin\alpha-p),\quad y-2\sin\alpha(x\cos\alpha+y\sin\alpha-p),\qquad (1.0)$$

so that, if z, Z are the values of $x+iy$ corresponding to the two points and z is $x-iy$,

$$Z=-ze^{2i\alpha}+2pe^{i\alpha}. \qquad (1.1)$$

The equation to the line may then be written $z = Z$, where Z is this linear integral function of z.

Now any function reduced to zero by the operator ∇^4 may be written as

$$f_1(z)+f_2(z)+zf_3(z)+zf_4(z) \tag{1.21}$$

and therefore as

$$f_1(z)+f_2(Z)+(z-Z)f_3(z)+(z-Z)f_4(Z), \tag{1.22}$$

with a change of notation.

If this function vanishes on the line $z = Z$, we have

$$f_1(z)+f_2(z) = 0 \tag{1.23}$$

for all values of z, since an analytic function vanishing on a segment of line must vanish identically. If it vanishes on the line $z = Z$ to the second order, then

$$f_1'(z)+f_3(z)+f_4(z) \tag{1.24}$$

must similarly vanish for all values of z, and by another change of notation the function may be written as

$$\phi(z)-\psi(z)-\phi(Z)+\psi(Z)+(Z-z)\{\phi'(z)-\psi'(Z)\} \tag{1.25}$$

or as

$$-\int_Z^z (z-t)\{\phi''(Z+z-t)+\psi''(t)\}\,dt. \tag{1.26}$$

When $Z = ze^{2i\gamma}$, so that the line goes through the origin and makes an angle γ with the real axis, we may use the form

$$e^{-2i\gamma}\phi(z)-e^{-2i\gamma}\phi(ze^{2i\gamma})-e^{2i\gamma}\psi(ze^{-2i\gamma})$$

$$+e^{2i\gamma}\psi(z)+(z-ze^{-2i\gamma})\phi'(z)+(z-ze^{2i\gamma})\psi'(z). \tag{1.27}$$

Let $G(z, z, z_0, z_0)$, or simply G, denote the deflection produced at any point (x, y) by unit load at a point (x_0, y_0) on a thin uniform isotropic rectangular plate bounded by the four lines

$$y = 0, \quad y = b, \quad x = 0, \quad x = a \tag{1.31}$$

or

$$z = z, \quad z = 2ib+z, \quad z = -z, \quad z = 2a-z, \tag{1.32}$$

and clamped along the edges. The determination of G, often called Green's function, is the main object of the present paper.

This function must vanish to the second order on each edge and can therefore be written in four ways, namely

$$\phi_1(z)-\psi_1(z)-\phi_1(\mathrm{z})+\psi_1(\mathrm{z})+(\mathrm{z}-z)\{\phi_1'(z)-\psi_1'(\mathrm{z})\} \tag{1.33}$$

$$\phi_2(z)-\psi_2(z-2ib)-\phi_2(\mathrm{z}+2ib)+\psi_2(\mathrm{z})+(\mathrm{z}+2ib-z)\{\phi_2'(z)-\psi_2'(\mathrm{z})\}, \tag{1.34}$$

$$-\phi_3(z)+\psi_3(-z)+\phi_3(-\mathrm{z})-\psi_3(\mathrm{z})+(\mathrm{z}+z)\{\phi_3'(z)+\psi_3'(\mathrm{z})\}, \tag{1.35}$$

$$-\phi_4(z)+\psi_4(2a-z)+\phi_4(2a-\mathrm{z})-\psi_4(\mathrm{z})+(\mathrm{z}+z-2a)\{\phi_4'(z)+\psi_4'(\mathrm{z})\}. \tag{1.36}$$

These four being equivalent, we find, by operating with ∇^2, that $\phi_r''(z)+\psi_r''(\mathrm{z})$ is the same for all the suffixes and thus

$$\phi_r(z)-\phi_s(z) = Az^2+Bz+C \quad (r, s = 1, 2, 3, 4),$$

$$\psi_r(\mathrm{z})-\psi_s(\mathrm{z}) = -A\mathrm{z}^2+E\mathrm{z}+F \quad (r, s = 1, 2, 3, 4),$$

where A, B, C, E, F are constants which may be taken as zero, since (1.25) is unchanged by the addition of such terms to ϕ and ψ. We may then drop the suffixes 1, 2, 3, 4, and we need to find functions ϕ, ψ regular as a rule over the plate and such that the expressions (1.33)–(1.36) are equal and represent a function of x, y with the singularity of $\frac{1}{2}(z-z_0)(\mathrm{z}-\mathrm{z}_0)\{\log(z-z_0)+\log(\mathrm{z}-\mathrm{z}_0)\}$. It is sometimes useful to remember that $G(z, \mathrm{z}, z_0, \mathrm{z}_0)$ is a symmetric function of the positions of the two points (x, y), (x_0, y_0).

2. Comparing (1.35) with (1.36), we find

$$\psi(-z)-\psi(2a-z)+2a\phi'(z) = \phi(2a-\mathrm{z})-\phi(-\mathrm{z})-2a\psi'(\mathrm{z}), \tag{2.01}$$

so that each of these has a constant value, say A.

Similarly, from (1.33) and (1.34),

$$\psi(z)-\psi(z-2ib)+2ib\phi'(z) = -\phi(\mathrm{z})+\phi(\mathrm{z}+2ib)+2ib\psi'(\mathrm{z}) = B, \tag{2.02}$$

another constant, and from (1.33) and (1.35)

$$2\phi(z)-\psi(z)-\psi(-z)-2z\phi'(z)$$
$$= -2\psi(\mathrm{z})+\phi(\mathrm{z})+\phi(-\mathrm{z})+2\mathrm{z}\psi'(\mathrm{z}) = C, \tag{2.03}$$

which is also constant.

By the addition to $\phi(z)$, $\psi(z)$ of the terms

$$-Az/4a-Bz/4ib \quad \text{and} \quad Az/4a-Bz/4ib+\tfrac{1}{2}C,$$

the constants A, B, C are reduced to zero and the expressions (1.33), etc., are not affected.

Hence the conditions to be satisfied by ϕ, ψ are

$$\psi(-z+2a)-\psi(-z) = 2a\phi'(z) \tag{2.11}$$

$$\phi(-z+2a)-\phi(-z) = 2a\psi'(z), \tag{2.12}$$

$$\psi(z-2ib)-\psi(z) = 2ib\phi'(z), \tag{2.13}$$

$$\phi(z+2ib)-\phi(z) = -2ib\psi'(z), \tag{2.14}$$

$$\psi(z)+\psi(-z) = 2\phi(z)-2z\phi'(z), \tag{2.15}$$

$$\phi(z)+\phi(-z) = 2\psi(z)-2z\psi'(z). \tag{2.16}$$

From (2.15), (2.16) it follows that ϕ, ψ cannot be uniform in the domain of the origin. For, if they were, $\phi(z)/z$ and $\psi(z)/z$ would be integrals of even functions, that is

$$\phi(z)-Az \quad \text{and} \quad \psi(z)-Bz$$

would be even functions, A, B being constant. Hence

$$\phi(z)-Az = \psi(z)-z\psi'(z), \quad \psi(z)-Bz = \phi(z)-z\phi'(z),$$

so that $$\phi(z) = Az+C+Ez^2, \quad \psi(z) = Bz+C-Ez^2,$$

C, E being constants. These forms are illusory. Thus ϕ, ψ cannot be uniform* in the domain of a corner of the plate. We therefore suppose the argument of ϕ to lie primarily in the two strips $0 < x < a$ and $0 < y < b$, formed by producing the sides of the plate indefinitely, and the argument of ψ in the strips $0 < x < a$ and $0 > y > -b$.

In (2.15) the logarithm of $-z$ is less by $i\pi$ than that of z, and accordingly $-z$ will be written $e^{-i\pi}z$, but in (2.16) $-z$ will be written $e^{i\pi}z$, since its logarithm exceeds that of z by $i\pi$. The two equations could therefore be satisfied by putting $\phi(z) = \Sigma Az^m$, $\psi(z) = \Sigma Bz^m$, where

$$B(1+e^{-mi\pi}) = 2A(1-m),$$

$$A(1+e^{mi\pi}) = 2B(1-m),$$

$$e^{mi\pi}+e^{-mi\pi}+2 = 4(1-m)^2. \quad \text{(See below, § 3.)}$$

Equations similar to (2.15), (2.16) may be found for the other corners. For $(a, 0)$ we have, by comparing (1.33) with (1.36) or by

* Hence, for instance, the value of (1.33) is not necessarily zero on the whole axis of x, but only on the segment $0 < x < a$.

adding (2.11) to (2.15) and (2.12) to (2.16),

$$\psi\{a+(z-a)e^{i\pi}\}+\psi(z) = 2\phi(z)-2(z-a)\phi'(z), \qquad (2.171)$$

$$\phi\{a+(\mathrm{z}-a)e^{-i\pi}\}+\phi(\mathrm{z}) = 2\psi(\mathrm{z})-2(\mathrm{z}-a)\psi'(\mathrm{z}), \qquad (2.172)$$

where z is primarily in the original rectangle and the argument of the first term in (2.171) is $2a-z$ reached from z by a rotation through $+\pi$ about a.

Since $a-\mathrm{z}$ is also in the original rectangle, we have

$$\psi(a-\mathrm{z}e^{i\pi})+\psi(a-\mathrm{z}) = 2\phi(a-\mathrm{z})+2\mathrm{z}\phi'(a-\mathrm{z}), \qquad (2.173)$$

$$\phi(a-ze^{-i\pi})+\phi(a-z) = 2\psi(a-z)+2z\psi'(a-z). \qquad (2.174)$$

The equations for the corner $(0, b)$ become, if we write $\mathrm{z}+ib$ for z and therefore $z-ib$ for z,

$$\psi(\mathrm{z}-ib)+\psi(e^{i\pi}\mathrm{z}-ib) = 2\phi(\mathrm{z}+ib)-2\mathrm{z}\phi'(\mathrm{z}+ib), \qquad (2.181)$$

$$\phi(z+ib)+\phi(e^{-i\pi}z+ib) = 2\psi(z-ib)-2z\psi'(z-ib). \qquad (2.182)$$

In the domain of the corner (a, b) we put $a+ib-z$ for z, and thus

$$\psi(a-ib-e^{-i\pi}z)+\psi(a-ib-z) = 2\phi(a+ib-z)+2z\phi'(a+ib-z), \qquad (2.191)$$

$$\phi(a+ib-e^{i\pi}\mathrm{z})+\phi(a+ib-\mathrm{z}) = 2\psi(a-ib-\mathrm{z})+2\mathrm{z}\psi'(a-ib-\mathrm{z}), \qquad (2.192)$$

Thus, if

$$\phi_{\alpha\beta}(z) = \phi(z)-\alpha\psi(a-z)-\beta\psi(z-ib)+\alpha\beta\phi(a+ib-z), \qquad (2.2)$$

where α, β are two independent symbols for ± 1,

$$\phi_{\alpha\beta}(a+ib-z) = \alpha\beta\phi_{\alpha\beta}(z), \qquad (2.3)$$

$$\begin{aligned}\phi_{\alpha\beta}(\mathrm{z})+\phi_{\alpha\beta}(\mathrm{z}e^{i\pi}) &= -2\alpha\phi_{\alpha\beta}(a-\mathrm{z})-2\alpha\mathrm{z}\phi'_{\alpha\beta}(a-\mathrm{z})\\ &= -2\beta\phi_{\alpha\beta}(\mathrm{z}+ib)+2\beta\mathrm{z}\phi'_{\alpha\beta}(\mathrm{z}+ib),\end{aligned} \qquad (2.4)$$

$$\begin{aligned}\alpha\phi_{\alpha\beta}(a-z)+\alpha\phi_{\alpha\beta}(a-ze^{-i\pi}) &= \beta\phi_{\alpha\beta}(ib+z)+\beta\phi_{\alpha\beta}(ib+ze^{-i\pi})\\ &= -2\phi_{\alpha\beta}(z)+2z\phi'_{\alpha\beta}(z).\end{aligned} \qquad (2.5)$$

Conversely, if $\phi_{\alpha\beta}$ satisfies the conditions (2.3), (2.4), (2.5), of which three are independent, then (2.11)–(2.16) are satisfied by putting $\phi(z) = \phi_{\alpha\beta}(z)$, $\psi(z) = -\alpha\phi_{\alpha\beta}(a-z)$. We have also

$$\phi_{\alpha\beta}(z+a)-\phi_{\alpha\beta}(z-a) = 2a\alpha\phi'_{\alpha\beta}(z), \qquad (2.61)$$

$$\phi_{\alpha\beta}(z+ib)-\phi_{\alpha\beta}(z-ib) = 2\beta ib\phi'_{\alpha\beta}(z). \qquad (2.62)$$

In all these equations z is primarily in the original rectangle, but the functions may presumably be continued with due precaution. For instance, (2.61) should hold throughout the strip $0 < y < b$ and (2.62) throughout the strip $0 < x < a$.

As to singularities, it is presumed that the deflection G has within the original rectangle only that of

$$\tfrac{1}{2}(z-z_0)(z-z_0)\{\log(z-z_0)+\log(z-z_0)\}, \tag{2.71}$$

so that $\partial^2 G/\partial z_0 \partial z_0$ has only the singularity of

$$\tfrac{1}{2}\log(z-z_0)+\tfrac{1}{2}\log(z-z_0). \tag{2.72}$$

This simpler deflection is derived from $\partial^2\phi/\partial z_0\partial z_0$ and $\partial^2\psi/\partial z_0\partial z_0$ as G from ϕ and ψ. Thus by comparing (2.72) with (1.33) we see that the term $\frac{1}{2}\log(z-z_0)$ must belong to $-\partial^2\psi(z)/\partial z_0\partial z_0$ and $\frac{1}{2}\log(z-z_0)$ to $-\partial^2\phi(z)/\partial z_0\partial z_0$. Thus $\partial^2\phi_{\alpha\beta}(z)/\partial z_0\partial z_0$ satisfies the conditions (2.3), (2.4), (2.5) and has the singularities of*

$$-\tfrac{1}{2}\log(z-z_0)+\tfrac{1}{2}\alpha\log(z+z_0-a)+\tfrac{1}{2}\beta\log(z-z_0+ib) \\ -\tfrac{1}{2}\alpha\beta\log(z+z_0-a+ib). \tag{2.73}$$

It has no other singularities within the rectangle $0 < x < a$, $-b < y < b$, and if its values are known over the whole of this rectangle they may be derived for the region formed by the crossed strips, and thence for other regions, by means of the conditions (2.3), (2.4), (2.5).

If $\chi(z)$ is written for $\partial^3\phi_{\alpha\beta}(z)/\partial z\partial z_0\partial z_0$, then $\chi(z)$ is uniform over the crossed strips and has the singularities of

$$-\frac{1}{2(z-z_0)}+\frac{\alpha}{2(z+z_0-a)}+\frac{\beta}{2(z-z_0+ib)}-\frac{\alpha\beta}{2(z+z_0-a+ib)}. \tag{2.74}$$

Since $\chi(z)$ satisfies the equation (2.62) it has within the strip $0 < x < a$ only the singularities of

$$-\frac{1}{2ib}\left\{\varpi_\beta\frac{z-z_0-ib}{ib}-\alpha\varpi_\beta\frac{z+z_0-a-ib}{ib} \right. \\ \left. -\beta\varpi_\beta\frac{z-z_0}{ib}+\alpha\beta\varpi_\beta\frac{z+z_0-a}{ib}\right\}, \tag{2.81}$$

* The signs of ib in the last two terms have been changed in virtue of (2.62).

where* $$\varpi_\beta(u) = -\int_{-\infty}^{\infty} \frac{e^{\lambda u}\, d\lambda}{e^{\lambda} - e^{-\lambda} - 2\beta\lambda}.$$

Similarly, by (2.61), $\chi(z)$ has within the strip $0 < y < b$ only the singularities of

$$\frac{1}{2a}\left\{\varpi_\alpha \frac{z+z_0-a}{a} - \alpha\varpi_\alpha \frac{z-z_0}{a} - \beta\varpi_\alpha \frac{z+z_0-a-ib}{a} + \alpha\beta\varpi_\alpha \frac{z-z_0-ib}{a}\right\}. \quad (2.82)$$

Also, from (2.3), (2.4), (2.5), we have

$$\chi(a+ib-z) = -\alpha\beta\chi(z), \quad (2.91)$$

$$\chi(z) - \chi(ze^{i\pi}) = 2az\chi'(a-z) = 2\beta z\chi'(z+ib). \quad (2.92)$$

$$-\alpha\chi(a-z) + \alpha\chi(a-ze^{-i\pi}) = \beta\chi(z+ib) - \beta\chi(ib+ze^{-i\pi}) = 2z\chi'(z). \quad (2.93)$$

The infinite strip and infinite sector.

3. When two opposite edges of the plate move to infinity in opposite directions, the limiting form is an infinite strip say $0 < x < a$. Then in the foregoing we omit (1.33), (1.34), (2.02), (2.03), (2.13) to (2.5), and also (2.62).

If $$\chi(z) = \frac{\partial^3}{\partial z\, \partial z_0 \partial z_0}\{\phi(z) - a\psi(a-z)\},$$

then $\chi(z)$ has the singularities of

$$\frac{1}{2a}\varpi_\alpha \frac{z+z_0-a}{a} - \frac{a}{2a}\varpi_\alpha \frac{z-z_0}{a} \quad (3.1)$$

and $$\chi(z+a) - \chi(z-a) = 2aa\chi'(z). \quad (3.2)$$

Hence $\chi(z)$ coincides with (3.1), and if

$$\rho_\alpha(u) = -\int_{-\infty}^{\infty} \frac{e^{\lambda u}\, d\lambda}{\lambda^3(e^{\lambda} - e^{-\lambda} - 2a\lambda)},$$

* See *Proc. London Math. Soc.* (2), 21 (1922), 277–8. The integral is to be taken along the two paths, one just above and the other just below the origin and the average of the results is $-\varpi_\beta(u)$. $\varpi_\beta(u)$ has the property that $\varpi_\beta(u+1) - \varpi_\beta(u-1) = 2\varpi'_\beta(u)$: its singularities are simple poles at ± 1 with residues 1, and the consequential poles at ± 2, ± 3, ...

so that $\rho_\alpha'''(u) = \varpi_\alpha(u)$, and if Σ denotes a summation over the two values of α,

$$\phi(z) = \tfrac{1}{4}\Sigma\left[a^2\left\{\rho_\alpha\frac{z-z_0-a}{a} - \rho_\alpha\frac{z+z_0-a}{a} + \alpha\rho_\alpha\frac{z-z_0}{a} - \alpha\rho_\alpha\frac{z+z_0}{a}\right\}\right.$$
$$\left. + a(z_0+z_0)\left\{\rho_\alpha'\frac{z+z_0-a}{a} + \alpha\rho_\alpha'\frac{z-z_0}{a}\right\}\right], \quad (3.31)$$

$$\psi(z) = \tfrac{1}{4}\Sigma\left[a^2\left\{\alpha\rho_\alpha\frac{z_0-z}{a} - \alpha\rho_\alpha\frac{-z-z_0}{a} + \rho_\alpha\frac{-z+z_0+a}{a} - \rho_\alpha\frac{-z-z_0+a}{a}\right\}\right.$$
$$\left. - a(z_0+z_0)\left\{\alpha\rho_\alpha'\frac{z_0-z}{a} + \rho_\alpha'\frac{-z-z_0+a}{a}\right\}\right]. \quad (3.32)$$

Then Green's function G is the result of substituting these in (1.35) or (1.36).

If a, b both tend to infinity the limiting form of the plate is an infinite quadrant, and the conditions to be satisfied by ϕ, ψ are (2.15), (2.16). More generally for an infinite sector* bounded by the lines of amplitudes 0, γ, the conditions are derived by equating (1.27) to (1.33). They are

$$\psi(z) - e^{2i\gamma}\psi(ze^{-2i\gamma}) = (1-e^{-2i\gamma})\{\phi(z) - z\phi'(z)\}, \quad (3.41)$$

$$\phi(z) - e^{-2i\gamma}\phi(ze^{2i\gamma}) = (1-e^{2i\gamma})\{\psi(z) - z\psi'(z)\}. \quad (3.42)$$

If we put $\chi(z)$, $\omega(z)$ for $\partial^3\phi(z)/\partial z\,\partial z_0\,\partial z_0$ and $\partial^3\psi(z)/\partial z\,\partial z_0\,\partial z_0$, we have

$$\chi(z) - \chi(ze^{2i\gamma}) = (e^{2i\gamma}-1)\,z\omega'(z), \quad (3.43)$$

$$\omega(z) - \omega(ze^{-2i\gamma}) = (e^{-2i\gamma}-1)\,z\chi'(z). \quad (3.44)$$

The singularities of $\chi(z)$ and $\omega(z)$ between the amplitudes $\pm\gamma$ are those of $-\{1/2(z-z_0)\}$ and $-\{1/2(z-z_0)\}$ respectively.

Hence
$$\chi(z) = \frac{1}{2z_0}\left\{\sigma\left(\frac{z}{z_0}\right) - \sigma\left(\frac{z}{z_0}\,e^{-2i\gamma}\right)\right\} + \frac{1-e^{2i\gamma}}{2z_0^2}\,z\sigma'\left(\frac{z}{z_0}\right), \quad (3.51)$$

$$\omega(z) = -\frac{1}{2z_0}\left\{\sigma\left(\frac{z}{z_0}\right) - \sigma\left(\frac{z}{z_0}\,e^{2i\gamma}\right)\right\} - \frac{1-e^{-2i\gamma}}{2z_0^2}\,z\sigma'\left(\frac{z}{z_0}\right). \quad (3.52)$$

where $\sigma(u)$ is a function such that

$$2\sigma(u) - \sigma(ue^{2i\gamma}) - \sigma(ue^{-2i\gamma}) = (2-e^{2i\gamma}-e^{-2i\gamma})\left(u\frac{d}{du}\right)^2\sigma(u), \quad (3.6)$$

and one which has the singularity of $(ue^{2i\gamma}-1)^{-1}$ at the amplitude -2γ,

* This is in substance the problem investigated in *Proc. London Math. Soc.* (2), 19 (1921), 373–386.

but is regular between the amplitudes $\pm 2\gamma$. That is, for amplitudes of u between $\pm 2\gamma$,

$$\sigma(u) = -\int_{-\infty i}^{\infty i} \frac{u^\lambda d\lambda}{e^{2\lambda i\gamma}+e^{-2\lambda i\gamma}-2+4\lambda^2 \sin^2\gamma}, \tag{3.7}$$

the path being a parallel to the lateral axis. For the proof of these properties of $\sigma(u)$, compare the discussion of $\theta(u)$ in § 8 below. The expressions for $\chi(z)$, $\phi(z)$ are derived from (3.7) by changing u^λ into

$$\tfrac{1}{2} z^\lambda \{ z_0^{-\lambda-1}(1-e^{-2\lambda i\gamma}) + \lambda z_0^{-\lambda-1}(1-e^{2i\gamma}) \}$$

and

$$\frac{z^{\lambda+1}}{2(\lambda-\lambda^3)} [\{ z_0^{1-\lambda} - \lambda z_0^{1-\lambda} - (1-\lambda) z_0 z_0^{-\lambda} \} (1-e^{-2\lambda i\gamma}) + \lambda \{ z_0^{1-\lambda} - \lambda z_0^{1-\lambda} - (1-\lambda) z_0 z_0^{-\lambda} \} (1-e^{2i\gamma})]$$

respectively; $\omega(z)$ and $\psi(z)$ are the complex quantities conjugate to $\chi(z)$ and $\phi(z)$. There will be a similar expression for Green's function (1.33), the subject of integration being a homogeneous function of z, z of degree $\lambda+1$, and by taking the path of integration to be the lateral axis itself we secure that the derivatives of the function vanish both at the origin and at infinity*, for in the expression for Green's function there are no poles of the integrand actually on the lateral axis and the path may therefore be displaced slightly to the right when $|z|$ is small, or to the left when $|z|$ is great.

4. Returning now to the rectangular plate we presume $\chi(z)$ to be bounded near the corners, since that is the case for the infinite quadrant. We take

$$\int \left\{ \varpi_a \left(\frac{t-z}{a} - 1 \right) \chi(t) + \varpi_a \left(\frac{t-z}{a} \right) \chi(t-a) \right\} dt \tag{4.0}$$

round the original rectangle, supposing z, z_0 to lie within the rectangle; then the integrand has poles at z, z_0, $a-z_0$, $ib+z_0$, $a+ib-z_0$, and the value of the integral is

$$2i\pi \left\{ a\chi(z) + \frac{a}{2} \varpi_a \frac{z-z_0}{a} - \tfrac{1}{2} \varpi_a \frac{z+z_0-a}{a} - \frac{a\beta}{2} \varpi_a \frac{z-z_0-ib}{a} + \frac{\beta}{2} \varpi_a \frac{z+z_0-a-ib}{a} \right\}. \tag{4.0}$$

* By this assignment of the conditions at infinity the symmetry of Green's function is preserved, though the function itself is not necessarily bounded at infinity.

This will still be true for positions of z, z_0 anywhere in the strip $0 < y < b$ if the path from a to $a+ib$ is suitably deformed and a corresponding change made in the path from ib to 0. The contribution of these parts of the path is

$$\int_0^{ib}\left[\varpi_\alpha\frac{t-z}{a}\{\chi(t+a)-\chi(t-a)\}+\chi(t)\left\{\varpi_\alpha\frac{t-z+a}{a}-\varpi_\alpha\frac{t-z-a}{a}\right\}\right]dt$$

$$=2a\int_0^{ib}\left\{\varpi_\alpha\frac{t-z}{a}.a\chi'(t)+\varpi'_\alpha\frac{t-z}{a}\chi(t)\right\}dt$$

$$=2a\alpha\left\{\varpi_\alpha\frac{ib-z}{a}\chi(ib)+\varpi_\alpha\frac{z}{a}\chi(0)\right\}. \qquad (4.1)$$

The contribution of the other sides is

$$\int_0^a\left[\varpi_\alpha\left(\frac{t-z}{a}-1\right)\chi(t)+\varpi_\alpha\frac{t-z}{a}\chi(t-a)\right.$$

$$\left.-\varpi_\alpha\frac{ib-t-z}{a}\chi(a+ib-t)-\varpi_\alpha\frac{a+ib-t-z}{a}\chi(ib-t)\right]dt. \qquad (4.20)$$

In the second term of (4.20) put $a-t$ for t and use (2.92), in the third term use (2.91), in the fourth term use (2.91) and then (2.93). Thus (4.20) becomes

$$\int_0^a\left[\left\{\varpi_\alpha\frac{t-z-a}{a}+\alpha\beta\varpi_\alpha\frac{ib-t-z}{a}\right\}\chi(t)+\varpi_\alpha\frac{a-t-z}{a}\{\chi(t)-2at\chi'(a-t)\}\right.$$

$$\left.+\alpha\beta\varpi_\alpha\frac{a+ib-t-z}{a}\{\chi(a-t)+2at\chi'(t)\}\right]dt, \qquad (4.21)$$

or again

$$\int_0^a\chi(t)\left\{\varpi_\alpha\frac{t-z-a}{a}+\alpha\beta\varpi_\alpha\frac{ib-t-z}{a}+\varpi_\alpha\frac{a-t-z}{a}-2a\varpi_\alpha\frac{t-z}{a}\right.$$

$$+2a\frac{a-t}{a}\varpi'_\alpha\frac{t-z}{a}+\alpha\beta\varpi_\alpha\frac{ib+t-z}{a}$$

$$\left.-2\beta\varpi_\alpha\frac{a+ib-t-z}{a}+2\beta\frac{t}{a}\varpi'_\alpha\frac{a+ib-t-z}{a}\right\}dt$$

$$+\left[-2a(a-t)\varpi_\alpha\frac{t-z}{a}\chi(t)+2\beta t\varpi_\alpha\frac{a+ib-t-z}{a}\chi(t)\right]_0^a. \qquad (4.22)$$

The terms at the limits in (4.22) are exactly equal and opposite to the

terms (4.1). Hence, when z, z_0 are in the strip $0 < y < b$,

$$\chi(z) = -\frac{a}{2a}\,\varpi_\alpha\frac{z-z_0}{a}+\frac{1}{2a}\,\varpi_\alpha\frac{z+z_0-a}{a}+\frac{a\beta}{2a}\,\varpi_\alpha\frac{z-z_0-ib}{a}$$

$$-\frac{\beta}{2a}\,\varpi_\alpha\frac{z+z_0-a-ib}{a}+\int_0^a\chi(t)\,K(t,z)\,dt, \quad (4.3)$$

where $2i\pi aK(t,z)$ is the factor in brackets under the integral sign in (4.22).

Similarly, by taking

$$\int\left\{\varpi_\beta\frac{t-z-ib}{ib}\,\chi(t)+\varpi_\beta\frac{t-z}{ib}\,\chi(t-ib)\right\}dt$$

round the rectangle we have, if z, z_0 are in the strip $0 < x < a$,

$$\chi(z) = \frac{\beta}{2ib}\,\varpi_\beta\frac{z-z_0}{ib}-\frac{1}{2ib}\,\varpi_\beta\frac{z-z_0-ib}{ib}-\frac{a\beta}{2ib}\,\varpi_\beta\frac{z+z_0-a}{ib}$$

$$+\frac{a}{2ib}\,\varpi_\beta\frac{z+z_0-a-ib}{ib}+\int_0^{ib}\chi(t)\,J(t,z)\,dt, \quad (4.4)$$

where*

$$-2\pi bJ(t,z) = -\varpi_\beta\frac{t-z-ib}{ib}+\varpi_\beta\frac{t+z-ib}{ib}+a\beta\varpi_\beta\frac{t+z-a}{ib}$$

$$-a\beta\varpi_\beta\frac{t-z+a}{ib}+2\beta\varpi_\beta\frac{t-z}{ib}-2a\varpi_\beta\frac{t+z-a-ib}{ib}$$

$$-2\beta\frac{ib-t}{ib}\,\varpi'_\beta\frac{t-z}{ib}-2a\,\frac{t}{ib}\,\varpi'_\beta\frac{t+z-a-ib}{ib}.$$

The values of z in (4.3) and t in (4.4) may be taken on the segment $(0, ib)$, those of z in (4.4) and t in (4.3) on the segment $(0, a)$; then

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* A question arises (§ 12 below) as to the behaviour of the right-hand expressions in (4.3), (4.4) when z tends to infinity. For instance, take the right side of (4.4) when $x \to -\infty$. If $\beta = -1$, the expansion of $\varpi_\beta(u)$ [*Proc. London Math. Soc.* (2), 21 (1923), 278], is

$$-\frac{i\pi}{4}-i\pi\Sigma\,\frac{\exp\lambda u}{\cosh\lambda+1} \quad \text{when } u \text{ is } \frac{z-s}{ib},$$

but

$$\frac{i\pi}{4}+i\pi\Sigma\,\frac{\exp(-\lambda u)}{\cosh\lambda+1} \quad \text{when } u \text{ is } \frac{s-z}{ib}.$$

Here all the values of λ lie above the real axis; s may be z_0, t, $t-ib$ or the like. Thus the right side of (4.4) tends to a constant limit independent of y when y is kept fixed and x tends to $-\infty$.

But when $\beta = +1$, there are terms (*loc. cit.*, 285)

$$\pm\tfrac{3}{2}i\pi\,(u^2-\tfrac{1}{10})$$

in the expansion, besides the exponential terms: the limiting form of the expression in (4.4) is however linear in z, for the term in z^2 vanishes, since $\beta = +1$.

(4.3), (4.4) are integral equations to be satisfied by the values of χ on these two segments. The nuclei J, K become infinite whenever t, z both have extreme values, and thus the integral equations cannot be solved by the usual methods without special preparation, as explained in §§ 5, 6 below. The single equation (4.3) must be satisfied by the values of $\chi(z)$ on the segment $(0, a)$, but is still more disadvantageous since the nucleus becomes infinite whenever $t = z$.

5. With a view to the solution of the equations (4.3), (4.4), let the row of four functions

$$\chi(\tfrac{1}{2}ax), \quad \chi(a-\tfrac{1}{2}ax), \quad \chi(\tfrac{1}{2}ibx), \quad \chi(ib-\tfrac{1}{2}ibx)$$

be denoted individually by $\phi_r(x)$ $(r = 1, 2, 3, 4)$ and collectively by $\phi(x)$. Then we may write (4.3), (4.4) as

$$\phi(x) = f(x) + \int_0^1 \{P(x, t) + Q(x, t)\}\, \phi(t)\, dt, \tag{5.1}$$

where $f(x)$ denotes a row of four functions $f_r(x)$ $(r = 1, 2, 3, 4)$ and

$$f_1(x) = f_2(2-x) = \frac{\beta}{2ib}\,\varpi_\beta\,\frac{ax-2z_0}{2ib} - \frac{1}{2ib}\,\varpi_\beta\,\frac{ax-2z_0-2ib}{2ib}$$

$$-\frac{a\beta}{2ib}\,\varpi_\beta\,\frac{ax+2z_0-2a}{2ib} + \frac{a}{2ib}\,\varpi_\beta\,\frac{ax+2z_0-2a-2ib}{2ib}, \tag{5.2}$$

$$f_3(x) = f_4(2-x) = -\frac{a}{2a}\,\varpi_\alpha\,\frac{ibx-2z_0}{2a} + \frac{1}{2a}\,\varpi_\alpha\,\frac{ibx+2z_0-2a}{2a}$$

$$+\frac{a\beta}{2a}\,\varpi_\alpha\,\frac{ibx-2z_0-2ib}{2a} - \frac{\beta}{2a}\,\varpi_\alpha\,\frac{ibx+2z_0-2a-2ib}{2a}, \tag{5.3}$$

while $P(x, t)$, $Q(x, t)$ are matrices* of the fourth order whose sum is

$$\begin{pmatrix} 0, & 0, & \tfrac{1}{2}ibJ(\tfrac{1}{2}ibt, \tfrac{1}{2}ax), & \tfrac{1}{2}ibJ(ib-\tfrac{1}{2}ibt, \tfrac{1}{2}ax) \\ 0, & 0, & \tfrac{1}{2}ibJ(\tfrac{1}{2}ibt, a-\tfrac{1}{2}ax), & \tfrac{1}{2}ibJ(ib-\tfrac{1}{2}ibt, a-\tfrac{1}{2}ax) \\ \tfrac{1}{2}aK(\tfrac{1}{2}at, \tfrac{1}{2}ibx), & \tfrac{1}{2}aK(a-\tfrac{1}{2}at, \tfrac{1}{2}ibx), & 0, & 0, \\ \tfrac{1}{2}aK(\tfrac{1}{2}at, ib-\tfrac{1}{2}ibx), & \tfrac{1}{2}aK(a-\tfrac{1}{2}at, ib-\tfrac{1}{2}ibx), & 0, & 0, \end{pmatrix} \tag{5.4}$$

* It may be well to state that $P(x, t)\,\phi(t)$ denotes the row of four functions

$$\sum_{s=1}^{4} P_{rs}(x, t)\,\phi_s(t) \quad (r = 1, 2, 3, 4),$$

where $P_{rs}(x, t)$ is the s-th constituent in the r-th row of the matrix $P(x, t)$ and also that $P(x, t) + Q(x, t)$ is a matrix in which the s-th constituent of the r-th row is $P_{rs}(x, t) + Q_{rs}(x, t)$.

In (5 . 1) x, t are real variables ranging from 0 to 1. The only singularities of the nucleus are at the origin, and they are singularities of a matrix of homogeneous functions of x, t of degree -1. This matrix of homogeneous functions will be denoted by $P(x, t)$ and the remainder $Q(x, t)$ is a matrix of regular functions. The actual value of $P(x, t)$ is

$$\frac{abx}{\pi}\begin{vmatrix} 0, & 0, & \dfrac{1}{a^2x^2+b^2t^2}, & \dfrac{-\beta}{(ax+ibt)^2} \\ 0, & 0, & \dfrac{\alpha}{(ax-ibt)^2}, & \dfrac{-\alpha\beta}{a^2x^2+b^2t^2} \\ \dfrac{1}{a^2t^2+b^2x^2}, & \dfrac{\alpha}{(at+ibx)^2}, & 0, & 0, \\ \dfrac{-\beta}{(at-ibx)^2}, & \dfrac{-\alpha\beta}{a^2t^2+b^2x^2}, & 0, & 0. \end{vmatrix} \tag{5 . 5}$$

6. Let ϕ and $(P)\phi$ be written for $\phi(x)$ and $\int_0^1 P(x, t)\,\phi(t)\,dt$, and so in other cases; then (5 . 1) may be written

$$\{1-(P)-(Q)\}\,\phi = f. \tag{6 . 1}$$

Assume a solution $$\phi = \psi-(R)\psi+\omega, \tag{6 . 2}$$

where (R) is another operation such as (P) or (Q) and ψ, ω are rows of functions of x. Then

$$\{1-(P)-(R)+(P)(R)-(Q)+(Q)(R)\}\,\psi = f-\omega+(P)\omega+(Q)\omega, \tag{6 . 3}$$

and if (R) is such that the nuclei of

$$(P)+(R)-(P)(R) \quad \text{and} \quad (Q)(R)$$

consist of bounded and continuous functions ψ can be found by the ordinary methods and then ϕ by substitution in (6 . 2).

A matrix that may be used for R is the solving matrix of the equation

$$\phi(x) = f(x)+\int_0^\infty P(x, t)\,\phi(t)\,dt, \tag{6 . 4}$$

which will now be investigated*: the purpose of introducing ω will be seen later.

* For the method compare *Proc. London Math. Soc.* (2), 22 (1924), 201–222, where a discussion is given of some matters of convergence.

7. From (6.4) we have, if the integrals are absolutely convergent,

$$\int_0^\infty \{\phi(x)-f(x)\}\, x^{-\lambda-1}\,dx = \int_0^\infty \left\{\int_0^\infty x^{-\lambda-1} P(x,t)\,\phi(t)\,dx\right\} dt$$

$$= \Lambda \int_0^\infty t^{-\lambda-1}\phi(t)\,dt, \qquad (7.1)$$

where $\Lambda = \int_0^\infty u^{-\lambda-1} P(u,1)\,du$, a matrix depending on λ only: this follows by the substitution $x = ut$, since $P(ut, t) = t^{-1}P(u,1)$. Hence

$$\int_0^\infty x^{-\lambda-1}\phi(x)\,dx = (1-\Lambda)^{-1}\int_0^\infty x^{-\lambda-1} f(x)\,dx, \qquad (7.2)$$

so that by the formula of Laplace (or Mellin)

$$\phi(x) = \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} x^\lambda \{(1-\Lambda)^{-1}\int_0^\infty t^{-\lambda-1} f(t)\,dt\}\,d\lambda \qquad (7.21)$$

$$= f(x) + \frac{1}{2i\pi}\int_{-i\infty}^{i\infty} x^\lambda \left\{\frac{\Lambda}{1-\Lambda}\int_0^\infty t^{-\lambda-1} f(t)\,dt\right\} d\lambda. \qquad (7.22)$$

The integrals involved in Λ all depend on $\int_0^\infty \frac{u^{-\lambda-1}\,du}{u-v}$, which, when the real part of λ is between 0 and -1, is known to be

$$2i\pi v^{-\lambda-1}/(1-e^{-2\lambda i\pi})$$

if the amplitude of v is between 0 and 2π, and that of u is 0 on the whole path of integration. Hence

$$(1-e^{-2\lambda i\pi})\,\Lambda$$

$$= \begin{pmatrix} 0, & 0, & \left(\frac{a}{b}\right)^\lambda (e^{-\frac{1}{2}\lambda i\pi} - e^{-\frac{3}{2}\lambda i\pi}), & -2\beta\lambda\left(\frac{a}{b}\right)^\lambda e^{-\frac{3}{2}\lambda i\pi} \\ 0, & 0, & -2a\lambda\left(\frac{a}{b}\right)^\lambda e^{-\frac{1}{2}\lambda i\pi}, & -a\beta\left(\frac{a}{b}\right)^\lambda (e^{-\frac{1}{2}\lambda i\pi} - e^{-\frac{3}{2}\lambda i\pi}) \\ \left(\frac{b}{a}\right)^\lambda (e^{-\frac{1}{2}\lambda i\pi} - e^{-\frac{3}{2}\lambda i\pi}), & 2\lambda a\left(\frac{b}{a}\right)^\lambda e^{-\frac{1}{2}\lambda i\pi}, & 0, & 0 \\ 2\lambda\beta\left(\frac{b}{a}\right)^\lambda e^{-\frac{3}{2}\lambda i\pi}, & -a\beta\left(\frac{b}{a}\right)^\lambda (e^{-\frac{1}{2}\lambda i\pi} - e^{-\frac{3}{2}\lambda i\pi}), & 0, & 0 \end{pmatrix}$$

(7.3)

The determinant of the matrix $1-\Lambda$ is

$$\Delta(e^{\lambda i\pi}-e^{-\lambda i\pi})^{-4},$$

where Δ or $\Delta(\lambda) = (e^{\lambda i\pi}+e^{-\lambda i\pi}-2+4\lambda^2)(e^{3\lambda i\pi}+e^{-3\lambda i\pi}-2+4\lambda^2)$, (7.4)

and $(1-\Lambda)^{-1}$ is $1-16U/\Delta$ (7.50)

where U is

$$\begin{pmatrix}
\frac{1}{16}\Delta-\sin^2\lambda\pi(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi-\lambda^2e^{\lambda i\pi}) & \alpha i\lambda\sin^3\lambda\pi & -(a/b)^\lambda\sin\lambda\pi\sin\frac{1}{2}\lambda\pi\times(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi+\lambda^2) & -i\beta(a/b)^\lambda\lambda\sin\lambda\pi e^{\frac{1}{2}\lambda i\pi}\times(e^{-\lambda i\pi}\sin^2\lambda\pi+\sin^2\frac{1}{2}\lambda\pi-\lambda^2) \\
-\alpha i\lambda\sin^3\lambda\pi & \frac{1}{16}\Delta-\sin^2\lambda\pi(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi-\lambda^2e^{-\lambda i\pi}) & -\alpha i(a/b)^\lambda\lambda\sin\lambda\pi e^{-\frac{1}{2}\lambda i\pi}\times(e^{\lambda i\pi}\sin^2\lambda\pi+\sin^2\frac{1}{2}\lambda\pi-\lambda^2) & \alpha\beta(a/b)^\lambda\sin\lambda\pi\sin\frac{1}{2}\lambda\pi\times(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi+\lambda^2) \\
-(b/a)^\lambda\sin\lambda\pi\sin\frac{1}{2}\lambda\pi\times(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi+\lambda^2) & i\alpha(b/a)^\lambda\lambda\sin\lambda\pi e^{\frac{1}{2}\lambda i\pi}\times(e^{\lambda i\pi}\sin^2\lambda\pi+\sin^2\frac{1}{2}\lambda\pi-\lambda^2) & \frac{1}{16}\Delta-\sin^2\lambda\pi(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi-\lambda^2e^{-\lambda i\pi}) & -i\beta\lambda\sin^3\lambda\pi \\
i\beta(b/a)^\lambda\lambda\sin\lambda\pi e^{\frac{1}{2}\lambda i\pi}\times(e^{-\lambda i\pi}\sin^2\lambda\pi+\sin^2\frac{1}{2}\lambda\pi-\lambda^2) & \alpha\beta(b/a)^\lambda\sin\lambda\pi\sin\frac{1}{2}\lambda\pi\times(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi+\lambda^2) & i\beta\lambda\sin^3\lambda\pi & \frac{1}{16}\Delta-\sin^2\lambda\pi(\sin^2\lambda\pi-\sin^2\frac{1}{2}\lambda\pi-\lambda^2e^{\lambda i\pi})
\end{pmatrix} \tag{7.51}$$

8. Thus $\phi(z)$ as given by (7.22) involves such integrals as

$$\int_{-i\infty}^{i\infty}\frac{z^\lambda\,d\lambda}{\Delta(\lambda)} \quad\text{or}\quad \int_{-i\infty}^{i\infty}\frac{z^\lambda\,d\lambda}{16(\sin^2\frac{1}{2}\lambda\pi-\lambda^2)(\sin^2\frac{3}{2}\lambda\pi-\lambda^2)}. \tag{8.0}$$

$\Delta(\lambda)$ vanishes to the fourth order when $\lambda=0$ and to the second order when $\lambda=\pm1$. We take the path to be the lateral axis, with deviations on both sides of the origin, and reckon the integral as the average of the values given by the two paths, to right and left. So long as the amplitude of z is between $\pm4\pi$, the integral is absolutely convergent and represents a regular analytic even function of $\log z$ for finite values of $\log z$. Call it $\theta(z)$.

To continue the function beyond the amplitude 4π we divide the path at any point, say i, and write

$$\theta(z)=\int_i^{i\infty}\frac{z^\lambda\,d\lambda}{\Delta(\lambda)}+\int_{-i\infty}^{i}\left\{\frac{z^\lambda}{\Delta(\lambda)}-z^\lambda e^{-4\lambda i\pi}\right\}d\lambda+\int_{-i\infty}^{i}z^\lambda e^{-4\lambda i\pi}\,d\lambda. \tag{8.1}$$

The first two parts of this expression represent a function which is regular for amplitudes from -4π to 5π, and the last term is $(ze^{-4i\pi})^i \div (\log z-4i\pi)$

so long as the amplitude of z is less than 4π. Putting this value for the last term in (8.1) we have an expression for $\theta(z)$ as continued to the amplitude 5π.

When the amplitude of z is greater than 4π, we have

$$(ze^{-4i\pi})^i \div (\log z - 4i\pi) = -\int_i^{i\infty} z^\lambda e^{-4\lambda i\pi}\, d\lambda, \tag{8.2}$$

and thus, for amplitudes from 4π to 5π,

$$\theta(z) = \int_{-i\infty}^{i\infty} z^\lambda \left\{\frac{1}{\Delta(\lambda)} - e^{-4\lambda i\pi}\right\} d\lambda. \tag{8.3}$$

Hence for amplitudes from 0 to π

$$\theta(ze^{4i\pi}) = \int_{-i\infty}^{i\infty} z^\lambda \left\{\frac{e^{4\lambda i\pi}}{\Delta(\lambda)} - 1\right\} d\lambda,$$

that is

$$\begin{aligned}&\theta(ze^{4i\pi}) + \theta(ze^{2i\pi}) + \theta(ze^{-2i\pi}) + \theta(ze^{-4i\pi})\\ &\quad + \left\{4\left(z\frac{d}{dz}\right)^2 - 2\right\}\left\{\theta(ze^{3i\pi}) + \theta(ze^{i\pi}) + \theta(ze^{-i\pi}) + \theta(ze^{-3i\pi})\right\}\\ &\quad + \left\{4\left(z\frac{d}{dz}\right)^2 - 2\right\}^2 \theta(z) = 0.\end{aligned} \tag{8.4}$$

This is the characteristic functional equation of $\theta(z)$: it may be written

$$\Delta\left(z\frac{d}{dz}\right)\theta(z) = 0.$$

It holds by continuation for all finite values of $\log z$.

The singularity of $\theta(z)$ at the amplitude 4π, apart from those at 0 and ∞, is that of (8.2) or of $1/(z-1)$. Thus from (8.4) there is a pole with residue -1 when $z = 1$ and $\log z = -4i\pi$. $\theta(z)$ is a uniform function of $\log z$, since the other singularities for which $\log z$ is finite are derived by means of (8.4) from these simple poles.

Now if in U (7.51) we put all the sines and cosines in their exponential forms and multiply out, the terms in each constituent are such as

$$A\lambda^n k^\lambda$$

where A, n, k are constants. If for this term we write in each case

$$\frac{8A}{i\pi t}\left(x\frac{d}{dx}\right)^n \theta\left(\frac{kx}{t}\right) \tag{8.5}$$

with the same constant values for A, n, k, and call the resulting matrix $R'(x, t)$, then we may write (7.22) as

$$\phi(x) = f(x) - \int_0^\infty R'(x, t) f(t)\, dt \tag{8.6}$$

and the matrices R', P are connected by the relations

$$R'(x, t) + P(x, t) = \int_0^\infty R'(x, u)\, P(u, t)\, du \tag{8.61}$$

$$= \int_0^\infty P(x, u)\, R'(u, t)\, du. \tag{8.62}$$

It is desirable to verify these equations strictly.

9. The constituents of $R'(x, u)\, P(u, t)$ and of $P(x, u)\, R'(u, t)$ consist of such terms as

$$\frac{1}{t-mu}\cdot\frac{1}{u}\,\theta\left(\frac{kx}{u}\right) \quad \text{and} \quad \frac{1}{u-mx}\cdot\frac{1}{t}\,\theta\left(\frac{ku}{t}\right), \tag{9.1}$$

and derivatives of these with respect to $\log x$ and $\log t$. To test the integrals in (8.61) and (8.62) for convergence we must ascertain the behaviour of $\theta(z)$ when $\log z$ tends to $\pm\infty$. It will be proved that when $|z| < 1$, $\theta(z)$ can be expanded in terms that are generally multiples of powers of z, not integral*, but arranged in the order of increasing real parts, and that the error at any stage is of the order of magnitude of the first term omitted: when $|z| > 1$ there is a corresponding expansion since $\theta(z)$ is an even function of $\log z$.

If the path of integration in (8.0) is taken to be the line $x = g$, where g is positive, the integral is equal to

$$\theta(z) + 2i\pi\Sigma z^\lambda / \Delta'(\lambda), \tag{9.20}$$

where the summation extends over the values of λ that lie in the strip

* Compare *Proc. London Math. Soc.* (2), 21 (1923), 278.

$0 < x < g$ between the two paths, and that satisfy the equation $\Delta(\lambda) = 0$*: the necessary modification is to be made for multiple roots. That is, the integral is equal to the error committed in taking $\theta(z)$ equal to

$$-2i\pi\Sigma z^{\lambda}/\Delta'(\lambda). \qquad (9.21)$$

This integral is at most of the order of z^g, so that, if $|z| < 1$, the error tends to zero when g increases indefinitely. When z is a small quantity z^g is small as compared with the last term included in the summation, and if this last term is also omitted the total error is of its order of magnitude, that is, of the order of the first term now omitted.

This ascending expansion of $\theta(z)$ may contain a constant term and multiples of $\log z$, $(\log z)^2$, $(\log z)^3$, z, $z \log z$, on account of the fourfold zero of Δ at the origin and the double zero at $+1$, but these terms do not appear in R' since U vanishes at 0 and ± 1 to the same orders as Δ. Hence when z is small we may treat $\theta(z)$ in R' as of the order of z^{λ_1} and when z is great as of the order of $z^{-\lambda_1}$ where λ_1 is that one of the other zeros of Δ which has the least positive real part. It is found by tracing the changes in $\log(\sin^2 \frac{3}{2}\lambda\pi - \lambda^2)$ and $\log(\sin^2 \frac{1}{2}\lambda\pi - \lambda^2)$ when λ travels round the rectangle whose vertices are $\pm 1 \pm ig$, where g is great, that Δ has eight zeros within the strip $-1 < x < 1$ and none on the boundary except at ± 1. Of the eight, four are at the origin and the others may be denoted by $\pm\lambda_1$, $\pm\lambda_2$. When the values $\frac{1}{3}$, $\frac{2}{3}$, $\frac{17}{18}$ are given to λ in $\sin^2 \frac{3}{2}\lambda\pi - \lambda^2$ the signs of the expression are $+$, $-$, $+$. Hence λ_1, λ_2 are real and

$$\tfrac{1}{3} < \lambda_1 < \tfrac{2}{3} < \lambda_2 < \tfrac{17}{18} \qquad (9.3)$$

if the greater positive value is taken for λ_2. The roots in order of ascending real parts are then 0, 0, 0, 0, λ_1, λ_2, 1, 1, λ_3, λ_4, ...

The typical terms (9.1) are now comparable with

$$\frac{1}{t-mu} \cdot \frac{1}{u} \left(\frac{u}{kx}\right)^{\lambda_1} \quad \text{and} \quad \frac{1}{u-mx} \cdot \frac{1}{t} \left(\frac{ku}{t}\right)^{\lambda_1} \quad \text{when } u \text{ is small,}$$

$$\frac{1}{t-mu} \cdot \frac{1}{u} \left(\frac{kx}{u}\right)^{\lambda_1} \quad \text{and} \quad \frac{1}{u-mx} \cdot \frac{1}{t} \left(\frac{t}{ku}\right)^{\lambda_1} \quad \text{when } u \text{ is great,}$$

so that the integrals in (8.61), (8.62) are absolutely convergent, and the

* It is supposed that no value of λ satisfying the equation $\Delta(\lambda) = 0$ lies on the line $x = g$.

integrals of the same subjects over arcs of circles about the origin tend to zero when the radii of those circles are indefinitely increased or diminished. Hence the path of integration, which is originally the real positive axis, may be altered to another infinite radius vector, at an angle γ with the axis, in the first quadrant, provided that no poles of the subjects occur in the space between the two paths. This is so, for all the poles of P lie on the lateral axis and R' has no pole between the amplitudes $\pm\frac{1}{2}\pi$, at which the terms involving $\theta(e^{\pm\frac{1}{2}i\pi}x/t)$ become infinite.

Now $\theta(kx/t)$ is reduced to zero by the operation $\Delta\{x(d/dx)\}$ and therefore $R'(x, t)$ may be written in another form by cancelling the terms $\frac{1}{16}\Delta$ in the diagonal of U, since these terms contribute zero to $R'(x, t)$: what remains is $U-\frac{1}{16}\Delta$, in which each constituent has the factor $1-e^{-2\lambda i\pi}$, so that

$$R'(x, t) = \frac{1}{t}\, R_1\left(\frac{x}{t}\right) - \frac{1}{t}\, R_1\left(\frac{xe^{-2i\pi}}{t}\right), \tag{9.41}$$

where $R_1(z)$ is a matrix derived from

$$(U-\tfrac{1}{16}\Delta) \div (1-e^{-2\lambda i\pi}) \tag{9.42}$$

by substituting throughout

$$\frac{8}{i\pi}\left(z\,\frac{d}{dz}\right)^n \theta(kz) \quad \text{for} \quad \lambda^n k^\lambda. \tag{9.43}$$

Then
$$\int_0^\infty P(x, u)\, R'(u, t)\, du = \int P(x, tz)\, R_1(z)\, dz, \tag{9.44}$$

where $z = u/t$ and the path of z is from the origin to infinity at vectorial angle γ in the first quadrant and back along the same line after a rotation through -2π. If ρ, σ are two positive quantities of which ρ decreases and σ increases without limit, the integral is the limit of one with the same subject along a closed contour consisting of a straight line from $\rho e^{i\gamma}$ to $\sigma e^{i\gamma}$, the circumference of the circle about the origin from $\sigma e^{i\gamma}$ to $\sigma e^{i(\gamma-2\pi)}$, the straight line from $\sigma e^{i(\gamma-2\pi)}$ to $\rho e^{i(\gamma-2\pi)}$, and then the circumference of the circle about the origin from $\rho e^{i(\gamma-2\pi)}$ to $\rho e^{i\gamma}$. Within this contour both P and R_1 become infinite.

Each constituent of the principal diagonal of $R_1(z)$ contains a term $-(1/2i\pi)\,\theta(ze^{4i\pi})$ which has the singularity of $-\{1/2i\pi(z-1)\}$ when $\log z = 0$; this pole falls within the contour and hence the value of the integral includes the residue at this pole multiplied by $-2i\pi$ (since the

contour is described negatively), that is

$$P(x, t). \tag{9.50}$$

There is no other pole of $R_1(z)$ within the contour: the rest of the expression consists of the residues at the poles of $P(x, tz)$ each multiplied by $-2i\pi$; the poles are

$$\pm i\frac{ax}{bt}, \quad \pm i\frac{bx}{at},$$

the amplitudes being all between 0 and -2π: the residues are

$$\frac{1}{2i\pi t}\begin{vmatrix} 0, & 0, & -1, & 0 \\ 0, & 0, & 2\alpha x\frac{d}{dx}, & \alpha\beta \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix} R_1\left(\frac{ax}{bt}\, e^{-\frac{1}{2}i\pi}\right), \tag{9.51}$$

$$\frac{1}{2i\pi t}\begin{vmatrix} 0, & 0, & 1, & 2\beta x\frac{d}{dx} \\ 0, & 0, & 0, & -\alpha\beta \\ 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \end{vmatrix} R_1\left(\frac{ax}{bt}\, e^{-\frac{3}{2}i\pi}\right), \tag{9.52}$$

$$\frac{1}{2i\pi t}\begin{vmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ -1, & -2\alpha x\frac{d}{dx}, & 0, & 0 \\ 0, & \alpha\beta, & 0, & 0 \end{vmatrix} R_1\left(\frac{bx}{at}\, e^{-\frac{1}{2}i\pi}\right), \tag{9.53}$$

$$\frac{1}{2i\pi t}\begin{vmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0 \\ 1, & 0, & 0, & 0 \\ -2\beta x\frac{d}{dx}, & -\alpha\beta, & 0, & 0 \end{vmatrix} R_1\left(\frac{bx}{at}\, e^{-\frac{3}{2}i\pi}\right). \tag{9.54}$$

The sum of these residues is what would be derived from

$$\frac{1}{2i\pi}\begin{vmatrix} 0, & 0, & \left(\frac{a}{b}\right)^{\lambda}(e^{-\frac{3}{2}\lambda i\pi}-e^{-\frac{1}{2}\lambda i\pi}), & 2\beta\lambda\left(\frac{a}{b}\right)^{\lambda}e^{-\frac{3}{2}\lambda i\pi} \\ 0, & 0, & 2a\lambda\left(\frac{a}{b}\right)^{\lambda}e^{-\frac{1}{2}\lambda i\pi}, & -a\beta\left(\frac{a}{b}\right)^{\lambda}(e^{-\frac{3}{2}\lambda i\pi}-e^{-\frac{1}{2}\lambda i\pi}) \\ \left(\frac{b}{a}\right)^{\lambda}(e^{-\frac{3}{2}\lambda i\pi}-e^{-\frac{1}{2}\lambda i\pi}), & -2a\lambda\left(\frac{b}{a}\right)^{\lambda}e^{-\frac{1}{2}\lambda i\pi}, & 0, & 0 \\ -2\beta\lambda\left(\frac{b}{a}\right)^{\lambda}e^{-\frac{3}{2}\lambda i\pi}, & -a\beta\left(\frac{b}{a}\right)^{\lambda}(e^{-\frac{3}{2}\lambda i\pi}-e^{-\frac{1}{2}\lambda i\pi}), & 0, & 0 \end{vmatrix}$$

$$\times(U-\tfrac{1}{16}\Delta)\div(1-e^{-2\lambda i\pi}) \quad (9.6)$$

in the same way (8.5) as $R'(x, t)$ was derived from U.

But (9.6) is $(1/2i\pi)(-\Lambda)(U-\frac{1}{16}\Delta)$ or $-(1/2i\pi)\,U$ by the original construction of U (7.50). Hence the sum of these residues is

$$-R'(x, t)\div 2i\pi$$

and

$$\int_0^{\infty} P(x, u)\,R'(u, t)\,du = P(x, t)+R'(x, t). \quad (8.62)$$

The proof of (8.61) is similar.

The functions ϕ satisfying (6.4) and the "solving matrix" $R'(x, t)$ are not unique. For let $\lambda = -\lambda_1$ or $-\lambda_2$ so that $\Delta(\lambda) = 0$ and let κ denote a row of four constants. Then in

$$\int_0^{\infty} P(x, t)\,\kappa t^{\lambda}\,dt$$

the integrals converge and the row is equal to

$$x^{\lambda}\int_0^{\infty} P(u, 1)\,\kappa u^{-\lambda-1}\,du \quad \text{or} \quad \Lambda\kappa x^{\lambda}. \quad (9.70)$$

But, since $\Delta(\lambda) = 0$, we have $\kappa = \Lambda\kappa$ if the row κ is proportional to any column in U, that is to a row of minors in Δ, which is the determinant of $(1-\Lambda)\,2i\sin\lambda\pi$. Hence, for such a row κ,

$$\int_0^{\infty} P(x, t)\,\kappa t^{\lambda}\,dt = \kappa x^{\lambda}. \quad (9.71)$$

Here λ may be $-\lambda_1$ or $-\lambda_2$ and the corresponding rows κ will be distinguished as κ_1, κ_2.

Thus, if c_1, c_2 are single constants, $\phi(z)$ may be altered by the addition of

$$c_1\kappa_1 z^{-\lambda_1}+c_2\kappa_2 z^{-\lambda_2}. \tag{9.8}$$

Moreover $R'(z, t)$ may be altered by the addition of matrices in which each column is of the form (9.8) and here c_1, c_2 may involve t. In particular, the terms in $z^{-\lambda_1}$ and $z^{-\lambda_2}$ of the expansion in descending complex powers of z may be subtracted; let $R'(z, t)$ as thus modified be called $R(z, t)$. $R(z, t)$ thus becomes infinite of the order $z^{-\lambda_2}$ (see 9.3) when z vanishes, but is of the order t^{λ_3-1} when t approaches zero. We have then

$$R(x, t)+P(x, t) = \int_0^\infty P(x, u)\, R(u, t)\, du, \tag{9.9}$$

and also, by the same reasoning as (9.50),

$$\int_0^\infty R(u, 1)\, du = 1, \tag{9.91}$$

$R(u, 1)$ being of the order $u^{-\lambda_3}$ when u is great, and $u^{-\lambda_2}$ when u is small.

10. From (9.9) it follows that the nucleus of $(P)+(R)-(P)(R)$ (§ 6), that is

$$P(x, t)+R(x, t)-\int_0^1 P(x, u)\, R(u, t)\, du\ .$$

is $$\int_1^\infty P(x, u)\, R(u, t)\, du \quad \text{or} \quad \int_0^1 P(xv, 1)\, R(1, tv)\, dv.$$

Call this $S_1(x, t)$, then since, for values of x, t from 0 to 1 inclusive, $P(x, 1)$, $R(1, t)$ are matrices of bounded and continuous functions, it follows without difficulty that the same is true of $S_1(x, t)$.

Again if

$$S_2(x, t) = \int_0^1 Q(x, u)\, R(u, t)\, du = \int_0^{1/t} Q(x, tv)\, R(v, 1)\, dv,$$

we have

$$S_2(x', t')-S_2(x, t)$$

$$= \int_0^{1/t} \{Q(x', t'v)-Q(x, tv)\}\, R(v, 1)\, dv+\int_{1/t}^{1/t'} Q(x', t'v)\, R(v, 1)\, dv. \tag{10.1}$$

Since the functions of x, t in $Q(x, t)$ are bounded and continuous, it follows, except when t approaches 0, that the functions in $S_2(x, t)$ are

bounded and continuous. When t approaches 0, we have, by help of (9.91),

$$S_2(x, t)-Q(x', 0) = \{Q(x, 0)-Q(x', 0)\}+\int_0^{1/\sqrt{t}} \{Q(x, tv)-Q(x, 0)\} R(v, 1)\, dv$$
$$+\int_{1/\sqrt{t}}^{1/t} \{Q(x, tv)-Q(x, 0)\} R(v, 1)\, dv$$
$$-\int_{1/t}^{\infty} Q(x, 0) R(v, 1)\, dv. \qquad (10.2)$$

Each part of this expression diminishes indefinitely when $x-x'$ and t do so, and therefore $S_2(x, t)$ is continuous even when t approaches 0, and

$$S_2(x, 0) = Q(x, 0). \qquad (10.3)$$

Thus (6.3) may be written in the form

$$\{1-(S)\}\psi = f-\omega+(P)\omega+(Q)\omega, \qquad (10.4)$$

where
$$S(x, t) = S_1(x, t)+Q(x, t)-S_2(x, t), \qquad (10.5)$$

which is bounded and continuous, so that the ordinary methods may be applied to (10.4) to find ψ. Compare the discussion of a simpler case in *Proc. London Math. Soc.* (2), 22 (1924), 220–222.

There is still a difficulty in finding ϕ from (6.2) for $R(x, t)$ and therefore probably $(R)\psi$ are infinite when $x = 0$, whereas ϕ should be bounded. To meet this point, take ω in (6.2) to be $c_1\kappa_1 x^{-\lambda_1}+c_2\kappa_2 x^{-\lambda_2}$, as at (9.8).

Then $(Q)\omega$ consists of such integrals as $\int_0^1 Q_{rs}(x, t)\, t^\lambda dt$ where $\lambda+1$ is positive and Q_{rs} is regular in both variables. Hence $(Q)\omega$ is bounded and continuous.

Also, when $\lambda = -\lambda_1$ or $-\lambda_2$, and

$$\int_0^\infty P(x, t)\kappa t^\lambda dt = \kappa x^\lambda,$$

we have
$$\kappa x^\lambda-(P)\kappa x^\lambda = \int_1^\infty P(x, t)\kappa t^\lambda dt$$
$$= \int_0^1 P(xu, 1)\kappa u^{-\lambda-1} du,$$

which is a row of bounded and continuous functions, since the index of u

is greater than -1, and since P is bounded and continuous at the limits and between them. Thus, with the proposed value of ω, the right side of (10.4) is a bounded and continuous function of x, and the same is true of ψ, as given by Fredholm's method applied to (10.4): ψ now contains linearly the constants c_1, c_2 and so will ϕ as given by (6.2). Thus the part of ϕ that is not bounded is $C_1\kappa_1 x^{-\lambda_1}+C_2\kappa_2 x^{-\lambda_2}$, where C_1, C_2 are linear functions of c_1, c_2 and may presumably be made to vanish by proper choice of c_1, c_2, so that ϕ is then bounded; but the method may fail if

(1) the equations $C_1 = 0$, $C_2 = 0$ for c_1, c_2 are inconsistent, or

(2) there is a row of functions ψ such that

$$\psi = (S)\psi,$$

so that (10.4) cannot be solved for ψ unless a condition is satisfied.

11. If the method does not fail, it gives us a row of four functions ϕ, such that

$$\phi(z) = f(z)+\int_0^1 \{P(z, t)+Q(z, t)\}\,\phi(t)\,dt, \tag{11.0}$$

and these are in the first place bounded and continuous for *real* values of z from 0 to 1 and consequently in the second place analytical functions of the complex variable z with no singularities but those of $f(z)$ for such values of z as can be reached from some initial point without causing the singularities of P and Q to approach or cross the path of integration. These singularities occur when

$$t = \pm\frac{az}{ib}+2n, \quad \pm\frac{a}{ib}(2-z)+2n, \quad \pm\frac{ib}{a}z+2n, \quad \pm\frac{ib}{a}(2-z)+2n, \tag{11.1}$$

where n is integral. Hence, for values of z within the strip $0 < x < 2$, $\phi(z)-f(z)$ is analytical and regular. If z is within this strip, so is $2-z$, and $\phi_2(z) = \phi_1(2-z)$, $\phi_4(z) = \phi_3(2-z)$. Then, putting $\chi_1(z)$ for $\phi_1(2z/a)$ and $\chi_3(z)$ for $\phi_3(2z/ib)$, we have

$$\chi_1(z) = f_1\left(\frac{2z}{a}\right)+\int_0^{ib} J(t, z)\,\chi_3(t)\,dt, \tag{11.2}$$

$$\chi_3(z) = f_3\left(\frac{2z}{ib}\right)+\int_0^{a} K(t, z)\,\chi_1(t)\,dt. \tag{11.3}$$

Here $\chi_1(z)$ is regular within the strip $0 < x < a$, except at its poles, and on account of the properties of f_1, J (§§ 4, 5),

$$\chi_1(a+ib-z) = -\alpha\beta\chi_1(z), \tag{11.41}$$

$$\chi_1(z+ib)-\chi_1(z-ib) = 2\beta ib\chi_1'(z); \tag{11.42}$$

$\chi_3(z)$ is regular within the strip $0 < y < b$, except at poles, and

$$\chi_3(a+ib-z) = -\alpha\beta\chi_3(z), \tag{11.51}$$

$$\chi_3(z+a)-\chi_3(z-a) = 2a\alpha\chi_3'(z). \tag{11.52}$$

It does not follow at once that $\chi_1(z)$, $\chi_3(z)$ are the same function; for the proof of this some further developments are needed.

12. The right side of (11.2), which is a uniform function of z, is equal to $\chi_1(z)$ only when z is in the strip $0 < x < a$. The singular part of $2i\pi J(t, z)$ that gives rise to change when z crosses the lateral axis between $-ib$ and $2ib$ is

$$-\frac{1}{t-z}+\frac{1}{t+z}-\frac{2\beta z}{(t-z-ib)^2}+\frac{1}{t+z-2ib}-\frac{2\beta(z-ib)}{(t-z+ib)^2}. \tag{12.0}$$

On account of the first term the left side of (11.2) must be changed to

$$\chi_1(z)-\chi_3(z), \tag{12.11}$$

when the crossing is between 0 and ib; on account of the second and third terms the left side changes to

$$\chi_1(z)-\chi_3(-z)-2\beta z\chi_3'(z+ib) \tag{12.12}$$

when the crossing is between 0 and $-ib$; on account of the fourth and fifth terms it changes to

$$\chi_1(z)-\chi_3(2ib-z)-2\beta(z-ib)\chi_3'(z-ib) \tag{12.13}$$

when the crossing is between ib and $2ib$.

Since, then, the right side has one value for $ze^{i\pi}$ and $ze^{-i\pi}$, and one for $ib-ze^{i\pi}$ and $ib-ze^{-i\pi}$, we have

$$\chi_1(ze^{i\pi})-\chi_3(ze^{i\pi}) = \chi_1(ze^{-i\pi})-\chi_3(z)+2\beta z\chi_3'(ib-z), \tag{12.21}$$

$$\chi_1(ib-ze^{-i\pi})-\chi_3(ib-ze^{-i\pi}) = \chi_1(ib-ze^{i\pi})-\chi_3(ib-z)-2\beta z\chi_3'(z). \tag{12.22}$$

From (11.41), (11.51), (12.22), we have also

$$\chi_1(a+ze^{-i\pi})-\chi_3(a+ze^{-i\pi}) = \chi_1(a+ze^{i\pi})-\chi_3(a+z)+2\alpha z\chi_3'(z). \quad (12.23)$$

In these three formulae z is to be taken primarily on the lateral axis between 0 and ib. Similarly* from (11.3)

$$\chi_3(ze^{-i\pi})-\chi_1(ze^{-i\pi}) = \chi_3(ze^{i\pi})-\chi_1(z)+2\alpha z\chi_1'(a-z), \quad (12.24)$$

$$\chi_3(a-ze^{i\pi})-\chi_1(a-ze^{i\pi}) = \chi_3(a-ze^{-i\pi})-\chi_1(a-z)-2\alpha z\chi_1'(z), \quad (12.25)$$

$$\chi_3(ib+ze^{i\pi})-\chi_1(ib+ze^{i\pi}) = \chi_3(ib+ze^{-i\pi})-\chi_1(ib+z)+2\beta z\chi_1'(z). \quad (12.26)$$

In these three z is primarily on the real axis between 0 and a, but all the six formulae may be extended to other values by continuation. For instance, in (12.21) put $ze^{i\pi}$ for z and add (12.24), then

$$\phi'(ze^{2i\pi})-\phi'(ze^{-i\pi}) = -2z\psi''(z), \quad (12.31)$$

if $$\phi'(z) = \chi_1(z)-\chi_3(z),$$

and $$\psi'(z) = \alpha\chi_1(a-z)-\alpha\chi_3(a-z),$$

for when z is on the segment $(0, a)$ $ib-ze^{i\pi}$ is the same as $ib+z$, and $\chi_3'(ib-ze^{i\pi}) = \alpha\beta\chi_3'(a-z)$.

Similarly, by adding (12.23) and (12.25),

$$\psi'(ze^{-2i\pi})-\psi'(ze^{i\pi}) = -2z\phi''(z). \quad (12.32)$$

From (12.31) and (12.32) we have, by integration with any assigned values of $\phi(0)$ and $\psi(0)$,

$$\phi(ze^{2i\pi})+\phi(ze^{-i\pi})+2z\psi'(z)-2\psi(z) = 2\phi(0)-2\psi(0)$$

$$= -\psi(ze^{-2i\pi})-\psi(ze^{i\pi})-2z\phi'(z)+2\phi(z), \quad (12.4)$$

so that

$$-\phi(z)+\phi(ze^{-i\pi})+\psi(ze^{i\pi})-\psi(z)+(z+z)\,\phi'(z)+(z+z)\,\psi'(z), \quad (12.41)$$

$$= \phi(z)-\phi(ze^{2i\pi})-\psi(ze^{-2i\pi})+\psi(z)+(z-z)\,\phi'(z)+(z-z)\,\psi'(z). \quad (12.42)$$

The two sides of this are the values of (1.27) when γ is replaced by $-\frac{1}{2}\pi$, π respectively, and therefore represent a possible deflection of a plate clamped along the lines of amplitudes $-\frac{1}{2}\pi$, π.

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* It is remarkable that the regular parts of J, K and the values of f_1, f_3 do not affect these relations so long as f_1, f_3 are uniform.

Since $\psi'(z) = -\beta\phi'(ib+z)$, we may write (12.42) as

$$\phi(z)-\phi(ze^{2i\pi})+\beta\phi(ib+ze^{-2i\pi})-\beta\phi(z+ib)$$
$$+(z-z)\{\phi'(z)+\beta\phi'(z+ib)\}, \quad (12.43)$$

and when z is in the half plane adjoining the original rectangle on the left of the lateral axis the other arguments that occur in (12.43) are also in that half-plane, so that in each case the expression for ϕ' is the right side of (11.2); it therefore follows from the property of ϖ_β given in the note to (2.81) that

$$\phi'(z+ib)-\phi'(z-ib) = 2\beta ib\phi''(z), \quad (12.431)$$

and hence by integration (12.43) only differs by a constant from

$$\phi(z)-\phi\{ib+(ib+z)e^{-2i\pi}\}+\beta\phi\{(z-ib)e^{2i\pi}\}$$
$$-\beta\phi(z+ib)+(z-z+2ib)\{\phi'(z)+\beta\phi'(z+ib)\}, \quad (12.44)$$

where the arguments still lie in the same half-plane.

When z tends to infinity in this half-plane, its real part being great and negative, we have seen in the note to (4.4) that $\phi'(z)$ tends to a definite constant value if $\beta = -1$, and tends to equality with an expression such as $A+Bz$ when $\beta = +1$. In each case it is then found that (12.43) and (12.44) both tend to zero, so that their difference, which is constant, must be zero. Hence, if $H(x, y)$ is written for (12.41) or (12.42), we have from the form (12.44) that

$$H(x, y) = -\beta H(x, b-y). \quad (12.51)$$

Similarly, $H(x, y)$ or (12.41) may be written

$$-\phi(z)+\phi(ze^{-i\pi})-a\phi(a-ze^{i\pi})+a\phi(a-z)$$
$$+(z+z)\{\phi'(z)+a\phi'(a-z)\}, \quad (12.45)$$

which differs by a constant from

$$-\phi(z)+\phi\{a-(a-z)e^{i\pi}\}-a\phi\{(a-z)e^{-i\pi}\}$$
$$+a\phi(a-z)+(z+z-2a)\{\phi'(z)+a\phi'(a-z)\} \quad (12.46)$$

in the half-plane below the real axis, since there

$$\phi'(z+a)-\phi'(z-a) = 2aa\phi''(z). \quad (12.461)$$

As before, the constant difference is zero, and thus

$$H(x, y) = -\alpha H(a-x, y) \tag{12.52}$$

$$= \alpha\beta H(a-x, b-y). \tag{12.53}$$

Hence, if the sides of the original rectangle are produced indefinitely, $H(x, y)$ vanishes to the second order on each of the produced parts, and since it is reduced to zero by ∇^4 it represents a possible deflection for a plate in the form of the crossed strips, clamped along the edges.

13. The existence of this deflection depends on that of $\phi'(z)$ or $\chi_1(z)-\chi_3(z)$ for which we have various expressions, valid in different regions. In the original rectangle it is the difference of the right sides of (11.2) and (11.3). In the half-planes $x < 0$ and $x > a$, as entered directly from the plate, it is the right side of (11.2); and in the half-planes $y < 0$ and $y > b$, as entered directly from the plate, it is the right side of (11.3) with reversed sign. No two of these half-planes have any region in common, since ϕ' is not uniform in the neighbourhood of a corner.

By (12.24) we have

$$\phi'(ze^{-2i\pi}) = -\chi_3(z)+\chi_1(ze^{-i\pi})+2\alpha z\chi_1'(a+z) \tag{13.01}$$

$$= -\chi_3(z)+\chi_1(ze^{-i\pi})+2\beta z\chi_1'(ib-z) \tag{13.02}$$

and if z is in the rectangle $0 < y < b$, $-a < x < 0$, entered directly from the plate, the arguments on the right in (13.02) are all in the strips where the expressions (11.2), (11.3) hold for χ_1, χ_3; the argument of ϕ' is in the rectangle $0 < y < b$, $-a < x < 0$, but entered by a three-quarter turn about the origin in the negative sense. If the same rectangle is entered by a three-quarter turn about $(0, b)$ in the positive sense, we write $ib+(z-ib)e^{2i\pi}$ for z and by (12.26)

$$\phi'\{ib+(z-ib)e^{2i\pi}\} = -\chi_3(z)+\chi_1\{ib+(z-ib)e^{i\pi}\}$$

$$+2\beta(z-ib)\chi_1'\{(z-ib)e^{i\pi}\}. \tag{13.1}$$

Here the arguments on the right are again such that (11.2), (11.3) hold good and by (11.42)

$$\phi'(ze^{-2i\pi}) = \phi'\{ib+(z-ib)e^{2i\pi}\}. \tag{13.2}$$

Thus a circuit starting in the original rectangle and going round $(0, b)$ positively and then round the origin positively is a closed circuit for ϕ', that is, the half-planes $y < 0$ and $y > b$ are connected not only through the original rectangle, but also through the rectangle $-a < x < 0$, $0 < y < b$. A similar result holds for the rectangle $a < x < 2a$, $0 < y < b$; it is most easily proved by means of (11.41) and (11.51), or the equation (12.461) may be used to continue the function ϕ' from either of the half-planes $y < 0$, $y > b$ into the strip $0 < y < b$, except the lines $x = na$ $(n = 0, \pm 1, \pm 2, \ldots)$ which must be considered as cuts in the plane. When these cuts are made $\phi'(z)$ is uniform in the plane.

Similarly, $\phi'(z)$ is uniform in the plane of the original rectangle if cuts are made along the lines $y = nb$ $(x = 0, \pm 1, \pm 2, \ldots)$ across the strip $0 < x < a$. These two regions of uniformity for $\phi'(z)$ have no common part but the original rectangle; in Ω_1, the first of them, (12.461) holds, and in Ω_2, the second, (12.431). There are no singularities of $\phi'(z)$ in the original rectangle or in the four half-planes entered directly from it, but in the rectangle $-a < x < 0$, $0 < y < b$ entered by a three-quarter turn about a corner as above, the expression (13.01) shows that $\phi'(z)$ has the singularities of

$$-\chi_3(z)+\chi_1(ze^{-i\pi})$$

only. The singularities of $\chi_3(z)$ in the strip $0 < y < b$ are those of $f_3(2z/ib)$ by (11.3), so that in the rectangle $-a < x < 0$, $0 < y < b$ it has just four simple poles at

$$z_0-a, \quad ib-z_0, \quad -z_0, \quad -a+ib+z_0, \tag{13.3}$$

with residues

$$-\tfrac{1}{2}\alpha, \quad -\tfrac{1}{2}\beta, \quad \tfrac{1}{2}, \quad \tfrac{1}{2}\alpha\beta. \tag{13.31}$$

Similarly, $\chi_1(z)$ has in the strip $0 < x < a$ only the singularities of $f_1(2z/a)$ by (11.2) and therefore in the rectangle $-b < y < 0$, $0 < x < a$ it has simple poles at

$$a-z_0, \quad z_0-ib, \quad z_0, \quad a-ib-z_0, \tag{13.4}$$

with residues

$$\tfrac{1}{2}\alpha, \quad \tfrac{1}{2}\beta, \quad -\tfrac{1}{2}, \quad -\tfrac{1}{2}\alpha\beta. \tag{13.41}$$

Thus the singularities of $\chi_1(ze^{-i\pi})$ in the rectangle $-a < x < 0$, $0 < y < b$ are the same as those of $\chi_3(z)$ and $\phi'(ze^{-2i\pi})$ has no singularity in that rectangle; the same thing holds for $\phi'(ze^{i\pi})$ by (12.21). It follows from (12.461) and (12.431) that $\phi'(z)$ has no singularity within the regions Ω_1, Ω_2 over which it has been proved uniform.

Now in (12.41) if we write $-\beta\phi'(ib+z)$ for $\psi'(z)$ the arguments that occur are z, $ze^{-i\pi}$, $ib+ze^{i\pi}$, $ib+z$, all of which fall within Ω_1 if z is in the strip $0 < x < a$. Hence $H(x, y)$ has no singularity in this strip, and similarly in the strip $0 < y < b$ by the use of the expression (12.42).

Towards infinity in any of the four directions $H(x, y)$ is exponentially small; see the note to (4.4) and the reasoning before (12.51).

In the same way a function such as $H(x, y)$ could be formed if there were two different functions χ_1, χ_3, such that

$$\chi_1(z) = \int_0^{ib} J(t, z)\,\chi_3(t)\,dt, \tag{13.5}$$

$$\chi_3(z) = \int_0^{a} K(t, z)\,\chi_1(t)\,dt, \tag{13.6}$$

both being bounded and continuous throughout the respective ranges of integration; the only change in the reasoning is that the functions f_1, f_3 are zero.

The properties proved for $H(x, y)$ are that it is bounded and has no singularity within the crossed strips, vanishes to the second order on their edges and exponentially at infinity, and satisfies the equation $\nabla^4 H = 0$. It will be proved in § 15 that such a deflection without load cannot exist, and hence $\phi'''(z) = 0$ and

$$\chi_1''(z) = \chi_3''(z). \tag{13.7}$$

We may then put $\chi_3(z) = \chi(z)$, $\chi_1(z) = \chi(z)+Az+B$, where A, B are constants. It will be found on examination that A must be zero unless $\alpha = \beta = +1$.

14. It now follows from (12.21), (12.22), (12.23), that

$$\chi(ze^{-i\pi})-\chi(z)+2\beta z\chi'(ib-z) = 0, \tag{14.11}$$

$$\chi(ib-ze^{i\pi})-\chi(ib-z)-2\beta z\chi'(z) = 0, \tag{14.12}$$

$$\chi(a+ze^{i\pi})-\chi(a+z)+2\alpha z\chi'(z) = 0, \tag{14.13}$$

and, by putting $ze^{i\pi}$ for z in (14.11), we have

$$\chi(z)-\chi(ze^{i\pi})-2\alpha z\chi'(a-z) = 0. \tag{14.14}$$

In each of these four formulae the argument of χ' may be brought within the original rectangle and the arguments of χ are then in rectangles entered directly from the original. Now let $\chi(z) = \phi'(z)$ and integrate (14.11) putting z for $ze^{-i\pi}$, thus

$$\phi(z)+\phi(ze^{i\pi})-2\beta z\phi'(ib+z)+2\beta\phi(ib+z) = 2\phi(0)+2\beta\phi(ib), \quad (14.21)$$

and from (14.12)

$$\phi(ib+z)+\phi(ib+ze^{-i\pi})-2\beta z\phi'(z)+2\beta\phi(z) = 2\beta\phi(0)+2\phi(ib), \quad (14.22)$$

so that

$$\phi(z)-\phi(z)+\beta\phi(ib+z)-\beta\phi(ib+z)+(z-z)\{\phi'(z)+\beta\phi'(ib+z)\} \quad (14.30)$$

$$= -\phi(z)+\phi(ze^{i\pi})-\beta\phi(ib+ze^{-i\pi})+\beta\phi(ib+z)$$

$$+(z+z)\{\phi'(z)-\beta\phi'(ib+z)\}. \quad (14.31)$$

Denote the common value of these by $F(x, y)$. Then if in (14.21) we write z for $ib+z$, in (14.22) $z+ib$ for z, we have

$$F(x, y) = \phi(z)-\phi(z+2ib)+\beta\phi(z-ib)$$

$$-\beta\phi(z+ib)+(z-z+2ib)\{\phi'(z)+\beta\phi'(ib+z)\}$$

$$= -\beta F(x, b-y), \quad (14.32)$$

so that $F(x, y)$ vanishes to the second order on the side of the plate $z = z+2ib$, as well as on the sides $z = \pm z$.

Again $\phi'(z) = -a\beta\phi'(a+ib-z)$, so that (14.30) and (14.31) may be written

$$\phi(z)-\phi(z)+a\phi(a-z)-a\phi(a-z)+(z-z)\{\phi'(z)-a\phi'(a-z)\}, \quad (14.33)$$

and

$$-\phi(z)+\phi(-z)-a\phi(a+z)+a\phi(a-z)+(z+z)\{\phi'(z)+a\phi'(a-z)\}, \quad (14.34)$$

new forms for $F(x, y)$ where the arguments are still in the plate or in the rectangles immediately adjoining. From (14.13), (14.14) we have, by integration

$$\phi(a-z)+\phi(a+z)-2az\phi'(z)+2a\phi(z) = 2\phi(a)+2a\phi(0)$$

$$= a\phi(z)+a\phi(ze^{i\pi})+2z\phi'(a-z)+2\phi(a-z). \quad (14.4)$$

In the first member of (14.4) put $a-\mathrm{z}$ for z and in the third $a-z$ for z, and we have by (14.33), (14.34)

$$F(x, y) = -aF(a-x, y),$$

so that $F(x, y)$ vanishes to the second order on the side $z = 2a-\mathrm{z}$ also.

Thus, if $F(x, y)$ is formed by means of functions satisfying (13.5), (13.6) it is a possible deflection for the original rectangular plate, clamped, but without load, since there is no singularity within the rectangle, and no discontinuity at its boundary. Such a deflection cannot exist on account of the lemma to be proved in § 15. Hence in (13.5), (13.6) $\chi_1(z)$, $\chi_3(z)$ can only be linear functions of z.

If the functions from which $F(x, y)$ is formed satisfy (11.2) and (11.3), then $F(x, y)$ is a possible deflection for the plate clamped at the edges, but with singularities in the rectangle, namely those of

$$-\phi(\mathrm{z})+a\phi(a-z),$$

that is, of

$$\begin{aligned}&\tfrac{1}{2}\log(\mathrm{z}-\mathrm{z}_0)-\tfrac{1}{2}a\log(\mathrm{z}+z_0-a)-\tfrac{1}{2}\beta\log(\mathrm{z}-z_0+ib)\\ &\quad+\tfrac{1}{2}a\beta\log(\mathrm{z}+\mathrm{z}_0-a+ib)+\tfrac{1}{2}\log(z-z_0)-\tfrac{1}{2}a\log(z+\mathrm{z}_0-a)\\ &\quad-\tfrac{1}{2}\beta\log(z-\mathrm{z}_0-ib)+\tfrac{1}{2}a\beta\log(z+z_0-a-ib),\end{aligned}\qquad(14.5)$$

it has no discontinuity at the edges or corners.

If the average is taken of the four expressions $F(x, y)$ with $a, \beta = \pm 1$, the result is a deflection with the singularity of

$$\tfrac{1}{2}\log(z-z_0)+\tfrac{1}{2}\log(\mathrm{z}-\mathrm{z}_0).$$

This must be $(\partial^2/\partial z_0\partial \mathrm{z}_0)\,G(z, \mathrm{z}, z_0, \mathrm{z}_0)$, and it may be written in the form

$$g''(z_0)+g_2''(\mathrm{z}_0),$$

since f_1, f_3 in (11.2), (11.3) may be resolved into two such parts, depending on z_0, z_0 respectively. In the rectangle $-b<y<0$, $0<x<a$, it has in addition the singularities of

$$-\tfrac{1}{2}\log(\mathrm{z}-z_0)-\tfrac{1}{2}\log(z-\mathrm{z}_0)+\tfrac{1}{2}(z-\mathrm{z})\left\{\frac{1}{z-\mathrm{z}_0}-\frac{1}{\mathrm{z}-z_0}\right\}.$$

arising from terms in (14.33) which have no singularity in the original rectangle.

Then, since Green's function depends symmetrically on the two points involved, we have

$$G(z, z, z_0, z_0) = g_1(z_0) - g_1(z_0) - g_2(z_0) + g_2(z_0) + (z_0 - z_0)\{g_1'(z_0) - g_2'(z_0)\}, \quad (14.6)$$

where $g_1(z_0)$ has the singularities of

$$\tfrac{1}{4}(z-z_0)^2 \log(z-z_0) - \tfrac{1}{4}(2z-z-z_0)(z-z_0)\log(z-z_0),$$

and $g_2(z_0)$ those of

$$\tfrac{1}{4}(z-z_0)^2 \log(z-z_0) - \tfrac{1}{4}(2z-z-z_0)(z-z_0)\log(z-z_0),$$

from which it may be verified that G has the singularity prescribed in § 1, and therefore supplies the solution of the problem proposed.

Lemma.

15. It has been assumed in the foregoing work (§§ 13, 14) that on a clamped plate, whether rectangular or in the form of the crossed strips, there can be no deflection without singularity, that is, without load; this follows from the fact that if U is a function of x and y, regular throughout the area, reduced to zero by the operator ∇^4 and vanishing on the boundary to the second order in general, at the corners to an order fractionally higher than the first and, in the case of the crossed strips, exponentially at infinity, then*

$$\iint \left\{ \left(\frac{\partial^2 U}{\partial x^2} - \frac{\partial^2 U}{\partial y^2} \right)^2 + 4 \left(\frac{\partial^2 U}{\partial x \partial y} \right)^2 \right\} dx\, dy = 0, \quad (15.0)$$

if taken over the given area. (15.0) can be verified in the usual way and leads to the conclusion that

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial y^2}, \quad \frac{\partial^2 U}{\partial x \partial y} = 0 \quad (15.1)$$

throughout the area. Thus U must be of the form

$$A(x^2+y^2) + Bx + Cy + D \quad (15.2)$$

where A, B, C, D are constants, and these constants must vanish if U vanishes to the second order along a segment of a straight line.

It follows that the deflection G is unique.

* I understand that this method is due to the late Lord Rayleigh and has been used by Ritz.

16. Thus, in conclusion, if G is Green's function for the rectangular plate, G is formed as in § 1 from functions ϕ, ψ, whose properties and singularities are given in § 2, and which depend on a function χ: in § 4, χ is found to satisfy certain integral equations, whose solution is discussed in §§ 5–10. The construction of G from the solution is discussed in §§ 11–15, and it follows from the Lemma of § 15 that the solution must be unique. In the case of an integral equation with a bounded and continuous nucleus, the uniqueness of the solution ensures its possibility; but when, as here, the nucleus is not bounded the theory of integral equations in its present state does not enable us to draw the same conclusion*. In other respects the mathematical determination of G depends on recognized operations, though there is scope for further discussion on the point mentioned and others.

17. The results of §§ 12, 13 enable us to find Green's function $[\mathbf{G}]$ for a plate of another form, namely that of the crossed strips, still clamped along the edges. If the functions f_3, f_4 of (5 . 1) and (11 . 3) are replaced by zero, the only part of the reasoning which is affected is that relating to the singularities of χ_1, χ_3 and ϕ': $\phi'(z)$ has the poles specified in (13 . 3) with the residues (13 . 31), and the singularities following from these by the equations (12 . 31), (12 . 32), (12 . 431), (12 . 461). Thus the singularities of $H(x, y)$ in the crossed strips are those of

$$\phi(ze^{-i\pi})-\beta\phi(ib+ze^{i\pi}),$$

that is, of (14 . 5). The average of the four expressions $H(x, y)$, with $\alpha, \beta = \pm 1$, is then

$$\frac{\partial^2}{\partial z_0 \partial z_0}[\mathbf{G}](z, z, z_0, z_0),$$

and $[\mathbf{G}]$ itself may be deduced as at (14 . 6).

* Compare *Proc. London Math. Soc.* (2), 22 (1924), 209, 210.

IDENTIFICATION OF THE COEFFICIENTS IN A TRIGONOMETRICAL INTEGRAL

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1. THEOREM I. *If, for all finite real values of x,*

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds = f(x) \neq \pm\infty, \tag{1.1}$$

and

$$A = \int_0^\infty |a_s|\, ds < \infty, \quad B = \int_0^\infty |b_s|\, ds < \infty, \tag{1.11}$$

then, for almost all positive values of s,

$$a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos sx\, dx, \quad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin sx\, dx, \tag{1.2}$$ (C 1)

the exceptional values of s, if existent, forming a set of measure zero.

By (1.1) is meant

$$\lim_{\nu\to\infty} \int_0^\nu (a_s \cos sx + b_s \sin sx)\, ds = f(x), \tag{1.3}$$

and by (1.2),

$$a_s = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda d\nu \int_{-\nu}^{\nu} f(x)\cos sx\, dx, \quad b_s = \lim_{\lambda\to\infty} \frac{1}{\lambda}\int_0^\lambda d\nu \int_{-\nu}^{\nu} f(x)\sin sx\, dx. \tag{1.4}$$

It is not possible to prove, in general, that (1.2) holds for all values of s, since an alteration in a_s, b_s at the points of a set of measure zero does not affect the value of the integral in (1.3) and therefore cannot affect the validity of (1.1).

In virtue of (1.11), the convergence of the integral in (1.1) is uniform, so that $f(x)$ is continuous and the integrals in (1.4) certainly exist. Also

$$|f(x)| \leqslant \int_0^\infty \{|a_s| + |b_s|\}\, ds \leqslant A + B.$$

If, in the integral,

$$\int_0^\lambda d\nu \int_{-\nu}^{\nu} \cos \sigma x \, dx \int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds,$$

we replace all integrands by their absolute values, the result is not greater than

$$2\nu\lambda(A+B) < \infty,$$

so that, by a well known theorem, we may change the order of integrations in any way we please. Thus

$$(1.5) \quad \frac{1}{\lambda}\int_0^\lambda d\nu \int_{-\nu}^{\nu} f(x) \cos \sigma x \, dx$$

$$= \frac{1}{\lambda}\int_0^\infty a_s\, ds \int_0^\lambda d\nu \int_{-\nu}^{\nu} \cos \sigma x \cos sx \, dx$$

$$+ \frac{1}{\lambda}\int_0^\infty b_s\, ds \int_0^\lambda d\nu \int_{-\nu}^{\nu} \cos \sigma x \sin sx \, dx$$

$$= \frac{1}{\lambda}\int_0^\infty a_s \frac{1-\cos\lambda(s+\sigma)}{(s+\sigma)^2}\, ds + \frac{1}{\lambda}\int_0^\infty a_s \frac{1-\cos\lambda(s-\sigma)}{(s-\sigma)^2}\, ds.$$

Suppose now $\sigma > 0$. Then

$$\left| \frac{1}{\lambda}\int_0^\infty a_s \frac{1-\cos\lambda(s+\sigma)}{(s+\sigma)^2}\, ds \right| \leqslant \frac{1}{\lambda}\int_0^\infty |a_s| \frac{2}{\sigma^2}\, ds = \frac{1}{\lambda}\cdot\frac{2A}{\sigma^2} \to 0.$$

For the other integral, we may write

$$\frac{1}{\lambda}\int_0^\infty a_s \frac{1-\cos\lambda(s-\sigma)}{(s-\sigma)^2}\, ds$$

$$= \frac{1}{\lambda}\int_0^{\sigma-\epsilon} + \frac{1}{\lambda}\int_{\sigma-\epsilon}^{\sigma+\epsilon} + \frac{1}{\lambda}\int_{\sigma+\epsilon}^\infty a_s \frac{1-\cos\lambda(s-\sigma)}{(s-\sigma)^2}\, ds$$

$$= I_1 + I_2 + I_3,$$

where ϵ is any positive number less than σ.

Plainly

$$|I_1| \leqslant \frac{1}{\lambda}\int_0^{\sigma-\epsilon} |a_s| \frac{2}{\epsilon^2}\, ds \leqslant \frac{1}{\lambda}\int_0^\infty |a_s| \frac{2}{\epsilon^2}\, ds = \frac{1}{\lambda}\cdot\frac{2A}{\epsilon^2} \to 0.$$

$$|I_3| \leqslant \frac{1}{\lambda}\int_{\sigma+\epsilon}^\infty |a_s| \frac{2}{\epsilon^2}\, ds \leqslant \frac{1}{\lambda}\int_0^\infty |a_s| \frac{2}{\epsilon^2}\, ds = \frac{1}{\lambda}\cdot\frac{2A}{\epsilon^2} \to 0.$$

The remaining integral I_2 is similar to that which occurs in the theory of summation of Fourier series. Its limit, when existent, is independent of ϵ and has the value πa_σ for almost all the values of σ concerned.

This establishes the first part of (1 . 2). The second part is similarly obtained by considering

$$\frac{1}{\lambda}\int_0^\lambda dv \int_{-v}^{v} f(x)\sin\sigma x\,dx.$$

2. It is known that $I_2 \to \pi a_\sigma$

for any value of σ such that, as $h \to 0$,

$$\frac{1}{2h}\int_{\sigma-h}^{\sigma+h} |a_s - a_\sigma|\,ds \to 0. \tag{2 . 1}$$

Thus (1 . 2) hold for $s = \sigma$ whenever (2 . 1) and

$$\frac{1}{2h}\int_{\sigma-h}^{\sigma+h} |b_s - b_\sigma|\,ds \to 0 \tag{2 . 2}$$

hold.

In particular, they hold for $s = \sigma$ whenever a_s, b_s are continuous at $s = \sigma$ or, more generally, whenever, as $t \to 0$,

$$\frac{a_{\sigma+t}+a_{\sigma-t}}{2} \to a_\sigma, \quad \frac{b_{\sigma+t}+b_{\sigma-t}}{2} \to b_\sigma.$$

Considerations such as these apply throughout the subsequent theory.

3. LEMMA. *If, for any finite positive value of c,*

$$\int_0^c (a_s\cos sx + b_s\sin sx)\,ds = g(x), \tag{3 . 1}$$

where

$$A_c = \int_0^c |a_s|\,ds < \infty, \quad B_c = \int_0^c |b_s|\,ds < \infty, \tag{3 . 11}$$

then, for almost all values of s in $0 < s < c$,

$$a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\cos sx\,dx, \quad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\sin sx\,dx, \tag{3 . 2}$$ (C 1)

and, for all values of $s > c$,

$$0 = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\cos sx\,dx, \quad 0 = \frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\sin sx\,dx. \tag{3 . 3}$$

For (3 . 2) we suppose a_s, b_s defined outside the range $(0, c)$ by assigning to them the value zero, when (3 . 2) becomes a particular case of Theorem I.

For (3 . 3), which goes a little further, in that it asserts

$$\lim_{\mu,\nu\to\infty}\int_{-\mu}^{\nu} g(x)\cos sx\,dx = 0, \quad \lim_{\mu,\nu\to\infty}\int_{-\mu}^{\nu} g(x)\sin sx\,dx = 0,$$

we may proceed as follows.

As before, since $A_c < \infty$, $B_c < \infty$,

$$\int_{-\mu}^{\nu} g(x)\cos\sigma x\,dx = \int_0^c a_s\,ds\int_{-\mu}^{\nu}\cos\sigma x\cos sx\,dx + \int_0^c b_s\,ds\int_{-\mu}^{\nu}\cos\sigma x\sin sx\,dx.$$

Now $$\int_0^c a_s\,ds\int_{-\mu}^{\nu}\cos\sigma x\cos sx\,dx = \tfrac{1}{2}(I_\nu - I_\mu + J_\nu - J_\mu),$$

where $$I_k = \int_0^c a_s\frac{\sin k(\sigma+s)}{\sigma+s}\,ds, \quad J_k = \int_0^c a_s\frac{\sin k(\sigma-s)}{\sigma-s}\,ds.$$

If $\sigma > c$,

$$\int_0^c \frac{|a_s|}{\sigma+s}\,ds \leqslant \frac{A_c}{c} < \infty,$$

$$\int_0^c \frac{|a_s|}{\sigma-s}\,ds \leqslant \frac{A_c}{\sigma-c} < \infty,$$

so that, by the theorem of Riemann-Lebesgue,

$$I_k \to 0, \quad J_k \to 0$$

as $k \to \infty$.

Thus, if $\sigma > c$,

$$\lim_{\mu,\nu\to\infty}\int_0^c a_s\,ds\int_{-\mu}^{\nu}\cos\sigma x\cos sx\,dx = 0.$$

Similarly, $$\lim_{\mu,\nu\to\infty}\int_0^c b_s\,ds\int_{-\mu}^{\nu}\cos\sigma x\sin sx\,dx = 0,$$

and (3 . 3) is established.

We may observe that, in virtue of (3 . 11), $g(x)$ is a continuous function of x. Also

$$|g(x)| \leqslant A_c + B_c.$$

4. It is convenient at this stage to state precisely the meaning which is to be attached to the sign of integration. We adopt the sense of Denjoy-Perron together with the convention that, whenever a function does not change sign, infinite values of its integral are permitted. Thus, if $f(x) \geqslant 0$, we define, for finite ranges (α, β),

$$\int_\alpha^\beta f(x)\,dx = \lim_{n\to\infty}\int_\alpha^\beta f_n(x)\,dx,$$

where $$f_n(x) = \min\{f(x), n\}.$$

If $f(x)$ changes sign, then the existence in (a, b) of

$$f_1(x) = \int_a^x f(x)\,dx$$

implies the existence of continuous functions $\phi(x)$, $\psi(x)$ such that

$$\phi(a) = 0, \quad \psi(a) = 0,$$

$$D\{f_1(x) - \phi(x)\} \geqslant 0, \quad D\{\psi(x) - f_1(x)\} \geqslant 0,$$

$$+\infty \neq D\phi(x) \leqslant f(x),$$

$$-\infty \neq D\psi(x) \geqslant f(x),$$

where $D\theta(x)$ refers to all the limits, as $h \to 0$, of the difference ratio

$$\frac{\theta(x+h) - \theta(x)}{h},$$

and, finally, $$\psi(b) - \phi(b) < \epsilon,$$

where ϵ is any positive number given in advance.

The advantage of this definition, from the present point of view, is that it enables us to prove readily that, if $F(x)$ is continuous in (a, b) and, as $h \to 0$,

$$\frac{F(x+h) - 2F(x) + F(x-h)}{h^2} \to f(x) \neq \pm\infty,$$

where $f(x)$ has a finite integral in (a, b), then, in (a, b),

$$F(x) = \int_a^x dx \int_a^x f(x)\,dx + px + q,$$

where p and q are constants*.

It is not necessary to assume that $f(x)$ is absolutely integrable.

5. THEOREM II. *If, for all finite real values of* x,

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\,dx = f(x) \neq \pm\infty, \tag{5.1}$$

* For details of the proof, see de la Vallée Poussin, *Cours d'analyse infinitésimale*, 1 (1914), 3 ed., 290, 291.

and, for some positive value of c,

$$(5.11) \qquad A_* = \int_c^\infty \frac{|a_s|}{s^2}\,ds < \infty, \quad B_* = \int_c^\infty \frac{|b_s|}{s^2}\,ds < \infty,$$

$$(5.12) \qquad A_c = \int_0^c |a_s|\,ds < \infty, \qquad B_c = \int_0^c |b_s|\,ds < \infty,$$

and further $f(x)$ *has a finite integral in every finite interval, then, for almost all positive values of* s,

$$(5.2) \qquad a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos sx\,dx, \quad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin sx\,dx. \qquad (C\,1)$$

We begin by assuming that, for $0 \leqq s < c$,

$$(5.21) \qquad a_s = 0, \quad b_s = 0.$$

Then the integral

$$\int_0^\infty \left(-\frac{a_s}{s^2}\cos sx - \frac{b_s}{s^2}\sin sx\right) ds$$

converges absolutely for all finite values of x and represents a continuous function of x, which we will denote by $F(x)$.

By Theorem 1, for almost all positive values of s,

$$(5.3) \qquad \begin{cases} -\dfrac{a_s}{s^2} = \dfrac{1}{\pi}\lim\limits_{\lambda\to\infty}\dfrac{1}{\lambda}\displaystyle\int_0^\lambda d\nu\int_{-\nu}^{\nu} F(x)\cos sx\,dx, \\ -\dfrac{b_s}{s^2} = \dfrac{1}{\pi}\lim\limits_{\lambda\to\infty}\dfrac{1}{\lambda}\displaystyle\int_0^\lambda d\nu\int_{-\nu}^{\nu} F(x)\sin sx\,dx. \end{cases}$$

Further, since, in virtue of (5.21)

$$\int_0^\infty \frac{|a_s|}{s^2}\,ds < \infty, \quad \int_0^\infty \frac{|b_s|}{s^2}\,ds < \infty,$$

we have, by the theorem of Riemann-Lebesgue, which extends without difficulty to an infinite range,

$$(5.4) \qquad F(x) \to 0$$

as $|x| \to \infty$.

From the known identities

$$\cos s(x+2a) - 2\cos sx + \cos s(x-2a) = -4\sin^2 sa\cos sx,$$

$$\sin s(x+2a) - 2\sin sx + \sin s(x-2a) = -4\sin^2 sa\sin sx,$$

we have, if $a > 0$,

$$(5.41)\qquad \frac{F(x+2a)-2F(x)+F(x-2a)}{4a^2} = \int_0^\infty (a_s \cos sx + b_s \sin sx)\left(\frac{\sin sa}{sa}\right)^2 ds.$$

Now, x being for the moment fixed, write

$$\int_0^s (a_s \cos sx + b_s \sin sx)\, ds = \Phi(s),$$

$$\int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds = \Phi.$$

By integration by parts

$$\int_0^\lambda (a_s \cos sx + b_s \sin sx)\left(\frac{\sin sa}{sa}\right)^2 ds$$

$$= \left[\Phi(s)\left(\frac{\sin sa}{sa}\right)^2\right]_0^\lambda - \int_0^\lambda \Phi(s)\,\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 ds$$

$$= \Phi(\lambda)\left(\frac{\sin \lambda a}{\lambda a}\right)^2 - \int_0^\lambda \Phi(s)\,\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 ds.$$

Make $\lambda \to \infty$. Then, by (5.41),

$$\frac{F(x+2a)-2F(x)+F(x-2a)}{4a^2} = -\int_0^\infty \Phi(s)\,\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 ds.$$

Now $$\int_0^\infty \frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 ds = \lim_{\lambda\to\infty}\left[\left(\frac{\sin sa}{sa}\right)^2\right]_0^\lambda = -1.$$

Thus

$$(5.42)\qquad \frac{F(x+2a)-2F(x)+F(x-2a)}{4a^2} - f(x) = \int_0^\infty \{\Phi - \Phi(s)\}\,\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 ds.$$

But $$\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 = \frac{2\sin sa}{sa}\cdot\frac{sa\cos sa - \sin sa}{s^2 a}$$

$$= -\frac{2\sin sa}{sa}\cdot \theta a \sin \theta sa$$

for some value of θ between 0 and 1. Thus

$$(5.421)\qquad \left|\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2\right| \leqslant 2a,$$

and, as $a \to 0$, $$\frac{d}{ds}\left(\frac{\sin sa}{sa}\right)^2 \to 0.$$

It follows that, for every finite positive value of k, as $\alpha \to 0$,

$$\int_0^k \{\Phi - \Phi(s)\} \frac{d}{ds}\left(\frac{\sin s\alpha}{s\alpha}\right)^2 ds \to 0. \tag{5.43}$$

Again, by writing $s\alpha = u$, we have

$$\int_0^\infty \left| \frac{d}{ds}\left(\frac{\sin s\alpha}{s\alpha}\right)^2 \right| ds = \int_0^\infty \left| \frac{2\sin u}{u} \cdot \frac{u\cos u - \sin u}{u^2} \right| du = G,$$

a constant independent of α.

Since $\Phi(s) \to \Phi$, corresponding to each positive ϵ, there is a k such that

$$|\Phi - \Phi(s)| < \epsilon/G \quad (s \geqslant k),$$

and so

$$\int_k^\infty \{\Phi - \Phi(s)\} \frac{d}{ds}\left(\frac{\sin s\alpha}{s\alpha}\right)^2 ds \leqslant \frac{\epsilon}{G}\int_k^\infty \left| \frac{d}{ds}\left(\frac{\sin s\alpha}{s\alpha}\right)^2 \right| ds$$

$$\leqslant \frac{\epsilon}{G}\int_0^\infty \left| \frac{d}{ds}\left(\frac{\sin s\alpha}{s\alpha}\right)^2 \right| ds = \epsilon.$$

Thus, in view of (5.42),

$$\left| \frac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2} - f(x) - \int_0^k \{\Phi - \Phi(s)\} \frac{d}{ds}\left(\frac{\sin s\alpha}{s\alpha}\right)^2 ds \right| \leqslant \epsilon.$$

Make $\alpha \to 0$. We obtain, in view of (5.43),

$$\overline{\lim_{\alpha \to 0}} \left| \frac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2} - f(x) \right| \leqslant \epsilon,$$

i.e.

$$\frac{F(x+2\alpha) - 2F(x) + F(x-2\alpha)}{4\alpha^2} \to f(x).$$

Since $f(x)$ has a finite integral in every finite interval, we have

$$F(x) = \int_0^x dx \int_0^x f(x)\, dx + px + q.$$

Substitute in (5.3) and integrate by parts.

$$\int_{-\nu}^{\nu} F(x)\cos sx\, dx$$

$$= \left[\frac{F(x)\sin sx}{s}\right]_{-\nu}^{\nu} + \left[\frac{F'(x)\cos sx}{s^2}\right]_{-\nu}^{\nu} - \frac{1}{s^2}\int_{-\nu}^{\nu} f(x)\cos sx\, dx$$

$$= \{F(\nu) + F(-\nu)\}\frac{\sin s\nu}{s} + \{F'(\nu) - F'(-\nu)\}\frac{\cos s\nu}{s^2} - \frac{1}{s^2}\int_{-\nu}^{\nu} f(x)\cos sx\, dx.$$

Apply to each of these terms the operation

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\int_0^\lambda d\nu.$$

As $\nu\to\infty$, by (5 . 4), $\qquad F(\nu)+F(-\nu)\to 0.$

Therefore $\qquad \{F(\nu)+F(-\nu)\}\,\dfrac{\sin s\nu}{s}\to 0,$

and so, by the usual consistency theorem,

$$\frac{1}{\lambda}\int_0^\lambda \{F(\nu)+F(-\nu)\}\,\frac{\sin s\nu}{s}\,d\nu\to 0.$$

Again,

$$\frac{1}{\lambda}\int_0^\lambda \{F'(\nu)-F'(-\nu)\}\,\frac{\cos s\nu}{s^2}\,d\nu$$
$$=\frac{1}{\lambda}\left[\{F(\nu)+F(-\nu)\}\,\frac{\cos s\nu}{s^2}\right]_0^\lambda+\frac{1}{\lambda}\int_0^\lambda \{F(\nu)+F(-\nu)\}\,\frac{\sin s\nu}{s}\,d\nu.$$

Of these terms the second tends to zero as above, while the first tends to zero with both $1/\lambda$ and $F(\lambda)$. Thus

$$\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda d\nu\int_{-\nu}^{\nu}F(x)\cos sx\,dx=-\frac{1}{s^2}\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda d\nu\int_{-\nu}^{\nu}f(x)\cos sx\,dx,$$

i.e. in virtue of (5 . 3), for almost all positive values of s,

$$a_s=\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda d\nu\int_{-\nu}^{\nu}f(x)\cos sx\,dx.$$

In exactly the same way, for almost all positive values of s,

$$b_s=\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda d\nu\int_{-\nu}^{\nu}f(x)\sin sx\,dx,$$

and we have (5 . 2).

All this, however, is on assumption (5 . 21). To obtain the general case, subtract from $f(x)$ the function $g(x)$ of the lemma. The preceding analysis applies to $f(x)-g(x)$, and an application of the lemma yields (5 . 2).

6. Similar considerations apply to trigonometrical integrals involving the cosine only, or the sine only. Thus we have :

THEOREM IIa. *If, for all positive real values of x,*

$$\int_0^\infty a_s\cos sx\,ds=f(x)\neq\pm\infty, \tag{6 . 1}$$

and, for some positive value of c,

$$A_* = \int_c^\infty \frac{|a_s|}{s^2}\,ds < \infty, \tag{6.11}$$

$$A_c = \int_0^c |a_s|\,ds < \infty, \tag{6.12}$$

and, further, $f(x)$ has a finite integral in every finite interval containing only positive values of x, then, for almost all positive values of s,

$$a_s = \frac{2}{\pi}\int_0^\infty f(x)\cos sx\,dx. \tag{6.2}$$ (C 1)

A similar theorem holds for the sine-integral. Both of these can be deduced from Theorem II. For define $f(x)$ for negative values of x by writing

$$f(-x) = f(x) \quad (x > 0). \tag{6.3}$$

Then (6 . 1) gives

$$\int_0^\infty a_s \cos sx\,ds = f(x) \neq \pm\infty$$

for *all* real values of x. We can apply Theorem II, finding, for almost all positive values of s,

$$a_s = \frac{1}{\pi}\int_{-\infty}^\infty f(x)\cos sx\,dx \tag{C 1}$$

$$= \frac{2}{\pi}\int_0^\infty f(x)\cos sx\,dx \tag{C 1}$$

by (6 . 3). For the sine-integral we similarly use $f(-x) = -f(x)$.

7. THEOREM IIc. *If, for all finite real values of x,*

$$\int_{-\infty}^\infty (a_s\cos sx + b_s \sin sx)\,ds = f(x) \neq \pm\infty, \tag{7.1}$$ (P)

and, for some positive value of c,

$$A^* = \int_c^\infty \frac{|a_s|+|a_{-s}|}{s^2}\,ds < \infty, \quad B^* = \int_c^\infty \frac{|b_s|+|b_{-s}|}{s^2}\,ds < \infty, \tag{7.11}$$

$$A'_c = \int_{-c}^c |a_s|\,ds < \infty, \quad B'_c = \int_{-c}^c |b_s|\,ds < \infty, \tag{7.12}$$

and, further, $f(x)$ has a finite integral in every finite interval, then, for almost all values of s,

$$(7.2)\quad a_s+a_{-s}=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos sx\,dx,\quad b_s-b_{-s}=\frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin sx\,dx. \quad (C\,1)$$

By (7 . 1) we mean

$$\lim_{\nu\to\infty}\int_{-\nu}^{\nu}(a_s\cos sx+b_s\sin sx)\,ds=f(x).$$

This result can also be deduced from Theorem II. For, replacing x by $-x$, we find from (7 . 1)

$$\int_{-\infty}^{\infty}(a_s\cos sx-b_s\sin sx)\,ds=f(-x). \quad (P)$$

Thus, if $\quad 2f_1(x)=f(x)+f(-x),\quad 2f_2(x)=f(x)-f(-x),$

$$\int_{-\infty}^{\infty} a_s\cos sx\,ds=f_1(x),\qquad \int_{-\infty}^{\infty} b_s\sin sx\,ds=f_2(x).$$

Now $\quad f_1(-x)=f_1(x),\quad f_2(-x)=-f_2(x).$

Thus the above become

$$\int_0^{\infty}(a_s+a_{-s})\cos sx\,ds=f_1(x),\quad \int_0^{\infty}(b_s-b_{-s})\sin sx\,ds=f_2(x),$$

which yield, for almost all positive values of s,

$$a_s+a_{-s}=\frac{1}{\pi}\int_{-\infty}^{\infty} f_1(x)\cos sx\,ds, \quad (C\,1)$$

$$b_s-b_{-s}=\frac{1}{\pi}\int_{-\infty}^{\infty} f_2(x)\sin sx\,ds. \quad (C\,1)$$

Further, as is easily verified,

$$0=\frac{1}{\pi}\int_{-\infty}^{\infty} f_2(x)\cos sx\,ds, \quad (C\,1)$$

$$0=\frac{1}{\pi}\int_{-\infty}^{\infty} f_1(x)\sin sx\,ds. \quad (C\,1)$$

These, combined with the above, give (7 . 2).

8. THEOREM IID. *If, for all finite real values of x,*

$$(8.1)\qquad \int_{-\infty}^{\infty} c_s e^{\pm isx}\,ds=f(x) \quad (P)$$

where $$|f(x)| \neq \infty,$$

and, for some positive value of c,

$$T = \int_c^\infty \frac{|c_s| + |c_{-s}|}{s^2}\, ds < \infty, \tag{8.11}$$

$$T_c = \int_{-c}^{c} |c_s|\, ds < \infty, \tag{8.12}$$

and, further, the real and imaginary parts of $f(x)$ *have finite integrals in every finite interval, then, for almost all values of* s,

$$c_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{\mp isx}\, dx. \tag{8.2}$$ (C 1)

Write $$c_s = a_s \mp i b_s,$$

the sign chosen being opposed to that in (8 . 1), and

$$f(x) = \phi(x) \pm i\psi(x).$$

Then (8 . 1) gives

$$\int_{-\infty}^{\infty} (a_s \cos sx + b_s \sin sx)\, ds = \phi(x), \tag{P}$$

$$\int_{-\infty}^{\infty} (-b_s \cos sx + a_s \sin sx)\, ds = \psi(x), \tag{P}$$

from which we obtain, by Theorem IIc, for almost all values of s,

$$a_s + a_{-s} = \frac{1}{\pi} \int_{-\infty}^{\infty} \phi(x) \cos sx\, dx, \quad b_s - b_{-s} = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(x) \sin sx\, dx, \tag{C 1}$$

$$-b_s - b_{-s} = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(x) \cos sx\, dx, \quad a_s - a_{-s} = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(x) \sin sx\, dx, \tag{C 1}$$

which are equivalent to (8 . 2).

9. This last result enables us to obtain conditions for the inversion formula of Mellin.

THEOREM A. *If, for a particular value of* a, *and for all positive values of* x,

$$G(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\omega)\, x^{-\omega}\, d\omega \tag{9.1}$$ (P)

$$|G(x)| < \infty,$$

and, for some positive value of c,

$$F = \int_c^\infty \frac{|F(a+iv)|+|F(a-iv)|}{v^2}\,dv < \infty, \tag{9.11}$$

$$F_c = \int_{-c}^{c} |F(a+iv)|\,dv < \infty, \tag{9.12}$$

and, further, $G(x)$ *has a finite integral in every finite interval consisting of positive values of* x, *then, for almost all values of* ω *whose real part is* a,

$$F(\omega) = \int_0^\infty G(x)\,x^{\omega-1}\,dx. \tag{9.2}$$ (L)

By (9.1) we mean

$$G(x) = \frac{1}{2\pi i}\lim_{\nu\to\infty}\int_{a-i\nu}^{a+i\nu} F(\omega)\,x^{-\omega}\,d\omega,$$

by (9.2)

$$F(\omega) = \lim_{\lambda\to\infty}\frac{1}{\log\lambda}\int_1^\lambda \frac{d\mu}{\mu}\int_{1/\mu}^{\mu} G(x)\,x^{\omega-1}\,dx.$$

Write

$$\omega = u+iv,$$

$$F(a+iv) = f(v),$$

$$x = e^t,$$

$$G(x) = g(t).$$

We find that (9.1) becomes

$$e^{at}g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\,e^{-ivt}\,dv, \tag{P}$$

which yields, by Theorem IIᴅ, for almost all values of v,

$$\begin{aligned} f(v) &= \lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda dv\int_{-\nu}^{\nu} e^{at}g(t)\,e^{ivt}\,dt \\ &= \lim_{\lambda\to\infty}\frac{1}{\lambda}\int_0^\lambda dv\int_{e^{-\nu}}^{e^{\nu}} G(x)\,x^{\omega-1}\,dx \quad (\Re\omega = a) \\ &= \lim_{\lambda\to\infty}\frac{1}{\lambda}\int_1^{e^\lambda}\frac{d\mu}{\mu}\int_{1/\mu}^{\mu} G(x)\,x^{\omega-1}\,dx \\ &= \lim_{\lambda\to\infty}\frac{1}{\log\lambda}\int_1^\lambda\frac{d\mu}{\mu}\int_{1/\mu}^{\mu} G(x)\,x^{\omega-1}\,dx, \end{aligned}$$

which is (9.2).

THEOREM B. *If, for all finite values of* ω *such that* $\Re\omega = a$,

$$F(\omega) = \int_0^\infty G(x)\, x^{\omega-1}\, dx, \tag{9.3}$$ (P)

and, for $0 < \epsilon < 1$,

$$G_1 = \int_{1+\epsilon}^\infty |G(x)|\, \frac{x^{a-1}}{\log^2 x}\, dx < \infty, \tag{9.31}$$

$$G_2 = \int_0^{1-\epsilon} |G(x)|\, \frac{x^{a-1}}{\log^2 x}\, dx < \infty, \tag{9.32}$$

$$G_3 = \int_{1-\epsilon}^{1+\epsilon} |G(x)|\, x^{a-1}\, dx < \infty, \tag{9.33}$$

and, further, the real and imaginary parts of $F(a+iv)$ *have finite integrals in every finite range of* v, *then, for almost all positive values of* x,

$$G(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\omega)\, x^{-\omega}\, d\omega. \tag{9.4}$$ (C 1)

By (9 . 3) is meant

$$F(\omega) = \lim_{\nu\to\infty} \int_{1/\nu}^{\nu} G(x)\, x^{\omega-1}\, dx \quad (\Re\omega = a).$$

With the substitutions previously given,

$$f(v) = \int_{-\infty}^\infty g(t)\, e^{at}\, e^{ivt}\, dt.$$ (P)

The conditions (9 . 31), (9 . 32), (9 . 33) become

$$\int_d^\infty \frac{|g(t)|\, e^{at}}{t^2}\, dt, \quad \int_{-\infty}^{-c} \frac{|g(t)|\, e^{at}}{t^2}\, dt, \quad \int_{-c}^{d} |g(t)|\, e^{at}\, dt < \infty,$$

where $-c = \log(1-\epsilon)$, $d = \log(1+\epsilon)$.

Then, by Theorem IID,

$$g(t)\, e^{at} = \frac{1}{2\pi} \int_{-\infty}^\infty f(v)\, e^{-ivt}\, dt,$$ (C 1)

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^\infty f(v)\, e^{-(a+iv)t}\, dt = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\omega)\, x^{-\omega}\, d\omega,$$ (C 1)

i.e.

$$G(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(\omega)\, x^{-\omega}\, d\omega,$$ (C 1)

which is (9 . 4).

It may be observed that the conditions of Theorem B are all satisfied

if the integral in (9 . 3) converges absolutely, *i.e.* if

$$G = \int_0^\infty |G(x)| x^{\omega-1} dx < \infty .$$

For then $$|F(\omega)| \leqslant \int_0^\infty |G(x) x^{\omega-1}| dx \leqslant G,$$

and $F(\omega)$ is certainly integrable as required.

10. It is possible to rearrange the preceding theory so as to yield conditions for Fourier's integral theorem in its direct form.

THEOREM III. *If*

$$(10.1) \qquad a_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos sx\, dx, \quad b_s = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin sx\, dx, \qquad (P)$$

and, for some positive value of c,

$$(10.11) \qquad \int_c^\infty \frac{|f(x)|+|f(-x)|}{x^2}\, dx < \infty ,$$

$$(10.12) \qquad \int_0^c \{|f(x)|+|f(-x)|\}\, dx < \infty ,$$

and, further, a_s, b_s *have finite integrals in every finite interval, then, for almost all values of* x,

$$(10.2) \qquad f(x) = \int_0^\infty (a_s \cos sx + b_s \sin sx)\, ds. \qquad (C\,1)$$

For (10 . 1) yields

$$a_s = \frac{1}{\pi}\int_0^\infty \{f(x)+f(-x)\}\cos sx\, dx, \quad b_s = \frac{1}{\pi}\int_0^\infty \{f(x)-f(-x)\}\sin sx\, dx,$$

so that, by Theorems IIA and IIB, for almost all values of x,

$$f(x)+f(-x) = 2\int_0^\infty a_s \cos sx\, ds, \quad f(x)-f(-x) = 2\int_0^\infty b_s \sin sx\, ds. \qquad (C\,1)$$

By addition, we find (10 . 2).

In the usual theory we assume

$$(10.3) \qquad \int_{-\infty}^{\infty} |f(x)|\, dx < \infty ,$$

when (10 . 1) follows at once, and also the integrability of a_s, b_s, these being bounded.

Theorem III shows that the integrability of a_s, b_s permits us to

replace (10 . 3) by the less stringent (10 . 11), (10 . 12). Incidentally, Theorem III includes the usual theorem.

11. The conditions (5 . 11), etc., have a form of which the following is typical :

$$\int_c^\infty \frac{|\phi(s)|}{s^2}\,ds < \infty, \tag{11.11}$$

$$\int_0^c |\phi(s)|\,ds < \infty. \tag{11.12}$$

If these hold for any particular positive value of c they also hold for all positive values of c.

The first (11 . 11) holds, in particular, as is easily shown, when any of the following hold :

$$|\phi(s)| \leqslant K; \tag{11.21}$$

$$\phi(s) = O(s^\epsilon), \tag{11.22}$$

as $s \to \infty$, where $\epsilon < 1$;

$$\phi(s) = O\left\{\frac{s}{l_1 s\, l_2 s \ldots (l_n s)^a}\right\} \tag{11.23}$$

as $s \to \infty$, where $a > 1$ and

$$l_1 s = \log s, \quad l_2 s = \log (l_1 s), \quad \ldots, \quad l_n s = \log (l_{n-1} s);$$

$$\int_0^\infty |\phi(s)|\,ds < \infty; \tag{11.24}$$

$$\int_0^\infty |\phi(s)|^p\,ds < \infty, \tag{11.25}$$

where $p > 1$.

12. THEOREM IV. *A finite function $f(x)$ cannot be represented for all values of x by two distinct trigonometrical integrals satisfying the conditions of Theorem II.*

For, if we have

$$\int_0^\infty (a'_s \cos sx + b'_s \sin sx)\,ds = f(x) \neq \pm\infty, \tag{12.1}$$

$$\int_0^\infty (a''_s \cos sx + b''_s \sin sx)\,ds = f(x) \neq \pm\infty, \tag{12.2}$$

then

$$\int_0^\infty \{(a'_s - a''_s)\cos sx + (b'_s - b''_s)\}\,ds = 0.$$

Further, this integral plainly satisfies the conditions of Theorem II, which gives, for almost all positive values of s,

$$a_s'-a_s''=0, \quad b_s'-b_s''=0,$$

so that the integrals in (12 . 1) and (12 . 2) are essentially the same.

13. THEOREM V. *If the additional restriction $a_s,\ b_s \to 0$ is made, then we may admit exceptional points at which the integrals cease to represent $f(x)$. These exceptional points will have no effect on the conclusion of Theorem IV if they form a reducible set.*

Write $$a_s = a_s'-a_s'', \quad \beta_s = b_s'-b_s''.$$

Then $$\int_0^\infty (a_s \cos sx+\beta_s \sin sx)\, ds = 0$$

save possibly at the points of a reducible set R. Also $a_s,\ \beta_s \to 0$ and satisfy the conditions of Theorem II.

Write further

$$\int_0^\infty \left(-\frac{a_s}{s^2}\cos sx-\frac{\beta_s}{s^2}\sin sx\right) ds = F(x). \tag{13 . 1}$$

Then $F(x)$ is continuous, and, by what has been proved in § 5, linear in any interval not containing points of R.

Let us say that $F(x)$ is linear at a point if it is linear in some open interval containing this point.

It is easily shown that the points of non-linearity form a closed set which is contained in $R'+R$ where R' is the derived set of R.

We proceed to show that this closed set contains no isolated points.

By a classical argument, it is sufficient to show that, as $\alpha \to 0$ by positive values,

$$\frac{F(x+\alpha)-2F(x)+F(x-2\alpha)}{2\alpha} \to 0. \tag{13 . 2}$$

For if, to the immediate left of ξ, $F(x)$ has the form $px+q$, while, to the immediate right of ξ, it has the form $rx+s$, we have, since $F(x)$ is continuous at ξ,

$$p\xi+q = r\xi+s. \tag{13 . 3}$$

Also $$\frac{F(\xi+2\alpha)-2F(\xi)+F(\xi-2\alpha)}{2\alpha} = r-p,$$

so that (13 . 2) gives $$p = r,$$

which, substituted in (13 . 3), gives

$$q = s,$$

so that ξ is a point of linearity, there being no break there in the form of $F(x)$.

Now

$$\frac{F(x+2\alpha)-2F(x)+F(x-2\alpha)}{2\alpha} = \int_0^\infty (\alpha_s \cos sx + \beta_s \sin sx)\left(\frac{2\sin^2 s\alpha}{s^2\alpha}\right) ds.$$

If k is finite and positive,

$$\left| \int_0^k (\alpha_s \cos sx + \beta_s \sin sx)\left(\frac{2\sin^2 s\alpha}{s^2\alpha}\right) ds \right|$$

$$\leqslant \int_0^k |\alpha_s \cos sx + \beta_s \sin sx| \cdot \left(\frac{\sin s\alpha}{s\alpha}\right)^2 \cdot 2\alpha\, ds$$

$$\leqslant 2\alpha \int_0^k \{|\alpha_s| + |\beta_s|\}\, ds \to 0.$$

Thus, for (13 . 2), we have only to show

$$\overline{\lim_{\alpha \to 0}} \left| \int_k^\infty (\alpha_s \cos sx + \beta_s \sin sx)\left(\frac{2\sin^2 s\alpha}{s^2\alpha}\right) ds \right| < \epsilon \tag{13 . 4}$$

for some value of k, which may be dependent on ϵ.

Choose k so that

$$|\alpha_s|, \quad |\beta_s| < \frac{\epsilon}{2\pi}.$$

Then

$$\left| \int_k^\infty (\alpha_s \cos sx + \beta_s \sin sx)\left(\frac{2\sin^2 s\alpha}{s^2\alpha}\right) ds \right| \leqslant \frac{2\epsilon}{\pi} \int_k^\infty \frac{\sin^2 s\alpha}{s^2\alpha}\, ds = \frac{2\epsilon}{\pi} \int_{k\alpha}^\infty \left(\frac{\sin s}{s}\right)^2 ds$$

$$< \frac{2\epsilon}{\pi} \int_0^\infty \left(\frac{\sin s}{s}\right)^2 ds = \frac{2\epsilon}{\pi} \cdot \frac{\pi}{2} = \epsilon,$$

which is sufficient for (13 . 4) and so for (13 . 2).

It follows that the points of non-linearity cannot be isolated, and so either do not exist or form a perfect set. The latter is not possible as R is reducible. It follows that the points of non-linearity do not exist and $F(x)$ is linear throughout. This gives by the argument of § 5, for almost all positive values of s,

$$\alpha_s = 0, \quad \beta_s = 0.$$

This is what had to be proved.

EULER'S ϕ-FUNCTION AND ITS CONNEXION WITH MULTIPARTITE NUMBERS

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Introduction.

This short paper is regarded by the author as a small contribution to the theory of multipartite numbers on classical lines. The theory of numbers has involved the ϕ-function to no small extent since the time of Euler. The prominent manner in which the function and its extensions enter the multipartite theory seems to suggest that other parts of the classical theory may prove to be similarly involved as soon as the suitable point of view has been reached.

1. *The generating function which enumerates multipartite numbers.*

In the case of unipartite numbers the function is

$$\prod_1^\infty{}_a (1-x^a)^{-1},$$

where the exponent a ranges over all the unipartite numbers. We write

$$F(x) = \prod_1^\infty{}_a (1-x^a)^{-1}.$$

For bipartite numbers we write

$$F(x, y) = \prod_0^\infty{}_a \prod_0^\infty{}_b (1-x^a y^b)^{-1},$$

wherein the exponents a, b are such that they range over every bipartite number $\overline{ab}$, but are *not* simultaneously zero.

If a, b be relatively prime unipartite numbers, each greater than or equal to 1,

$$F(x^a y^b)$$

is a factor of $F(x, y)$, and we write

$$F(x, y) = F(x)\,F(y)\,\Pi F(x^a y^b).$$

Thus, for a few factors,

$$F(x, y) = F(x)\,F(y)\,F(xy)\,F(x^2y)\,F(xy^2)\,F(x^3y)\,F(xy^3)\ldots$$

Similarly in the case of tripartite numbers

$$F(x, y, z) = F(x)\,F(y)\,F(z)\,\Pi F(x^a y^b)\,\Pi F(x^a z^c)\,\Pi F(y^b z^c)\,\Pi F(x^a y^b z^c),$$

where, in the four infinite products,

$F(x^a y^b)$, a, b are relatively prime and each is equal to or greater than 1,

$F(x^a z^c)$, a, c ,, ,, ,, ,,

$F(y^b z^c)$, b, c ,, ,, ,, ,,

$F(x^a y^b z^c)$, a, b, c ,, ,, ,, ,,

and so on for quadripartite, etc., numbers.

A question that arises in dealing with the generating function in these forms, which involve m letters $x_1, x_2, \ldots, x_m$ is as to the enumeration of the relatively prime sets of numbers

$$a_1,\ a_2,\ \ldots,\ a_m,$$

which are such that $a_1+a_2+\ldots+a_m = n,$

where n is a given integer, and $a_s \geqslant 1$; $s = 1, 2, \ldots, m$. In other words, we require the enumeration of the multipartite numbers

$$\overline{a_1\,a_2\ldots a_m},$$

where the constituents $a_1, a_2, \ldots, a_m$ are relatively prime, the sum

$$a_1+a_2+\ldots+a_m = n,$$

a given integer, and $a_s \geqslant 1$ $(s = 1, 2, \ldots, m)$.

The constituents $a_1, a_2, \ldots$ may be alike or different, and we must consider them in all permutations, since a different multipartite number is in correspondence with each permutation.

In the case of bipartite numbers we find that the enumeration is given by the ϕ-function $\phi(n)$ of Euler.

For consider the particular case $n = 12$; the bipartite numbers which have relatively prime constituents are

$$\overline{11\ 1}$$

$$\overline{75}$$

$$\overline{57}$$

$$\overline{1\ 11}$$

If a, b be relatively prime and $a+b = n$, the relative primeness of a and $n-a$ involves the relative primeness of a and n and also of $n-a$ and n. It follows that the numbers in the first column of the bipartites, as above written, are relatively prime to n $(= 12)$ and that the enumeration of the bipartites is of necessity given by $\phi(12)$. Thence it is obvious that the enumeration is, in general, given by $\phi(n)$.

We have here, in fact, a new definition of $\phi(n)$. That of Euler is "$\phi(n)$ denotes the number of integers not exceeding n which are relatively prime to n".

In this definition, and throughout this paper, two or more unipartite numbers are said to be relatively prime if they contain no common factor greater than 1. The new definition that is suggested is "$\phi(n)$ denotes the number of bipartite numbers

$$\overline{a,\ n-a},$$

which are such that a, $n-a$ are relative primes and both greater than or equal to 1". The merit of the new definition is that the idea may be extended to multipartite numbers in general.

In view of the many generalizations of the ϕ-function* that have been given it is somewhat difficult to decide upon an appropriate notation. The $\phi(n)$ of Euler I write $\mu_1(n)$.

2. *First generalization of the ϕ-function.*

Consider the application of the second definition of § 1 to tripartite numbers.

"$\mu_2(n)$ denotes the number of tripartite numbers

$$\overline{a,\ b,\ n-a-b},$$

which are such that a, b, and $n-a-b$ are relative primes and each greater than or equal to 1".

* L. E. Dickson, *History of the Theory of Numbers*, 1 (1919), 140 *et seq.*

Take the particular case, $n = 6$, and write down the tripartites in numerical order :

41	411
32	321
31	312
23	231
21	213
14	141
13	132
12	123
11	114

and to the left of them the bipartites which arise from deletion of the third constituent of each tripartite.

Since a, b, $n-a-b$ contain no common divisor greater than 1, we see that a, b contain no divisor greater than 1, which is also a divisor of n. Thence the bipartites which have been reached by deletion from the tripartites are such that the highest common divisor of the two constituents is prime to n. Moreover, we obtain thus a complete set of bipartites thus defined; for suppose a bipartite $\overline{\alpha\beta}$ were not included we could derive a tripartite $\overline{\alpha, \beta, n-\alpha-\beta}$ which is not included in the tripartite set, and this is contrary to the definition which gives rise to the tripartites.

Hence we have also the first definition of $\mu_2(n)$.

"$\mu_2(n)$ denotes the number of bipartite numbers

$$\overline{ab}$$

such that $a+b < n$ and the constituents a, b are such that their highest common divisor is a relative prime to n and each of them greater than or equal to 1".

In general the two definitions of $\mu_k(n)$ are as follows :—

(i) "$\mu_k(n)$ denotes the number of k-partite numbers

$$\overline{a_1 a_2 \ldots a_k},$$

such that $a_1+a_2+\ldots+a_k < n$, and the constituents a_1, a_2, ..., a_k such that their highest common divisor is a relative prime to n and each of them greater than or equal to 1".

(ii) "$\mu_k(n)$ denotes the number of $k+1$-partite numbers,

$$\overline{a_1, a_2 \ldots a_k, n-\Sigma a},$$

which are such that a_1, a_2, ... a_k, $n-\Sigma a$ are relatively prime positive integers, each of which is greater than or equal to 1''.

Before discussing the properties of $\mu_k(n)$ it should be remarked that this new function has arisen in a natural manner from a discussion of multipartite numbers. Of other generalizations due to Schemmel*, C. Jordan†, W. E. Story‡ a few words may be in place, as they are concerned with *sets* of integers. Schemmel considered the $\Phi(m)$ sets of n consecutive integers each less than m and relatively prime to m.

C. Jordan, in connexion with the study of linear congruence groups, defined $J_k(n)$ as denoting the number of different sets of k (equal or distinct) positive integers less than or equal to n whose highest common divisor is prime to n. These sets involve one whose content is $nk-1$. The theory has given rise to much work by other mathematicians, but has no connexion with the present paper.

W. E. Story's generalization is concerned with ''sets of k integers $\leqslant n$, not all divisible by any factor of n, such that we do not distinguish between two sets differing only by permutation of their numbers''. This research also, it is clear, has no connexion with that before us.

3. *The properties of* $\mu_k(N)$.

The first property is $\mu_1(N) = \phi(N)$.

The second definition of $\mu_k(N)$ involves $k+1$-partite numbers of content N. These, from another point of view, are partitions of the unipartite number N, the order in which the parts are written being essential. If we discard the condition of relative primeness of parts the whole number of such *ordered* partitions (called by me in another place ''compositions'') is well known to be

$$\binom{N-1}{k} \tag{3.1}$$

If the distinct primes which are contained in N be p_1, p_2, p_3, ... a well known principle thence establishes that

$$\mu_k(N) = \binom{N-1}{k} - \Sigma\binom{(N/p_1)-1}{k} + \Sigma\binom{(N/p_1p_2)-1}{k} - \dots \tag{3.2}$$

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* *Journal für Math.*, 70 (1869), 191–2.

† *Traité des substitutions*, Paris (1870), 95–97.

‡ *Johns Hopkins Univ. Circulars*, 1 (1881), 132, *ib.* 51.

which, for $k = 1$, since the sum of the coefficients in

$$(1-x)^s \quad (s \text{ a positive integer}),$$

is zero, reduces to

$$\phi(N) = N - \Sigma \frac{N}{p_1} + \Sigma \frac{N}{p_1 p_2} - \ldots$$

as it should.

It is very easy to show, from the known properties of binominal coefficients, that

$$(3.3) \qquad \mu_k(N) + \mu_{k-1}(N) = \binom{N}{k} - \Sigma \binom{N/p_1}{k} + \Sigma \binom{N/p_1 p_2}{k} - \ldots,$$

and, in general,

$$(3.33) \qquad \mu_k(N) + \binom{s}{1} \mu_{k-1}(N) + \binom{s}{2} \mu_{k-2}(N) - \ldots$$

$$= \binom{N+s-1}{k} - \Sigma \binom{(N/p_1)+s-1}{k} + \Sigma \binom{(N/p_1 p_2)-1}{k} - \ldots.$$

I write, for brevity, the expression found for $\mu_k(N)$ in the form

$$\mu_k(N) = \left[\binom{N-1}{k}\right].$$

Then, considering $\mu_k(N/d)$ for every divisor d of N,

$$\mu_k(N) + \Sigma \mu_k\left(\frac{N}{p_1}\right) + \Sigma \mu_k\left(\frac{N}{p_1^2}\right) + \ldots$$

$$+ \Sigma \mu_k\left(\frac{N}{p_1 p_2}\right) + \ldots$$

$$= \left[\binom{N-1}{k}\right] + \Sigma \left[\binom{(N/p_1)-1}{k}\right] + \Sigma \left[\binom{(N/p_1^2)-1}{k}\right] + \ldots$$

$$+ \Sigma \left[\binom{(N/p_1 p_2)-1}{k}\right] + \ldots.$$

If we develop the expressions in square brackets, the right-hand side of this identity vanishes, with the exception of

$$\binom{N-1}{k}.$$

So that

$$\Sigma \mu_k\left(\frac{N}{d}\right) = \binom{N-1}{k},$$

where d is any divisor of N except N itself, or

$$\Sigma\mu_k(d) = \binom{N-1}{k} \quad (3.4)$$

[see also *post* the method of (5.3)],

where d is any divisor of N except unity.

This is the analogue of the formula of Gauss.

It will be recalled that Euler* obtained the result $\phi(m)\,\phi(n) = \phi(mn)$, where m and n are relatively prime. In seeking for some analogue in the case of the new functions I have been led to some interesting expressions for $\mu_k(n)$, which involve $\mu_1(n)$ as an *algebraical* factor. From the expression for $\mu_2(N)$ above

$$\mu_2(p_1^{\pi_1}) = \tfrac{1}{2}p_1^{\pi_1-1}(p_1-1)\{p_1^{\pi_1-1}(p_1+1)-3\},$$

or writing $p_1^{\pi_1} = N$,

$$\mu_2(N) = \tfrac{1}{2}\mu_1(N)\left\{\frac{N}{p_1}\,\sigma_1(p_1)-3\right\},$$

where $\sigma_1(m)$ = sum of the divisors of m; and in general, if $N = p_1^{\pi_1}p_1^{\pi_2}\ldots$ and $m = p_1p_2\ldots$, we are led to

$$\mu_2(N) = \tfrac{1}{2}\mu_1(N)\left\{\frac{N}{m}\,\sigma_1(m)-3\right\}.$$

Similarly we find

$$\mu_3(N) = \frac{1}{3!}\,\mu_1(N)\left\{\left(\frac{N}{m}\right)^2\sigma_1(m^2)-6\left(\frac{N}{m}\right)\sigma_1(m)+11\right\},$$

where $1, -6, 11$ are the first three coefficients which occur in the development of $(x-1)\,(x-2)\,(x-3)$.

It is easy to establish the final result

$$\mu_k(N) = \frac{1}{k!}\,\mu_1(N)\left\{\left(\frac{N}{m}\right)^{k-1}\sigma_1(m^{k-1})-a_2\left(\frac{N}{m}\right)^{k-2}\sigma_1(m^{k-2})+\ldots+(-)^{k+1}a_k\right\},$$

where $N = p_1^{\pi_1}p_2^{\pi_2}\ldots$, $m = p_1p_2\ldots$, and

$$1,\quad -a_2,\quad \ldots,\quad (-)^{k+1}a_k$$

are the k coefficients in the development of $(x-1)\,(x-2)\ldots(x-k)$.

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* If M be the highest common divisor of $N_1, N_2, \ldots, N_s$ it is easy to show that

$$\phi(N_1)\,\phi(N_2)\ldots\phi(N_s) = \phi\{(M)\}^{s-1}\,\phi\left(\frac{N_1N_2\ldots N_s}{M^{s-1}}\right).$$

4. *Further extensions of* $\phi(m)$.

Sections 1, 2, and 3 have dealt with extensions in regard to a unipartite number N as put in evidence by the notation $\mu_k(N)$. I now consider

$$\phi(\overline{N_1 N_2 \dots N_s}), \tag{4.1}$$

where the argument is a multipartite number and the ϕ function is defined as enumerating the multipartite numbers which are $< \overline{N_1 N_2 \dots N_s}$ and prime relatively to it.

The multipartite number $\overline{N_1' N_2' \dots N_s'}$ is defined as being less than $\overline{N_1 N_2 \dots N_s}$ if

$$N_r' \leqslant N_r \quad (r = 1, 2, \dots, s)$$

and

$$\sum_1^s N_r' < \sum_1^s N_r.$$

If M be the highest common divisor of $N_1, N_2, \dots, N_s$,

and M' „ „ „ $N_1', N_2', \dots, N_s'$,

the two multipartite numbers are defined as being relatively prime if M, M' have no common divisor > 1. In all cases *zero constituents are excluded* from the multipartites.

Taking first the bipartite theory, and discarding the condition of relative primeness, it will be noticed that the whole number of bipartites which are $\leqslant$ than $\overline{N_1 N_2}$ is $N_1 N_2$.

Let M, the highest common divisor of N_1, N_2, be $p_1^{\pi_1} p_2^{\pi_2} \dots$.

Of the bipartites $\leqslant \overline{N_1 N_2}$, there are $(N_1 N_2)/p_1^2$ which contain the divisor p_1.

Thence an appeal to the well known principle of Möbius gives

$$\phi(\overline{N_1 N_2}) = N_1 N_2 - \Sigma \frac{N_1 N_2}{p_1^2} + \Sigma \frac{N_1 N_2}{p_1^2 p_2^2} - \dots$$

$$= N_1 N_2 \left(1 - \frac{1}{p_1^2}\right)\left(1 - \frac{1}{p_2^2}\right) \dots .$$

By the same argument it is evident that

$$\phi(\overline{N_1 N_2 \dots N_s}) = N_1 N_2 \dots N_s \left(1 - \frac{1}{p_1^s}\right)\left(1 - \frac{1}{p_2^s}\right) \dots, \tag{4.2}$$

where $p_1^{\pi_1} p_1^{\pi_2} \dots$ is the highest common divisor of the unipartites $N_1, N_2, \dots, N_s$.

Let $\overline{L_1 L_2 \dots L_s}$ be another multipartite such that the highest common divisor of $L_1, L_2, \dots L_s$ contains none of the primes $p_1, p_2, \dots$, and put

$$L_r N_r = P_r \quad (r = 1, 2, \dots, s).$$

Then

$$\phi(\overline{L_1 L_2 \dots L_s}) \cdot \phi(\overline{N_1 N_2 \dots N_s}) = \phi(\overline{P_1 P_2 \dots P_s}) \tag{4.3}$$

is an obvious corollary.

Ex. gr.
$$\overline{N_1 N_2} = \overline{82}; \quad \overline{L_1 L_2} = \overline{63},$$
$$\phi(\overline{82}) = 8 \cdot 2 \cdot (1-\tfrac{1}{4}) = 12,$$
$$\phi(\overline{63}) = 6 \cdot 3 \cdot (1-\tfrac{1}{9}) = 16,$$
$$\phi(\overline{48\ 6}) = 48 \cdot 6(1-\tfrac{1}{4})(1-\tfrac{1}{9}) = 192 = 12 \cdot 16.$$

If d denote any common divisor of $N_1, N_2, \dots N_s$,

$$\Sigma\phi\left(\overline{\frac{N_1}{d} \cdot \frac{N_2}{d} \dots \frac{N_s}{d}}\right) = N_1 N_2 \dots N_s.$$

Ex. gr.
$$\phi(\overline{24\ 12}) = 192,$$
$$\phi(\overline{12\ 6}) = 48$$
$$\phi(\overline{84}) = 24$$
$$\phi(\overline{63}) = 16$$
$$\phi(\overline{42}) = 6$$
$$\phi(\overline{21}) = 2$$
$$\Sigma\left(\overline{\frac{24}{d} \cdot \frac{12}{d}}\right) = \overline{288} = 24 \times 12.$$

To establish this the first remark is that the result shows that, if M be the highest common divisor of $N_1, N_2, \dots N_s$ and $N_r = MN'_r$, $r = 1, 2, \dots, s$.

$$\Sigma\phi\left(\overline{\frac{N_1}{d}\,\frac{N_2}{d} \dots \frac{N_s}{d}}\right) = \phi(\overline{N'_1 N'_2 \dots N'_s})\,\Sigma\phi\left(\overline{\frac{M}{d} \cdot \frac{M}{d} \dots s \text{ factors}}\right),$$

because N'_r and M/d are relatively prime for $r = 1, 2, \dots, s$. Now $\phi(N'_1 N'_2 \dots N'_s) = N'_1 N'_2 \dots N'_s$ and $\phi(\overline{M/d \cdot M/d \dots})$ bears the same relation to M^s and d^s that $\phi(M/d)$ does to M and d; so that, since

$$\Sigma\phi\left(\frac{M}{d}\right) = M,$$

we have $$\Sigma\phi\left(\overline{\frac{M}{d}\,\frac{M}{d}\,\ldots\, s \text{ factors}}\right) = M^s.$$

Thence

$$\Sigma\phi\left(\overline{\frac{N_1}{d}\,\frac{N_2}{d}\,\ldots\,\frac{N_s}{d}}\right) = N_1' N_2' \ldots N_s' M^s = N_1 N_2 \ldots N_s, \tag{4.4}$$

as was required to be established.

(See also *post* the method of 5 . 3.)

The function $\phi(\overline{N_1 N_2 \ldots N_s})$ cannot be extended on lines parallel to $\mu_k(N)$ because, as it has been defined, it introduces zero constituents into the multipartites when attempts are made to extend it. We require a new definition in order to proceed in the direction indicated.

4 . 5. I define the multipartite $\overline{N_1' N_2' \ldots N_s'}$ to be s-fold less than $\overline{N_1 N_2 \ldots N_s}$ if

$$N_r' < N_r \quad (r = 1, 2, \ldots, s).$$

and I take $\mu_1(\overline{N_1 N_2 \ldots N_s})$ to enumerate the multipartites which are s-fold $< \overline{N_1 N_2 \ldots N_s}$ and relatively prime to it, two multipartites being relatively prime when the highest common divisors associated with them are relatively prime.

If relative primeness be ignored the enumeration of the s-fold lesser multipartites is given by

$$(N_1-1)(N_2-1)\ldots(N_s-1). \tag{4.51}$$

For the present take bipartites, so that $s = 2$.

Of these lesser bipartites the number which involve the factor p_1 in each constituent, $M = p_1^{\pi_1} p_2^{\pi_2} \ldots$ being the highest common divisor of N_1, N_2, is

$$\left(\frac{N_1}{p_1}-1\right)\left(\frac{N_2}{p_1}-1\right).$$

Thence we are led to the relation

$$\begin{aligned}\mu_1(\overline{N_1 N_2}) &= (N_1-1)(N_2-1) - \Sigma\left(\frac{N_1}{p_1}-1\right)\left(\frac{N_2}{p_1}-1\right) \\ &\qquad + \Sigma\left(\frac{N_1}{p_1 p_2}-1\right)\left(\frac{N_2}{p_1 p_2}-1\right) - \ldots \\ &= N_1 N_2\left(1-\frac{1}{p_1^2}\right)\left(1-\frac{1}{p_2^2}\right) - (N_1+N_2)\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \\ &= \phi(\overline{N_1 N_2}) - \phi(N_1) - \phi(N_2).\end{aligned} \tag{4.52}$$

Similarly we find that

$$\mu_1(\overline{N_1N_2N_3}) = \phi(\overline{N_1N_2N_3}) - \phi(\overline{N_1N_2}) + \phi(N_1) - \phi(\overline{N_1N_3}) + \phi(N_2) - \phi(\overline{N_2N_3}) + \phi(N_3),$$

which we may write

$$\mu_1(\overline{N_1N_2N_3}) = \phi(\overline{N_1N_2N_3}) - \Sigma\phi(\overline{N_1N_2}) + \Sigma\phi(N_1),$$

and in general

(4.53) $$\mu_1(\overline{N_1N_2\ldots N_s}) = \phi(\overline{N_1N_2\ldots N_s}) - \Sigma\phi(\overline{N_1N_2\ldots N_{s-1}}) + \ldots + (-)^{s+1}\Sigma\phi(N_1).$$

From this series we readily deduce the series

(4.54) $$\phi(\overline{N_1N_2\ldots N_s}) = \mu_1(\overline{N_1N_2\ldots N_s}) + \Sigma\mu_1(\overline{N_1N_2\ldots N_{s-1}}) + \ldots + \Sigma\mu_1(N_1).$$

I now return to the bipartite case and take as an example

$$N_1 = 8, \quad N_2 = 4, \quad M = 4.$$

The number of 2-fold lesser and relatively prime bipartites is

$$\mu_1(\overline{84}) = \phi(84) - \phi(8) - \phi(4) = 8 \cdot 4 \cdot (1-\tfrac{1}{4}) - 8(1-\tfrac{1}{2}) - 4(1-\tfrac{1}{2}) = 24 - 6 = 18.$$

4.55. These are set forth in the adjoined left-hand double column and the subtraction of these from $\overline{84}$ in the right-hand double column. If a bipartite to the left be $\overline{ab}$, that to the right is $\overline{N_1-a,\ N_2-b}$. If M contain a factor δ and such a factor be common to a and b it must be also common to N_1-a and N_2-b. Hence the bipartites $\overline{ab}$, $\overline{N_1-a,\ N_2-b}$ are relative primes and are a partition of $\overline{N_1N_2}$ into two parts which are relatively prime.

73	11
72	12
71	13
63	21
61	23
53	31
52	32
51	33
43	41
41	43
33	51
32	52
31	53
23	61
21	63
13	71
12	72
11	73

Hence we have before us a new definition of $\mu_1(\overline{N_1N_2})$ which as in the unipartite case leads to the following two definitions of $\mu_k(\overline{N_1N_2})$.

First definition.

4.6. "$\mu_k(\overline{N_1N_2})$ enumerates sets of k bipartite numbers, free from zero constituents,

$$\overline{N_1^{(1)}N^{(1)}}\ \ \overline{N_1^{(2)}N_1^{(2)}}\ \ \ldots\ \ \overline{N_1^{(k)}N_2^{(k)}},$$

the sum of which is 2-fold less than $\overline{N_1 N_2}$, and which contain no common divisor which is also a divisor of $\overline{N_1 N_2}$; and where account is taken of the permutations in which the bipartites, which make up a given combination, can be written".

Second definition.

4 . 61. "$\mu_k(\overline{N_1 N_2})$ enumerates partitions of $\overline{N_1 N_2}$ into $k+1$ parts which are relatively prime to one another. Account is taken of the different permutations of the bipartite parts of these partitions and zero constituents of the bipartite parts are not permitted".

In general for the s-partite numbers we have

First definition.

4 . 7. "$\mu_k(\overline{N_1 N_2 \ldots N_s})$ enumerates sets of k s-partite numbers, free from zero constituents,

$$\overline{N_1^{(1)} N_2^{(1)} \ldots N_s^{(1)}}\ \overline{N_1^{(2)} N_2^{(2)} \ldots N_s^{(2)}}\ \ldots\ \overline{N_1^{(s)} N_2^{(s)} \ldots N_s^{(s)}},$$

the sum of which is s-fold less than $\overline{N_1 N_2 \ldots N_s}$; and which contain no common divisor which is also a divisor of $\overline{N_1 N_2 \ldots N_s}$; and where account is taken of the permutations in which the s-partites, which make a given combination, can be written".

Second definition.

4 . 71. "$\mu_k(\overline{N_1 N_2 \ldots N_s})$ enumerates partitions of $\overline{N_1 N_2 \ldots N_s}$ into $k+1$ parts which are relatively prime to one another. Account is taken of the different permutations of the s-partite parts of these partitions and zero constituents of the s-partite parts are not permitted".

5. *Properties of* $\mu_k(\overline{N_1 N_2 \ldots N_s})$.

In order to evaluate this arithmetical function we have to enumerate the forms which are specified in one of its two definitions. It is fortunate that the second definition provides a ready means for accomplishing this. It can be established that, if we ignore the condition of relative primeness, the number of *ordered* partitions into $k+1$ parts, which are s-fold less than $\overline{N_1 N_2 \ldots N_s}$ and have no zero constituents, of the multipartite number $\overline{N_1 N_2 \ldots N_s}$ is

$$\binom{N_1-1}{k}\binom{N_2-1}{k}\ldots\binom{N_s-1}{k}.$$

For consider the $k+1$ parts of such a partition

$$A_1B_1C_1\dots S_1,\ A_2B_2C_2\dots S_2,\ \dots,\ A_{k+1}B_{k+1}C_{k+1}\dots S_{k+1}.$$

The constituents $A_1\ A_2\dots A_{k+1}$ constitute an ordered partition of the unipartite number N_1 into $k+1$ non-zero parts. It is one of the commonplaces of combinatory analysis, as in 3.1 above, that the enumeration of these is given by

$$\binom{N_1-1}{k}$$

We have similar enumerations for the constituents $B_1B_2\dots B_{k+1}$, $C_1C_2\dots C_{k+1}$, etc., ..., $S_1S_2\dots S_{k+1}$, and combining them we proceed to the desired enumeration

$$\binom{N_1-1}{k}\binom{N_2-1}{k}\dots\binom{N_s-1}{k}. \tag{5.1}$$

If M, the highest common divisor of $N_1, N_2, \dots, N_s$, be $p_1^{\pi_1}p_2^{\pi_2}\dots$, we proceed, as usual, to the formula

$$\begin{aligned}\mu_k(\overline{N_1N_2\dots N_s}) = {} & \binom{N_1-1}{k}\binom{N_2-1}{k}\dots\binom{N_s-1}{k} \\ & -\Sigma\binom{(N_1/p_1)-1}{k}\binom{(N_2/p_1)-1}{k}\dots\binom{(N_s/p_1)-1}{k} \\ & +\Sigma\binom{(N_1/p_1p_2)-1}{k}\binom{(N_2/p_1p_2)-1}{k}\dots\binom{(N_s/p_1p_2)-1}{k} \\ & -\dots.\end{aligned} \tag{5.2}$$

5.3. It can be shown that

$$\Sigma\mu_k\left(\overline{\frac{N_1N_2\dots N_s}{d\ d\ \dots d}}\right) = \binom{N_1-1}{k}\binom{N_2-1}{k}\dots\binom{N_s-1}{k},$$

where d ranges over every divisor of

$$M = p_1^{\pi_1}p_2^{\pi_2}\dots p_m^{\pi_m}$$

where M is the highest common divisor of $N_1, N_2, \dots, N_s$.

For write the formula for $\mu_k(\overline{N_1N_2\dots N_s})$ in the manner

$$\mu_k(\overline{N_1N_2\dots N_s}) = A_0-\Sigma A_{p_1}+\Sigma A_{p_1p_2}-\Sigma A_{p_1p_2p_3}+\dots+(-)^s\Sigma A_{p_1p_2\dots p_s}$$

and suppose that, in succession, we substitute for N_r $(r=1, 2, 3, \dots)$ N_r/d, where d runs through every divisor of M.

When d is the particular divisor

$$d' = p_1^{\phi_1} p_2^{\phi_2} \dots p_i^{\phi_i}$$

consider the occurrence of the term

$$\binom{(N_1/d')-1}{k}\binom{(N_2/d')-1}{k}\dots\binom{(N_s/d')-1}{k}$$

on the right-hand side of the identities. I say that it will be found,

in A_0, once with the positive sign when N_r/d' is substituted for N_r;

in ΣA_{p_1}, i times with the negative sign when $\frac{p_1}{d'} N_r$ or $\frac{p_2}{d'} N_r$ or ... are substituted for N_r;

in $\Sigma A_{p_1 p_2}$, $\binom{i}{2}$ times with the positive sign when $\frac{p_1 p_2}{d'} N_r$, $\frac{p_1 p_3}{d'} N_r$, ... are substituted for N_r;

... ...

and finally

in A_i, $\binom{i}{i}$ times with the sign $(-)^i$ when $\frac{p_1 p_2 \dots p_i}{d'} N_r$ is substituted for N_r.

The term in question will thus occur on the right-hand sides in the case of $1+\binom{i}{1}+\binom{i}{2}+\dots+\binom{i}{i}$ divisors of M; *i.e.* in the case of 2^i divisors of M; and since, on proceeding to the sum

$$\Sigma \mu_k \left(\frac{\overline{N_1 N_2 \dots N_s}}{d\, d \dots d}\right),$$

the coefficient of the term is

$$1-\binom{i}{1}+\binom{i}{2}-\dots+(-)^i\binom{i}{i} = 0,$$

it is clear that the term in question will not occur at all in such sum.

Thus it is manifest that every term in the sum except

$$\binom{N_1-1}{k}\binom{N_2-1}{k}\dots\binom{N_s-1}{k}$$

will vanish, and we are brought to the result

$$\Sigma\mu_k\left(\frac{N_1N_2\dots N_s}{d\,d\dots d}\right)=\binom{N_1-1}{k}\binom{N_2-1}{k}\dots\binom{N_s-1}{k}. \qquad \text{Q.E.D.}$$

The table of values of $\mu_k(N)$.

Since $$\mu_k(N)=\binom{N-1}{k}-\Sigma\binom{(N/p_1)-1}{k}+\dots$$

$p_1, p_2, \dots$ being the prime divisors of N, we see that

$$\mu_k(N)=\binom{N-1}{k} \text{ when } N \text{ is prime for all values of } k.$$

Also, if p_1 be the least prime divisor of N,

$$\mu_k(N)=\binom{N-1}{k} \quad \text{when} \quad k>\frac{N}{p_1}-1.$$

These values, being simple binomial coefficients, are not tabulated.

The value of $\mu_k(N)$ when less than or equal to N/p_1-1 is found at the intersection of the horizontal through N and the vertical through μ_k.

Thus $\mu_4(25)$ is tabulated because $4=\frac{25}{5}-1$, k in this case not exceeding $\frac{25}{5}-1$.

The results of § 3 have been freely used to verify the numbers entered.

Values of $\mu_k(N)$, when $k\leqslant(N/P_1)-1$, P_1 being the least prime divisor of N.

N	$\mu_1(N)$	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}
4	2										
6	2	9									
8	4	18	34								
9	6	27									
10	4	30	80	125							
12	4	42	154	325	461						
14	6	63	266	700	1281	1715					
15	8	84	360	1000							
16	8	84	420	1330	2982	4998	6434				
18	6	99	614	2305	6131	12348	19440	24039			
20	8	132	884	3750	11502	27048	50352	75573	92377		
21	12	174	1120	4830	15498	38759					
22	10	165	1210	5775	20097	54054	116160	203445	293920	352715	
24	8	180	1572	8490	33166	100478	244826	490149	817135	1144055	1352077
25	20	270	2020	10625							

ZUM WARINGSCHEN PROBLEM

Von E. LANDAU.

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Einleitung.

Über den major arc brauchen die Herren Hardy und Littlewood in ihrer 6. Abhandlung "Some problems of 'Partitio Numerorum'", *Math. Zeitschrift*, 23 (1925), 1–37, lediglich ihre Formel (4.11), die sie rasch aus folgendem *Satz* ableiten:

Es sei $k > 2$ *ganz,* $a = (1/k)$, $0 < \delta < a$, $a_1 = a - \delta$, $n > 0$ *ganz,* q *ganz,* $0 < q \leqq n^{a_1}$, $Y = q^k\{(1/n) + \vartheta i\}$, $-(2\pi/qn^{1-a_1}) < \vartheta < (2\pi/qn^{1-a_1})$, $0 \leqq t < 1$,

$$g(Y) = \sum_{m=0}^{\infty} e^{-Y(m+t)^k} = g(Y;\ k,\ t).$$

Dann ist

$$\left| g(Y) - \Gamma(1+a)\, Y^{-a} - \sum_{\beta=0}^{c_3} \sum_{\gamma=0}^{c_4} c_{\beta\gamma\delta}\, Y^{\beta} t^{\gamma} \right| < \frac{c_\delta}{n}.$$

Hierbei bezeichne ich mit c Zahlen, die nur von k und ihren Indices abhängen.

Den Hardy-Littlewoodschen Beweis dieses Satzes werde ich durch einen einfacheren ersetzen.

1. *Hilfssatz.*

Es sei $k > 2$ *ganz,* $a = (1/k)$, $0 \leqq t < 1$, $l > 0$ *ganz,* $u \geqslant 0$,

$$\Phi(u) = \frac{1}{l!} \sum_{0 \leqq m \leqq u^a - t} \left(u - (m+t)^k\right)^l = \Phi(u;\ k,\ t,\ l).$$

Dann ist

$$\left| \Phi(u) - \frac{\Gamma(1+a)}{\Gamma(l+1+a)}\, u^{l+a} - \sum_{\lambda=0}^{l} \sum_{\gamma=0}^{c_l} c_{\lambda\gamma l}\, u^{\lambda} t^{\gamma} \right| < c_l (u^{(1-a)l} + 1). \tag{1}$$

Beweis.—Ohne Beschränkung der Allgemeinheit sei $u > t^k$; denn bei dem hier zu ermittelnden Polynom $\sum_{\lambda,\gamma} c_{\lambda\gamma l}\, u^\lambda t^\gamma$ ist für $0 \leqq u \leqq t^k$ wegen $\Phi(u) = 0$ gewiss die linke Seite von (1) kleiner als c_l.

Es seien $\psi_0(z)$, $\psi_1(z)$, ... die Funktionen der Periode 1, welche nebst den Konstanten c_λ successive durch

$$\psi_0(z) = z - [z] - \tfrac{1}{2}, \quad \psi_{\lambda+1}(z) = \int_0^z \psi_\lambda(u)\, du + c_\lambda,$$

$$\int_0^1 \psi_{\lambda+1}(z)\, dz = 0 \quad (\lambda \geqslant 0)$$

definiert sind {so dass $|\psi_{\lambda+1}(z)| < c_\lambda$}. Dann ergibt sich durch partielle Integration,

$$F(z) = \left(u - (z+t)^k\right)^l \quad \text{für} \quad 0 \leqq z \leqq u^a - t$$

gesetzt,

$$l!\ \Phi(u) = \sum_{0 \leqq m \leqq u^a - t} F(m) = \int_0^{u^a - t} F(z)\, dz + \tfrac{1}{2} F(0) + \int_0^{u^a - t} \psi_0(z)\, F'(z)\, dz$$

$$(2) \quad = \int_0^{u^a - t} F(z)\, dz + \tfrac{1}{2} F(0) - \sum_{\rho=1}^{kl} (-1)^\rho \left(\psi_\rho(u^a - t)\, F^{(\rho)}(u^a - t) - \psi_\rho(0)\, F^{(\rho)}(0) \right).$$

Hierin ist für $0 \leqq \rho \leqq kl$

$$(3) \qquad F^{(\rho)}(0) = \sum_{\lambda=0}^{l} \sum_{\gamma=0}^{c} c_{\lambda\gamma l\rho}\, u^\lambda t^\gamma,$$

ferner, da $F(z)$ die l fache Wurzel $z = u^a - t$ hat,

$$(4) \qquad F^{(\rho)}(u^a - t) = 0 \quad \text{für} \quad 0 \leqq \rho < l;$$

für $l \leqq \rho \leqq kl$ ist, weil $F^{(\rho)}(z)$ in u^a, z ein Polynom des Grades $kl - \rho \leqq (k-1)\, l$ ist,

$$(5) \qquad |\psi_\rho(u^a - t)\, F^{(\rho)}(u^a - t)| < c_l (u^{(1-a)l} + 1).$$

Endlich ist

$$\int_0^{u^a - t} F(z)\, dz = \int_0^{u^a} (u - z^k)^l\, dz - \int_0^t (u - z^k)^l\, dz$$

$$(6) \qquad = u^{l+a}\, a \int_0^1 (1-v)^l v^{a-1}\, dv + \sum_{\lambda=0}^{l} \sum_{\gamma=0}^{c_l} c_{\lambda\gamma l}\, u^\lambda t^\gamma$$

und hierin

$$(7) \quad a \int_0^1 (1-v)^l v^{a-1}\, dv = a\mathrm{B}(l+1, a) = a\, \frac{\Gamma(l+1)\,\Gamma(a)}{\Gamma(l+1+a)} = l!\, \frac{\Gamma(a+1)}{\Gamma(l+1+a)}.$$

Aus (2) bis (7) folgt die Behauptung (1).

2. *Beweis des Satzes.*

$$g(Y) = Y^{l+1}\int_0^\infty \Phi(u)\, e^{-uY}\, du$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(l+1+\alpha)}\; Y^{l+1}\int_0^\infty u^{l+\alpha} e^{-uY}\, du + \sum_{\lambda=0}^{l}\sum_{\gamma=0}^{c_l} c_{\lambda\gamma}\, Y^{l+1}\, t^\gamma \int_0^\infty u^\lambda e^{-uY}\, du + R$$

$$= \Gamma(1+\alpha)\, Y^{-\alpha} + \sum_{\lambda=0}^{l}\sum_{\gamma=0}^{c_l} c_{\lambda\gamma l}\, Y^{l-\lambda}\, t^\gamma + R,$$

wo, wegen

$$\frac{n}{q^k} \geqslant 1, \quad |Y| < q^k \frac{c}{q n^{1-\alpha_1}},$$

$$|R| < c_l |Y|^{l+1}\left(\int_0^\infty u^{(1-\alpha)l} e^{-u(q^k/n)}\, du + \int_0^1 du\right) = c_l |Y|^{l+1}\left\{c_l\left(\frac{n}{q^k}\right)^{(1-\alpha)l+1} + 1\right\}$$

$$< c_l \frac{q^{k(l+1)}}{q^{l+1} n^{(1-\alpha_1)(l+1)}} \frac{n^{(1-\alpha)l+1}}{q^{k((1-\alpha)l+1)}} = c_l \frac{1}{q}\, n^{\alpha_1 - l\delta} \leqslant c_l\, n^{1-l\delta} < \frac{c_\delta}{n}$$

bei passendem $l(\delta)$.

Göttingen, den 30. Juni 1925.

THE ANALYSIS OF AN EMPIRIC FUNCTION INTO ITS QUASI-PERIODIC CONSTITUENTS

By H. LEVY.

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1. The methods of periodogram analysis suffice in general to derive the periodic constituents of which a set of observed data is composed, or to extract from a mass of data a hidden periodic term. In few natural phenomena, however, are the observations of a purely periodic nature, for, in virtue of absorption of energy in some form or other, oscillations must almost invariably be of a decaying nature. It becomes a matter of some importance, therefore, to devise a practical method of analysing any given mass of data composed of oscillating factors of a quasi-periodic nature into its constituent elements. In effect, this implies that the given observational data $y(x)$ are to be represented in the form

$$y(x) = e^{\lambda_1 x}\phi_1(x) + e^{\lambda_2 x}\phi_2(x) + \ldots + e^{\lambda_n x}\phi_n(x),$$

where λ_r is a number and $\phi_r(x)$ is a purely periodic function of x, not necessarily continuous itself, or its derivatives. In the present paper it is proposed to deal with the case in which the periods of all the functions $\phi_r(t)$ are equal, and known*.

2. In the field of pure mathematics the problem presents itself from a different angle. Take, for example, Hill's equation†,

$$\frac{d^2y}{dx^2} + (a_0 + a_1 \cos 2x + a_2 \cos 4x + \ldots)\, y = 0, \qquad (1)$$

which arises from a study of departures from the periodic orbit due to the action of certain disturbing forces, x being the time variable, and a_0, a_1, a_2, ... numerical coefficients. Floquet‡ has shown that the

* The method may be regarded as an extension of Prony's interpolation by exponentials. Whittaker and Robinson's *Calculus of observations*, 369.

† *Acta Math.*, 8 (1886), 1–36.

‡ *Annales de l'Ecole Normale Supérieure* (2), 12 (1883), 47–88.

general solution of (1) is of the form

$$y = C_1 e^{\lambda x}\phi_1(x) + C_2 e^{-\lambda x}\phi_2(x) \tag{2}$$

where C_1 and C_2 are arbitrary constants, λ is a constant depending on the coefficients a_0, a_1, ..., and $\phi_1(x)$ and $\phi_2(x)$ are periodic functions of x of period equal to that of the coefficient of y in (1).

For the determination of λ in (2), Hill introduced into analysis the infinite determinant. Now (2) is a very simple case of the representation of the function $y(x)$ in terms of two quasi-periodic constituents of the type already considered, and therefore if *any* particular solution of (1) can be obtained in numerical form, an analysis of the data so derived should provide the two constituent elements in (2), *i.e.* the value of λ and the form of $\phi_1(x)$ and $\phi_2(x)$ as purely periodic functions. The latter could then be represented as a Fourier series, and accordingly the *general* solution of (1) would be completely determined.

A numerical solution of (1) can, of course, easily be derived over any range to any required degree of accuracy. An application will be made in the sequel to Mathieu's equation, a particular case of (1).

3. Let a be the period of the function $\phi(x)$, and let E represent the operation which transforms $f(x)$ into $f(x+a)$, then

$$(E - e^{\lambda a})\, e^{\lambda x}\phi(x) = 0. \tag{3}$$

Similarly

$$(E - e^{\mu a}).e^{\lambda x}\phi(x) = (e^{\lambda a} - e^{\mu a})\, e^{\lambda x}\phi(x).$$

Let it be assumed that a given function $y(x)$ may be completely represented in the form

$$y(x) = \sum_{r=1}^{n} e^{\lambda_r x}\phi_r(x), \tag{4}$$

where λ_r is real and may be zero, and $\phi_r(x)$ is a periodic function, not necessarily continuous, of period a. Operating on both sides of (4) with $(E - e^{\lambda_1 a})(E - e^{\lambda_2 a})\ldots(E - e^{\lambda_n a})$ we obtain a difference equation which must be satisfied by $y(x)$, viz. :

$$\prod_{r=1}^{n} (E - e^{\lambda_r a})\, y(x) = 0, \tag{5}$$

or

$$[E^n + A_1 E^{n-1} + \ldots + A_n]\, y(x) = 0, \tag{6}$$

i.e. $$y(x+na)+A_1y(x+\overline{n-1}a)+\ldots+A_ny(x)=0, \qquad (7)$$

where $$A_1=-\sum_{r=1}^{n}e^{\lambda_r a}; \quad A_2=\sum_{r,s=1;\,r\neq s}^{r,s=n}e^{(\lambda_r+\lambda_s)a}. \qquad (8)$$

4. *Determination of n.*

It follows that if $y(x)$ satisfies a relation of this type for a range of values of x, then for that range $y(x)$ must be capable of being represented in terms of quasi-periodic terms of type (4). A direct test of data by means of this criterion cannot, of course, be applied, since the coefficients A_r are functions of the unknown constants λ_r, but an equivalent test may be made which simultaneously provides us with information regarding the number of terms of the type $e^{\lambda x}\phi(x)$ that requires to be taken in the analysis. Let $y(x)$ be known for a series of values of $x=a_1$, a_2, ..., a_r, ..., where r is any integer presumed, for the moment, greater than (or equal to) $n+1$, then from (7), on inserting these values, a system of linear equation in $A_1 \ldots A_n$ is derived of the type

$$y(a_r+na)+A_1y(a_r+\overline{n-1}a)+\ldots+A_ny(a_r)=0, \qquad (9)$$

$$r=1, 2, \ldots, n+1, \ldots.$$

The elimination of A_1, A_2, ..., A_n from any block of $n+1$ of these will involve that determinants of the type

$$|\,y(a_r+na),\quad y(a_s+\overline{n-1}a),\ \ldots\ y(a_\omega)\,|=0 \qquad (10)$$

must vanish. It will be possible to construct this determinant if $y(x)$ is known over n periods of the constituent functions $\phi_r(x)$ and at $n+1$ values of x in the $(n+1)$-th period. It follows therefore that, by constructing determinants of the above type from the data, for successive values of n, a value of the latter will at last be found such that this determinant and all higher order determinants of this type will vanish. This will then determine n, the number of terms in (4) which has to be assumed in order to represent the data completely.

The next step is to proceed to the determination of the numbers λ.

5. *Determination of the λ's.*

Returning to equations (5) and (6), it is evident that $e^{\lambda_1 a}, e^{\lambda_2 a}, \ldots, e^{\lambda_n a}$ are the n roots of the equation

$$z^n+A_1z^{n-1}+\ldots+A_n=0. \qquad (11)$$

Associating this with n equations of the set (9), or with n equations derived from them by linear combinations of systems of them if the

data are sufficiently ample to enable this to be done, we may eliminate $A_1, A_2, \ldots, A_n$ and thus obtain an algebraic equation in z of the nth degree whose roots are the n quantities $e^{\lambda_1 a}, e^{\lambda_2 a}, \ldots, e^{\lambda_n a}$. The coefficients in the equation are obtained from the data provided for analysis. Thus, for example :

$$\begin{vmatrix} z^n & z^{n-1} & \ldots & z & 1 \\ {}_nY_1 & {}_{n-1}Y_1 & \ldots & {}_1Y_1 & {}_0Y_1 \\ {}_nY_2 & {}_{n-1}Y_2 & \ldots & {}_1Y_2 & {}_0Y_2 \\ \ldots & \ldots & \ldots & \ldots & \\ {}_nY_u & {}_{n-1}Y_u & \ldots & {}_1Y_u & {}_0Y_u \end{vmatrix} = 0, \tag{12}$$

where the quantities ${}_{n-r}Y_s$ are given by

$${}_{n-r}Y_s = Py(a_p + \overline{n-r}a) + Qy(a_q + \overline{n-r}a) + \ldots,$$

$P, Q, \ldots$, being arbitrary constants.

This provides the numbers λ, and the method so far adopted is closely analogous to that used in the Lagrange method of periodogram analysis. The advantage in taking linear combinations of the members of the set of equations (9) instead of n individual equations chosen at random lies, of course, in the fact that all the data provided is thus utilized in the determination of the λ's and not merely that over a portion of the range.

6. *Determination of* $\phi_r(x)$.

Operate on equation (4) with $\prod_{r=1}^{n} (E - e^{\lambda_r a})$, $(r \neq \sigma)$, where one of the factors in the previous operator is absent. Then every term on the right-hand side of (4) will vanish with the exception of $e^{\lambda_\sigma x}\phi_\sigma(x)$ which will merely be multiplied by $\prod_{r=1}^{n} (e^{\lambda_\sigma a} - e^{\lambda_r a})$, $r \neq \sigma$. The left-hand side, viz. $\prod_{r=1}^{n} (E - e^{\lambda_r a}).y(x)$, $r \neq \sigma$, will on expansion be expressible in terms of $y(x)$, $y(x+a)$, $\ldots$, $y(x+\overline{n-1}a)$. Thus

$$e^{\lambda_\sigma x}\phi_\sigma(x) = \frac{y(x+\overline{n-1}a) + \beta_1 y(x+\overline{n-2}a) \ldots + \beta_{n-1}y(x)}{\prod_{\substack{r=1 \\ r \neq \sigma}}^{n} (e^{\lambda_\sigma a} - e^{\lambda_r a})}. \tag{13}$$

The numbers which appear on the right of this equation are either data initially provided or numbers such as λ_r which have already been calculated, while $e^{\lambda_\sigma x}$ is also known. Thus this equation determines the required periodic functions $\phi(x)$. It follows that, if the data represent-

ing a function composed of a system of n quasi-periodic terms be known over a range of $(n+1)a$ for the independent variable x where a is the known period, the data may be extrapolated to cover any desired range of x. It may be remarked that, corresponding to a root $z=1$ of equation (12), there will be present among the so-called quasi-periodic terms a purely periodic one.

7. Let the expression for $y(x)$ be of the form

$$y(x) = \sum_{r=1}^{n} e^{\lambda_r x}\phi_r(x) + F(x), \tag{14}$$

where $F(x)$ is a polynomial in x of degree m.

Let $$\Delta f(x) = f(x+a) - f(x) = (E-1)f(x),$$

then $\Delta y(x)$, $\Delta^2 y(x)$, ..., $\Delta^{m+1} y(x)$ may be derived at once from the given data by simple differencing at intervals of one period.

Now $$\Delta e^{\lambda x}\phi(x) = (E-1)\, e^{\lambda x}\phi(x) = (e^{\lambda a}-1)\, e^{\lambda x}\phi(x),$$

so that $$\Delta^{m+1} e^{\lambda x}\phi(x) = (e^{\lambda a}-1)^{m+1} e^{\lambda x}\phi(x)$$

and $$\Delta^{m+1} F(x) = 0.$$

It follows that

$$Y(x) \equiv \Delta^{m+1} y(x) = \sum_{r=1}^{n} (e^{\lambda_r a}-1)^{m+1} e^{\lambda_r x}\phi_r(x), \tag{15}$$

and if the data have been sufficiently accurate to allow of this differencing it will now be possible to analyse the derived data $Y(x)$ in the manner already explained. This will determine each function $(e^{\lambda_r a}-1)^{m+1} e^{\lambda_r x}\phi_r(x)$ and λ_r and consequently the quasi-periodic constituents in the original $y(x)$ will have been determined. If $F(x)$ is required it may then be derived directly from (14). It should be remarked, however, that, if m be large those terms for which λ is negative will become exceeding small in the expression for $Y(x)$ in (15), in comparison with those for which λ is positive. The utility of the method in this respect will depend consequently almost entirely on the accuracy of the original data.

If $y(x)$ ought to be represented purely in quasi-periodic terms, but if it be known that a *systematic* error is present, then, if we may assume that the systematic error may be expressed as a polynomial of the form $F(x)$ above, a differencing of the terms before commencing the analysis will assist in clearing out the error. It is important to note, however, that, if a purely periodic term of period a be present, this process will eliminate it also. The latter would require therefore to be obtained initially by the ordinary processes of periodogram analysis.

8. For simplicity an application will be made in the present paper to the case of Mathieu's equation, a particular case of (1), viz. :

$$\frac{d^2y}{dx^2}+(a+b\cos 2x)\,y=0. \tag{16}$$

From the general theory of linear differential equations with periodic coefficients it is known that any solution of (16) is of the form

$$y=e^{\lambda x}\phi_1(x)+e^{-\lambda x}\phi_2(x),$$

where λ is a constant, and ϕ_1 and ϕ_2 are periodic functions. The determination of λ and of the forms of ϕ_1 and ϕ_2 involves intricate and cumbersome analysis if the methods hitherto adopted are applied. In the particular case where λ is zero, one of the solutions of (16) becomes periodic, the other being of a quasi-periodic nature. This occurs for particular relations between a and b, in which case the purely periodic solution is known*. It will be seen that by the process of analysis adopted in the present paper these special considerations need not be entered into. The particular case of a purely periodic solution would be dealt with in the ordinary treatment.

Example : $$\frac{d^2y}{dx^2}+(1{\cdot}3158+0{\cdot}800\cos 2x)\,y=0. \tag{17}$$

Any numerical solution to this equation is first derived by any convenient standard method. To avoid tabulating unnecessary data the following selected values are given, at intervals of x equal to $\pi/6$ from $x=0$ to $x=6\pi$ to five significant figures :—

x	y	x	y	x	y
0	+0·0080				
$\pi/6$	+1·0449	$13\pi/6$	+0·9947	$25\pi/6$	+1·0337
$\pi/3$	+1·6486	$7\pi/3$	+1·9391	$13\pi/3$	+3·0078
$\pi/2$	+1·8553	$5\pi/2$	+2·4168	$9\pi/2$	+3·9480
$2\pi/3$	+1·7642	$8\pi/3$	+2·5093	$14\pi/3$	+4·2612
$5\pi/6$	+1·2214	$17\pi/6$	+1·9617	$29\pi/6$	+3·4892
π	+0·1644	3π	+0·5752	5π	+1·2168
$7\pi/6$	−0·9723	$19\pi/6$	−1·1146	$31\pi/6$	−1·7042
$4\pi/3$	−1·7101	$10\pi/3$	−2·3579	$16\pi/3$	−3·9519
$3\pi/2$	−2·0365	$7\pi/2$	−3·0341	$11\pi/2$	−5·2492
$5\pi/3$	−2·0369	$11\pi/3$	−3·2270	$17\pi/3$	−5·7120
$11\pi/6$	−1·5175	$23\pi/6$	−2·5983	$35\pi/6$	−4·7222
2π	−0·3524	4π	−0·8542	6π	−1·6987

* Whittaker, *Proc. Internat. Cong. of Math.*, 1 (1912), 366–371.

Equation (12) to determine $z_1 = e^{2\pi\lambda}$ and $z_2 = e^{-2\pi\lambda}$ is immediately written down :

$$\begin{vmatrix} z^2 & z & 1 \\ -0{\cdot}8542 & -0{\cdot}3524 & 0{\cdot}0080 \\ -1{\cdot}6987 & -0{\cdot}8542 & -0{\cdot}3524 \end{vmatrix} = 0,$$

giving $$z_1 = 1{\cdot}8650 \quad \text{and} \quad z_2 = 0{\cdot}5363,$$

and we note that $z_1 z_2 = 1{\cdot}0002$.

Thus $\lambda = 0{\cdot}0992$. Had one of the solutions of this equation been $z = 1$ there would have been present a purely periodic function in the solution of the differential equation. The other root of this equation in z would have determined the exponent of the quasi-periodic term, and the subsequent computation deriving the forms of the functions ϕ_1 and ϕ_2 would not have been affected. It may be remarked that, in the present case, since it is known that the roots of the equation in z are $e^{2\pi\lambda}$ and $e^{-2\pi\lambda}$, a simpler equation for the determination of λ can be found which does not involve obtaining a numerical solution of the differential equation over a greater range than two periods, viz. $0—4\pi$. For

$$\begin{aligned} y(0) &= \phi_1(0) + \phi_2(0), \\ y(2\pi) &= e^{2\pi\lambda}\phi_1(0) + e^{-2\pi\lambda}\phi_2(0), \\ y(4\pi) &= e^{4\pi\lambda}\phi_1(0) + e^{-4\pi\lambda}\phi_2(0), \end{aligned}$$

and accordingly

$$\begin{vmatrix} y(0) & 1 & 1 \\ y(2\pi) & e^{2\pi\lambda} & e^{-2\pi\lambda} \\ y(4\pi) & e^{4\pi\lambda} & e^{-4\pi\lambda} \end{vmatrix} = 0.$$

This gives

$$\begin{aligned} \cosh 2\pi\lambda &= [y(0)+y(4\pi)]/2y(2\pi) \\ &= 1{\cdot}2008, \end{aligned}$$

i.e. $\lambda = 0{\cdot}0992$ as before.

This method will not apply, of course, if one of the solutions be purely periodic.

Proceeding now to the determination of ϕ_1 and ϕ_2, equations (13) become

$$e^{\lambda x}\phi_1(x) = \frac{y(x+2\pi) - e^{-2\pi\lambda}y(x)}{e^{2\pi\lambda} - e^{-2\pi\lambda}},$$

$$e^{-\lambda x}\phi_2(x) = \frac{y(x)\,e^{2\pi\lambda} - y(x+2\pi)}{e^{2\pi\lambda} - e^{-2\pi\lambda}},$$

The calculation of $\phi_1(x)$ and $\phi_2(x)$ then proceeds as below for a complete period :—

x	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
$e^{\lambda x}\phi_1(x)$	−0·2684	+0·3269	+0·7941	+1·0700	+1·1765	+0·9834	+0·3665
$\phi_1(x)$	−0·2684	+0·3104	+0·7157	+0·9157	+0·9557	+0·7585	+0·2684
$e^{-\lambda x}\phi_2(x)$	+0·2764	+0·7180	+0·8545	+0·7853	+0·5877	+0·2380	−0·2021
$\phi_2(x)$	+0·2764	+0·7563	+0·9481	+0·9177	+0·7235	+0·3086	−0·2764

x		$7\pi/6$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
$e^{\lambda x}\phi_1(x)$		−0·4464	−1·0844	−1·4615	−1·6065	−1·3432	−0·5006
$\phi_2(x)$		−0·3104	−0·7157	−0·9157	−0·9557	−0·7585	−0·2684
$e^{-\lambda x}\phi_2(x)$		−0·5259	−0·6257	−0·5750	−0·4304	−0·1743	+0·1483
$\phi_2(x)$		−0·7563	−0·9481	−0·9177	−0·7235	−0·3086	+0·2764

These results, when represented in periodic form, determine $\phi_1(x)$ and $\phi_2(x)$ as follows :

$$\begin{aligned}\phi_1(x) = \quad & 0.9659 \sin x - 0{\cdot}2588 \cos x \\ & + 0{\cdot}0511 \sin 3x - 0{\cdot}0095 \cos 3x \\ & + 0{\cdot}0009 \sin 5x - 0{\cdot}0001 \cos 5x \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\end{aligned}$$

and

$$\begin{aligned}\phi_2(x) = \quad & 0{\cdot}9659 \sin x + 0{\cdot}2588 \cos x \\ & + 0{\cdot}0491 \sin 3x + 0{\cdot}0173 \cos 3x \\ & + 0{\cdot}0009 \sin 5x + 0{\cdot}0003 \cos 5x \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\end{aligned}$$

Thus the solution of Mathieu's equation (17) is completely represented for *any* range of values of x.

ON GENERALIZED ALMOST PERIODIC FUNCTIONS

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H. BOHR, in his papers in the *Acta Mathematica**, has constructed the theory of almost periodic functions. He has developed his theory for uniformly continuous functions only, and has not considered the problem of defining the widest class of functions to which his methods and results are applicable. This problem was considered by W. Stepanoff†. He did not confine the discussion to continuous and bounded functions, but, by changing H. Bohr's definition in a very natural way, he generalized the theory to a greater class of functions.

In the present paper I consider the question, whether the Riesz-Fischer theorem is true for this class of functions. I prove that the theorem does not hold for the Stepanoff class. This is shown by means of a series to which no function of this class corresponds. I then proceed to show how this class may be enlarged, so that the Riesz-Fischer theorem may hold for it. Finally, I consider the question whether in the class of almost periodic functions there exists an analogue to Cantor's theorem.

1. To construct the above-mentioned example, let us consider the series

$$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x} = G_1+G_2+G_3+\ldots, \tag{1}$$

* *Acta Math.*, 45 (1924), 29–127; 46 (1925), 101–214; 47 (1926), 237–281.

† *Comptes rendus*, 181 (1925), 90-92.

where G_1 is the sum of some number of the first terms of the series, G_2 is the sum of some number of the following terms, and so on. All coefficients a_n are positive; all coefficients of G_i for each i are equal, their sum is equal to s_i, and the sum of their squares is equal to l_i. We may evidently assume, if we suppose the number of terms in G_i sufficiently large, that s_i is as large and l_i as small as we please. We shall define the values of s_i and l_i so that the series

$$\sum_{i=1}^{\infty} l_i = \sum_{n=1}^{\infty} a_n^2$$

converges, and the series Σs_i diverges.

Let us introduce the notation

$$H_i = G_1 + G_2 + \ldots G_i,$$

$$L_i = l_{i+1} + l_{i+2} + \ldots,$$

$$S_i = s_1 + s_2 + \ldots + s_i.$$

The coefficients a_n will evidently be completely defined when the values of l_i are determined. Denoting by a_i and n_i the common value of the coefficients of G_i and the number of its terms respectively, we have $n_i a_i = s_i$ and $n_i a_i^2 = l_i$; thus we may give to l_i an arbitrary value under the single condition that the quotient $s_i^2/l_i = n_i$ should be an integer. For the exponents λ_n we take an arbitrary fixed sequence of numbers, all belonging to the interval $\{-(2\pi/3), +(2\pi/3)\}$ so that $|H_i(x)| \geqslant \frac{1}{2}S_i$ when x lies in the interval $(-\frac{1}{2}, +\frac{1}{2})$.

All the H_i are almost periodic functions. Let $\tau_i(1) = \tau_i$ be a translation-number of H_i corresponding to $\epsilon = 1$; then we have, for all values of x,

$$|H_i(x+\tau_i) - H_i(x)| \leqslant 1.$$

Let us denote by d_i a number such that every interval of the length d_i contains at least one interval of the form $(\tau_i - \frac{1}{2}, \tau_i + \frac{1}{2})$, in which the inequality

$$|H_i(x)| \geqslant \tfrac{1}{2}S_i - 1 \tag{2}$$

is true. Let us define now in order the functions G_1, G_2, G_3, ... Giving to l_1 an arbitrary possible value, we shall determine the functions $G_1(x)$ and $H_1(x) = G_1(x)$ and the number d_1. Giving to l_2 an arbitrary pos-

sible value satisfying also the inequality

$$l_2 < \frac{(\frac{1}{4}S_1)^2}{2d_1} \tag{3}$$

we shall determine the functions G_2 and $H_2 = G_1 + G_2$ and the number d_2. To l_3 we give an arbitrary possible value satisfying the inequalities

$$l_3 < \frac{(\frac{1}{4}S_1)^2}{2^2 d_1} \quad \text{and} \quad l_3 < \frac{(\frac{1}{4}S_2)^2}{2d_2}. \tag{4}$$

We shall define the functions G_3, H_3, and the number d_3; further we define l_4 subject to the conditions

$$l_4 < \frac{(\frac{1}{4}S_1)^2}{2^3 d_1}, \quad l_4 < \frac{(\frac{1}{4}S_2)^2}{2^2 d_2}, \quad l_4 < \frac{(\frac{1}{4}S_3)^2}{2d_3}, \tag{5}$$

and so on. In this way we shall define completely the series (1). On account of (3), (4), (5), ... we shall have

$$L_1 = l_2 + l_3 + {}_4 + \ldots < \frac{(\frac{1}{4}S_1)^2}{d_1},$$

$$L_2 = l_3 + l_4 + \ldots < \frac{(\frac{1}{4}S_2)^2}{d_2},$$

and generally

$$L_i < \frac{(\frac{1}{4}S_i)^2}{d_i}. \tag{6}$$

Let us now suppose that there is either in the class of Bohr almost periodic functions or in the class (III) of Stepanoff almost periodic functions a function $f(x)$ which has (1) as a Fourier series and for which, therefore, Parseval's theorem holds. We know that the integral

$$\int_a^{a+1} |f(x)|\,dx$$

is a bounded function of a. Then for some i we shall have

$$\int_a^{a+1} |f(x)|\,dx < \tfrac{1}{4}S_i - 1. \tag{7}$$

Let us consider now the integral

$$\int_a^{a+d_i} |f(x) - H_i(x)|^2 dx,$$

and choose an interval $(\tau_i-\frac{1}{2},\ \tau_i+\frac{1}{2})$ included in the interval $(a,\ a+d_i)$. Using Schwarz's inequality we obtain

$$\int_{\tau_i-\frac{1}{2}}^{\tau_i+\frac{1}{2}} |f(x)-H_i(x)|^2\,dx > \left[\int_{\tau_i-\frac{1}{2}}^{\tau_i+\frac{1}{2}} |f(x)-H_i(x)|\,dx\right]^2$$

$$\geqslant \left[\int_{\tau_i-\frac{1}{2}}^{\tau_i+\frac{1}{2}} |H_i(x)|\,dx-\int_{\tau_i-\frac{1}{2}}^{\tau_i+\frac{1}{2}} |f(x)|\,dx\right]^2$$

whence, from (2) and (7),

$$\int_{\tau_i-\frac{1}{2}}^{\tau_i+\frac{1}{2}} |f(x)-H_i(x)|^2 > (\tfrac{1}{4}S_i)^2,$$

so that *a fortiori*

$$\int_a^{a+d_i} |f(x)-H_i(x)|^2\,dx > (\tfrac{1}{4}S_i)^2,$$

and therefore

$$M\{|f(x)-H_i(x)|^2\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} |f(x)-H_i(x)|^2\,dx > \frac{(\frac{1}{4}S_i)^2}{d_i}. \tag{8}$$

On the other hand, by Parseval's theorem we have

$$M\{|f(x)-H_i(x)|^2\} = L_i, \tag{9}$$

but plainly from (6) the inequality (8) and the inequality (9) are contradictory. Thus there exists no almost periodic function with (1) as its Fourier series.

2. Let $f(x)$ be a positive function summable in any finite interval. We denote by $\bar{M}\{f(x)\}$ the number

$$\limsup_{T\to\infty} \frac{\int_0^T f(x)\,dx}{T}.$$

We call a complex function $f(x)$ of a real variable x *a generalized almost periodic function** if there exists a set of almost periodic Bohr's functions

$$s_1(x),\ s_2(x),\ \ldots,\ s_n(x),\ \ldots,$$

such that $$\lim_{n\to\infty} \bar{M}\{|f(x)-s_n(x)|^2\} = 0.$$

* We consider in what follows functions defined only for $0 < x < \infty$. (Of course we could treat in the same way the case of the interval $-\infty < x < \infty$.)

We show first that

$$M\{|f(x)|^2\} \quad \text{and} \quad M\{|f(x)-s_n(x)|^2\}$$

exist, and that consequently in our definition of generalized almost periodic functions we may replace $\bar{M}$ by M. It is sufficient, of course, to prove the existence of the first expression, because the function $f(x)-s_n(x)$ is related to the set of functions

$$s_1(x)-s_n(x), \quad s_2(x)-s_n(x), \quad \ldots, \quad s_m(x)-s_n(x), \quad \ldots$$

in the same way as the function $f(x)$ is to the set of functions

$$s_1(x), \quad s_2(x), \quad \ldots, \quad s_m(x), \quad \ldots.$$

By our definition there exist two sets of positive numbers

$$\epsilon_1, \quad \epsilon_2, \quad \ldots, \quad \epsilon_n, \quad \ldots \quad (\epsilon_n \to 0),$$

and

$$T_1, \quad T_2, \quad \ldots, \quad T_n, \quad \ldots,$$

such that for every n we have the inequality

$$\int_0^T |f(x)-s_n(x)|^2 dx < \epsilon_n^2 T \quad \text{for} \quad T \geqslant T_n,$$

or, denoting $f(x)-s_n(x)$ by $h_n(x)$, the inequality

$$\int_0^T |h_n(x)|^2 dx < \epsilon_n^2 T \quad \text{for} \quad T \geqslant T_n.$$

We have

$$\int_0^T |f(x)|^2 dx = \int_0^T |s_n(x)|^2 dx + \theta\left(2\int_0^T |s_n(x)|\,|h_n(x)|\,dx + \int_0^T |h_n(x)|^2 dx\right)$$

$$(-1 \leqslant \theta \leqslant +1).$$

From the inequality of Schwarz we deduce

$$\left\{\frac{1}{T}\int_0^T |s_n(x)|\,|h_n(x)|\,dx\right\}^2 \leqslant \frac{1}{T}\int_0^T |s_n(x)|^2 dx \cdot \frac{1}{T}\int_0^T |h_n(x)|^2 dx,$$

and therefore, if $T > T_n$,

$$\phi(T) = \frac{1}{T}\int_0^T |f(x)|^2 dx$$

$$= \frac{1}{T}\int_0^T |s_n(x)|^2 dx + \theta_1\left\{2\epsilon_n\sqrt{\left(\frac{1}{T}\int_0^T |s_n(x)|^2 dx\right)} + \epsilon_n\right\}$$

$$(-1 \leqslant \theta \leqslant +1).$$

Denoting now $M\{|s_n|^2\}$ by m_n^2, we have for an arbitrary fixed ϵ and a sufficiently great value of T,

$$\phi(T) = m_n^2 + \theta_1(2m_n\epsilon_n + \epsilon_n^2) + \epsilon.$$

This shows that the set of numbers

$$m_1, \ m_2, \ \ldots, \ m_n, \ \ldots$$

is bounded. Therefore there exists a positive number a, and a set of positive numbers

$$T_1', \ T_2', \ \ldots, \ T_n', \ \ldots,$$

such that for every n and $T > T_n'$, we have

$$\phi(T) = m_n^2 + \tfrac{1}{2}\theta_2 a\epsilon_n \quad (-1 \leqslant \theta_2 \leqslant +1).$$

Hence, for any $T' > T_n'$ and $T'' > T_n'$, we have

$$|\phi(T') - \phi(T'')| \leqslant a\epsilon_n.$$

Thus there exists

$$M\{|f(x)|^2\} = \lim_{T\to\infty} \phi(T)$$

and

$$M\{|f(x)|^2\} = \lim_{n\to\infty} m_n^2. \tag{1}$$

Let us now remark that, from

$$\frac{1}{T}\int_0^T |h(x)|^2 dx < \epsilon^2,$$

it follows by Schwarz's inequality that

$$\frac{1}{T}\int_0^T |h(x)|\, dx < \epsilon.$$

Hence, for $T > T_n$, we have

$$\left|\frac{1}{T}\int_0^T [f(x) - s_n(x)]\, e^{-i\lambda x} dx\right| \leqslant \frac{1}{T}\int_0^T |f(x) - s_n(x)|\, dx < \epsilon_n,$$

so that $$M\{f(x)\,e^{-i\lambda x}\}$$

exists and is equal to $$\lim_{n\to\infty} M\{s_n(x)\,e^{-i\lambda x}\}.$$

Let $$\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_m,\ \ldots$$

be all values of λ, for which the inequality

$$M\{s_n(x)\,e^{-i\lambda x}\} \neq 0$$

is satisfied for at least one value of n. Evidently the inequality

$$M\{f(x)\,e^{-i\lambda x}\} \neq 0$$

can hold only for these values of λ. Writing

$$M\{s_n(x)\,e^{-i\lambda_m x}\} = a_m^{(n)}, \quad M\{f(x)\,e^{-i\lambda_m x}\} = a_m,$$

we have on account of (1)

$$M\{|f(x)|^2\} = \lim_{n\to\infty} M\{|s_n(x)|^2\} = \lim \sum_{m=1}^{\infty} |a_m^{(n)}|^2 \qquad (2)$$

and (by Bessel's inequality, which evidently holds also for generalized almost periodic functions)

$$\sum_{m=1}^{\infty} |a_m - a_m^{(n)}|^2 \leqslant M\{|f(x)-s_n(x)|^2\} < \epsilon_n^2;$$

and, denoting $a_m - a_m^{(n)}$ by $h_m^{(n)}$,

$$\sum_{m=1}^{\infty} |a_m|^2 = \sum_{m=1}^{\infty} |a_m^{(n)}|^2 + \theta\left\{2\sum_{m=1}^{\infty} |a_m^{(n)}|\,|h_m^{(n)}| + \sum_{m=1}^{\infty} |h_m^{(n)}|^2\right\}$$

$$(-1 \leqslant \theta \leqslant +1).$$

But, from the inequality of Schwarz,

$$\sum_{m=1}^{\infty} |a_m^{(n)}|\,|h_m^{(n)}| \sqrt{\left(\sum_{m=1}^{\infty} |a_m^{(n)}|^2 \sum_{m=1}^{\infty} |h_m^{(n)}|^2\right)} \leqslant m_n \epsilon_n,$$

so that $$\sum_{m=1}^{\infty} |a_m|^2 = \sum_{m=1}^{\infty} |a_m^{(n)}|^2 + \theta_1(2m_n\epsilon_n + \epsilon_n^2) \quad (-1 \leqslant \theta_1 \leqslant +1),$$

whence $$\lim_{n\to\infty} \sum_{m=1}^{\infty} |a_m^{(n)}|^2 = \Sigma |a_m|^2$$

and, from (2), $$M\{|f(x)|^2\} = \sum_{m=1}^{\infty} |a_m|^2.$$

Thus *for generalized almost periodic functions Parseval's theorem is true.*

3. We now proceed to show that, to every series

$$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}, \tag{1}$$

for which the series $\sum_{n=1}^{\infty} |a_n|^2$ is convergent (while the exponents are quite arbitrary real numbers), there corresponds a generalized almost periodic function. Writing

$$s_j = \sum_{n=1}^{j} a_n e^{i\lambda_n x}, \quad l_j = \sum_{n=1}^{j} |a_n|^2,$$

there exists, corresponding to an arbitrarily chosen sequence,

$$\epsilon_1 > \epsilon_2 > \ldots \quad (\epsilon_n \to 0),$$

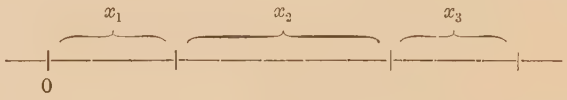
a set of positive numbers (>1)

$$T_1, \ T_2, \ \ldots, \ T_n, \ \ldots$$

such that for every j and $k = 0, 1, 2, \ldots, j-1$, and for $T \geqslant T_j$ and for all x, we have

$$\left| \frac{1}{T} \int_x^{x+T} |s_j - s_k|^2 dx - (l_j - l_k) \right| < \epsilon_j \quad (s_0 = l_0 = 0). \tag{2}$$

On the real axis we construct segments $x_1, x_2, x_3, \ldots$ in the following way.



Since

$$\frac{1}{x} \int_0^x |s_1|^2 dx \to l_1 \quad (x \to +\infty)$$

and

$$\int_x^{x+\xi} |s_2|^2 dx \leqslant \int_x^{x+T_2} |s_2|^2 dx < T_2(l_2 + \epsilon_2) \quad \text{for} \quad \xi \leqslant T_2,$$

we can choose $x_1 \geqslant T_1$, so that for $\xi \leqslant T_2$ the inequality

$$\left| \frac{\int_0^{x_1} |s_1|^2 dx + \int_{x_1}^{x_1+\xi} |s_2|^2 dx}{x_1 + \xi} - l_1 \right| < \epsilon_1 \tag{3}$$

is true.

For $\xi \geqslant T_2$ we have

$$\left| \frac{\int_0^{x_1} |s_1|^2 dx + \int_{x_1}^{x_1+\xi} |s_2|^2 dx}{x_1+\xi} - \frac{x_1 l_1 + \xi l_2}{x_1+\xi} \right| < \epsilon_1. \tag{4}$$

Since

$$\frac{1}{x} \int_{x_1}^{x_1+x} |s_2 - s_k|^2 \to l_2 - l_k \quad (k = 0, 1)$$
$$x \to +\infty$$

and

$$\int_x^{x+\xi} |s_3 - s_k|^2 dx < T_3(l_3 - l_k + \epsilon_3)$$

for $\xi \leqslant T_3$, we can choose $x_2 \geqslant T_2$, so that for $\xi \leqslant T_3$ (and $k = 0, 1$) the inequality

$$\left| \frac{\int_0^{x_1} |s_1 - s_k|^2 dx + \int_{x_1}^{x_1+x_2} |s_2 - s_k|^2 dx + \int_{x_1+x_2}^{x_1+x_2+\xi} |s_3 - s_k|^2 dx}{x_1+x_2+\xi} - (l_2 - l_k) \right| < \epsilon_2. \tag{5}$$

is true. For $\xi > T_3$ we have [by using (5) for $\xi = 0$ and (2) for $j = 3$]

$$\left| \frac{\int_0^{x_1} |s_1 - s_k|^2 dx + \int_{x_1}^{x_1+x_2} |s_2 - s_k|^2 dx + \int_{x_1+x_2}^{x_1+x_2+\xi} |s_3 - s_k|^2 dx}{x_1+x_2+\xi} - \frac{(x_1+x_2)(l_2-l_k) + \xi(l_3-l_k)}{x_1+x_2+\xi} \right| < \epsilon_2. \tag{6}$$

In the same way, since for $\xi \leqslant T_4$,

$$\int_x^{x+\xi} |s_4 - s_k|^2 dx < T_4(l_4 - l_k + \epsilon_4) \quad (k = 0, 1, 2),$$

and

$$\frac{1}{x} \int_{x_1+x_2}^{x_1+x_2+x} |s_3 - s_k| dx \to l_3 - l_k,$$
$$x \to +\infty$$

we can choose $x_3 \geqslant T_3$, so that for $\xi \leqslant T_4$ the inequality

$$\left| \frac{\int_0^{x_1} |s_1 - s_k|^2 dx + \int_{x_1}^{x_1+x_2} |s_2 - s_k|^2 dx + \int_{x_1+x_2}^{x_1+x_2+x_3} |s_3 - s_k|^2 dx + \int_{x_1+x_2+x_3}^{x_1+x_2+x_3+\xi} |s_4 - s_k|^2 dx}{x_1+x_2+x_3+\xi} - (l_3 - l_k) \right| < \epsilon_3 \tag{7}$$

is true. For $\xi > T_4$ we have

$$\left| \frac{\int_0^{x_1} |s_1 - s_k|^2 dx + \int_{x_1}^{x_1+x_2} |s_2 - s_k|^2 dx + \int_{x_1+x_2}^{x_1+x_2+x_3} |s_3 - s_k|^2 dx + \int_{x_1+x_2+x_3}^{x_1+x_2+x_3+\xi} |s_4 - s_k|^2 dx}{x_1+x_2+x_3+\xi} - \frac{(x_1+x_2+x_3)(l_3-l_k)+\xi(l_4-l_k)}{x_1+x_2+x_3+\xi} \right| < \epsilon_3. \quad (8)$$

and so on.

We define now the function $f(x)$ in the following way:

$$f(x) = s_i(x) \quad \text{for} \quad x_1+x_2+\ldots+x_{i-1} \leqslant x < x_1+x_2+\ldots+x_i.$$

We shall show that

$$\frac{1}{x} \int_0^x |f(x)-s_k(x)|^2 dx \underset{x \to +\infty}{\to} l-l_k, \quad \text{where} \quad l = |a_1|^2+|a_2|^2+\ldots.$$

For $k = 0$ we have to study the integral

$$\frac{1}{x} \int_0^x |f(x)|^2 dx.$$

Its value for $T_1 < x \leqslant x_1$ is included by (2) in the interval $(l-\epsilon_1,\ l_1+\epsilon_1)$:

for $x_1 < x \leqslant x_1+T_2$ by (3) in the same interval $(l_1-\epsilon_1,\ l_1+\epsilon_1)$;

for $x_1+T_2 < x \leqslant x_1+x_2$ by (4) in the interval $(l_1-\epsilon_1,\ l_2+\epsilon_1)$;

for $x_1+x_2 < x \leqslant x_1+x_2+T_3$ by (5) in the interval $(l_2-\epsilon_2,\ l_2+\epsilon_2)$;

for $x_1+x_2+T_3 < x \leqslant x_1+x_2+x_3$ by (6) in the interval $(l_2-\epsilon_2,\ l_3+\epsilon_2)$;

and so on. Since

$$l_j \to l \quad \text{and} \quad \epsilon_j \to 0 \quad \text{for} \quad j \to +\infty,$$

we have

$$\frac{1}{x} \int_0^x |f(x)|^2 dx \underset{x \to +\infty}{\to} l.$$

In the same way we prove that

$$\frac{1}{x} \int_0^x |f(x)-s_k(x)|^2 dx \underset{x \to +\infty}{\to} l-l_k.$$

These equalities show that $f(x)$ is a generalized almost periodic function, and that its Fourier series is

$$\Sigma a_n e^{i\lambda_n x}.$$

Let $h(x)$ be a function satisfying the condition

$$\frac{1}{x}\int_0^x |h(x)|^2 dx \underset{x\to\infty}{\to} 0.$$

Then for an arbitrary generalized almost periodic function $f(x)$, the function $f(x)+h(x)$ will, as it is easy to see, also be a generalized almost periodic function with the same Fourier series.

Conversely, if two generalized almost periodic functions $f_1(x)$ and $f_2(x)$ have the same Fourier series

$$\Sigma a_n e^{i\lambda_n x}$$

with convergent $\Sigma |a_n|^2$, then

$$\frac{1}{x}\int_0^x |f_1(x)-f_2(x)|^2 dx \underset{x\to+\infty}{\to} 0.$$

For we have

$$\frac{1}{x}\int_0^x |f_1(x)-f_2(x)|^2 dx \leqslant \frac{2}{x}\int_0^x |f_1(x)-s_n(x)|^2 dx + \frac{2}{x}\int_0^x |f_2(x)-s_n(x)|^2 dx.$$

The expression on the right-hand side of this inequality converges to $4(l-l_n)$, when x tends to infinity; hence it immediately follows (since $l_n \to l$) that

$$\frac{1}{x}\int_0^x |f_1(x)-f_2(x)|^2 dx \underset{x\to+\infty}{\to} 0. \tag{9}$$

Thus, if we do not consider functions $f_1(x)$ and $f_2(x)$ which satisfy the condition (9) as different, we have this theorem.

To every generalized almost periodic function $f(x)$ corresponds a Fourier series, and to every series

$$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

with convergent $\Sigma |a_n|^2$ corresponds one generalized almost periodic function $f(x)$.

4. In the same way we may prove that the series

$$F(z) = \sum_{j=1}^{\infty} \frac{z}{z-b_j} a_j e^{i\lambda_j z} \quad (z = \sigma + it)$$

for rapidly increasing real values of b_j represents, when $\sigma = 0$, a generalized almost periodic function of t with Fourier series

$$\sum_{j-1}^{\infty} a_j e^{i\lambda_j t}.$$

For some values of b_j the series representing $F(z)$ is convergent in the whole plane, except at the points b_j, and represents an analytic function with poles at the points b_j and regular at all other points.

Moreover, if there is a strip $\alpha < \sigma < \beta$, where the series

$$\Sigma |a_j|^2 e^{2\lambda_j \sigma}$$

is convergent, then it is possible to choose b_j so that the function $F(z)$ is regular in this strip and represents a generalized almost periodic function in the strip with

$$\Sigma a_j e^{\lambda_j z}$$

as its Fourier series.

5. Let us consider now another question, whether, in the case of almost periodic functions, there exists an analogue to du Bois-Reymond's theorem. According to this theorem, if the series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

converges towards the bounded function $f(x)$, then it is the Fourier series of this function.

Let the series $$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}$$

converge to some almost periodic function. Is then the series the Fourier series of the function? This question we answer in the negative, showing that every function satisfying some general conditions is representable in infinitely many ways by series of the form

$$\sum_{n=1}^{\infty} a_n e^{i\lambda_n x}.$$

This is the idea of our proof.

We take the sequence of intervals $d_1, d_2, \ldots, d_n, \ldots$ such that every one of them includes the preceding one and that $d_n \to (-\infty, +\infty)$; we take also the sequence of increasing positive numbers $p_1, p_2, \ldots, p_n, \ldots$, such that $p_n > d_n$ (we denote by d_n also the length of the interval d_n for all values of n).

We construct now the function $f_1(x)$ with the period p_1, equal to the given function $F(x)$ in the interval d_1, the function $f_2(x)$ with the period p_2 equal to $F(x)-f_1(x)$ in the interval d_2, the function $f_3(x)$ with the period p_3 equal to $F(x)-f_1(x)-f_2(x)$ in the interval d_3, and so on. Thus we shall have $F(x)$ represented in the whole interval

$$-\infty < x < +\infty$$

by the series of periodic functions:

$$F(x) = f_1(x)+f_2(x)+f_3(x)+\ldots.$$

Expanding every function $f_n(x)$ in a trigonometric series we shall get also representation of $F(x)$ by a trigonometric series. The actual construction is somewhat more complicated, but the idea is essentially the same.

Let $F(x)$ be a bounded continuous function defined in the whole interval $-\infty < x < +\infty$, with bounded variation in every finite interval. Let $|F(x)| \leqslant 1$. We shall show that the function $F(x)$ may be represented by a trigonometric series (which is, of course, a series of the form $\Sigma a_n e^{i\lambda_n x}$). The proof will show that this representation may be effected in infinitely many ways. The demonstration is also applicable to the case where only

$$\lim_{x\to\infty} \frac{F(x)}{x} = 0.$$

Let

$$(0, 2);\ (1, 3);\ (2, 4);\ (3, 5);\ \ldots;\ (-1, 1);\ (-2, 0);\ (-3, -1);\ \ldots$$

be the intervals, to which correspond the functions

$$f_1(x),\ f_3(x),\ f_5(x),\ f_7(x),\ \ldots,\ f_2(x),\ f_4(x),\ f_6(x),\ \ldots,$$

defined in the following way:

Let $l[x, (a, a+1), \phi]$ denote the linear function in the interval $(a, a+1)$ equal to $\phi(a)$ for $x = a$ and to 0 for $x = a+1$. We define the

periodic function $f_1(x)$ with the period $2p_1 > 2$ in the following manner :

$$\begin{aligned} f_1(x) &= F(x) - l[x, (0, 1), F] && \text{for} \quad 0 \leqslant x \leqslant 1, \\ &= l[x, (1, 2), F] && \text{for} \quad 1 \leqslant x \leqslant 2, \\ &= 0 && \text{for} \quad 2 \leqslant x \leqslant 2p_1. \end{aligned}$$

Let $\phi(x)$ be an even positive function greater than 2, monotonic in the interval $(0, +\infty)$, and such that

$$\phi(x) \to +\infty \quad \text{and} \quad \frac{\phi(x)}{x} \to 0 \quad \text{when} \quad x \to +\infty . \tag{1}$$

We have $|F(x) < \frac{1}{2}\phi(x)$ and $|f_1(x)| \leqslant 2 < \phi(x)$.

We write $F(x) - f_1(x) = F_1(x)$.

We suppose $2p_1$ so large that

$$|F_1(x)| < \tfrac{1}{2}\phi(x) \text{ outside the interval } (0, 2).$$

In the same way we define the periodic function $f_2(x)$ with the period $2p_2 > 4$:

$$\begin{aligned} f_2(x) &= F_1(x) - l[x, (-1, 0), F_1] && \text{for} \quad -1 \leqslant x \leqslant 0, \\ &= l[x, (0, 1), F_1] && \text{for} \quad 0 \leqslant x \leqslant 1, \\ &= 0 && \text{for} \quad 1 \leqslant x \leqslant -1 + 2p_2. \end{aligned}$$

Writing $F_1(x) - f_2(x) = F_2(x)$,

we have $|f_2(x)| < \phi(x)$.

Now suppose $2p_2$ so large that

$$|F_2(x)| < \tfrac{1}{2}\phi(x) \text{ outside the interval } (-1, 2).$$

We define the periodic function $f_3(x)$ with the period $2p_3 > 6$ by the equations

$$\begin{aligned} f_3(x) &= F_2(x) - l[x, (1, 2), F_2] && \text{for} \quad 1 \leqslant x \leqslant 2, \\ &= l[x, (2, 3), F_2] && \text{for} \quad 2 \leqslant x \leqslant 3, \\ &= 0 && \text{for} \quad 3 \leqslant x \leqslant 1 + 2p_3. \end{aligned}$$

Writing $$F_2(x)-f_3(x) = F_3(x),$$

we have $$|f_3(x)| < \phi(x),$$

and we may suppose $2p_3$ so large that

$$|F_3(x)| < \tfrac{1}{2}\phi(x) \text{ outside the interval } (-1, 3),$$

and so on.

Thus, we obtain

$$F(x) = f_1(x)+f_2(x)+\ldots+f_n(x)+F_n(x)$$

where $$F_{2k}(x) = 0 \quad \text{for} \quad -k+2 \leqslant x \leqslant k+1,$$

$$F_{2k+1}(x) = 0 \quad \text{for} \quad -k+1 \leqslant x \leqslant k+1,$$

so that $$F(x) = f_1(x)+f_2(x)+f_3(x)+\ldots. \tag{2}$$

All the functions $f_1(x)$, $f_2(x)$, ... are continuous and of bounded variation in every finite interval. We suppose that p_i/p_j is irrational for all $i \neq j$.

Let us develop every function $f_1(x)$, $f_2(x)$, ... as a Fourier series. We denote by $S^{(i)}(x, 0, n)$ the sum of the first $2n+1$ terms of the series corresponding to the function $f_i(x)$, and by $S^{(i)}(x, n, n_1)$ the sum of the following $2(n_1-n)$ terms. Then

$$S^{(i)}(x, 0, n) = \frac{1}{p_i}\int_0^{p_i/2} \frac{\sin(2n+1)(\pi/p_i)\,t}{\sin(\pi/p_i)\,t}\,[f_i(x+2t)+f_i(x-2t)]\,dt. \tag{3}$$

Every function $f_i(x)$ is different from zero only in an interval of length 2 in every period, and there is always such an interval included in the interval $(-i, +i)$, $(i > 1)$, and therefore we have

$$|f_i(x)| < \phi(i).$$

Denoting by d the minimum distance from the point x of the interval $(-p_i, +p_i)$ to the interval where $f(x)$ is different from zero, we have from (3) for any n

$$|S^i(x, 0, n)| < \frac{1}{p_i}\int_0^{p_i/2} \frac{1}{\sin(\pi d/2p_i)}\,|f_i(x+2t)+f_i(x-2t)|\,dt < \frac{\phi(i)}{d}. \tag{4}$$

We now take a set of numbers

$$\epsilon_1 > \epsilon_2 > \epsilon_3 > \ldots \quad (\epsilon_n \to 0) \tag{5}$$

and construct the series

$$\begin{aligned} & S^{(1)}(x, 0, n_1^{(1)}) \\ & + S^{(1)}(x, n_1^{(1)}, n_2^{(1)}) + S^{(2)}(x, 0, n_2^{(2)}) \\ & + S^{(1)}(x, n_2^{(1)}, n_3^{(1)}) + S^{(2)}(x, n_2^{(2)}, n_3^{(2)}) + S^{(3)}(x, 0, n_3^{(3)}) \\ & + S^{(1)}(x, n_3^{(1)}, n_4^{(1)}) + S^{(2)}(x, n_3^{(2)}, n_4^{(2)}) + S^{(3)}(x, n_3^{(3)}, n_4^{(3)}) + S^{(4)}(x, 0, n_4^{(4)}) \\ & + \ldots \qquad \ldots \end{aligned} \tag{6}$$

where the numbers $n_i^{(1)}, n_i^{(2)}, \ldots, n_i^{(i)}$ are defined so that

$$|S^{(j)}(x, 0, n) - f_j(x)| < \frac{\epsilon_i}{i} \tag{7}$$

for $n \geqslant n_i^{(j)}$, $j = 1, 2, 3, \ldots, i$, and for all values of x. This is possible, because all the functions $f_k(x)$ are representable by uniformly convergent Fourier series. If in the series (6) we consider every line as a single term, the series will converge to the function $F(x)$. For the sum of the first i lines is, as it is easy to see, equal to

$$S^{(1)}(x, 0, n_i^{(1)}) + S^{(2)}(x, 0, n_i^{(2)}) + \ldots + S^{(i)}(x, 0, n_i^{(i)}),$$

and by (7) this sum is equal to

$$f_1(x) + f_2(x) + \ldots + f_i(x) + \theta_i \epsilon_i \quad (|\theta_i| < 1). \tag{8}$$

This proves our statement, since the sum

$$f_1(x) + f_2(x) + \ldots + f_i(x)$$

converges to the function $F(x)$ and $\epsilon_i \to 0$.

We shall now show that, if we consider every term of the sums $S^{(k)}(x, \ldots, \ldots)$ as a term of the series (6), then the series (6) also will be convergent [to the same function $F(x)$, of course]. Let x be any fixed number and ϵ an arbitrarily small positive number. We define an integer i_0 such that

$$i_0 > 6|x| + 12, \quad \epsilon_{i_0 - 1} < \frac{\epsilon}{2}, \quad \frac{3\phi(i)}{i} < \frac{\epsilon}{2} \quad \text{for} \quad i \geqslant i_0. \tag{9}$$

Let us now consider any sum σ of terms of the series (6) and let i be the number of the line, to which the last term of the sum σ belongs. We assume that the number of terms of the sum σ is so great that $i \geqslant i_0$. There are two cases to consider :

(1°) the last term of σ belongs to the sum $S^{(j)}(x, n_{i-1}^{(j)}, n_i^{(j)})$ $(j < i)$,

(2°) the last term of σ belongs to the sum $S^{(i)}(x, 0, n_i^{(i)})$.

In the first case we have

$$\sigma = S^{(1)}(x, 0, n_i^{(1)}) + S^{(2)}(x, 0, n_i^{(2)}) + \ldots + S^{(j)}(x, 0, n) + S^{(j+1)}(x, 0, n_{i-1}^{(j+1)}) + \ldots + S^{(i-1)}(x, 0, n_{i-1}^{(i-1)}), \qquad (10)$$

where $n_{i-1}^{(j)} < n \leqslant n_i^{(j)}$.

It may happen that the second line of (10) contains only one term or no term.

In the same way as before we see that

$$\sigma = f_1(x) + f_2(x) + \ldots + f_{i-1}(x) + \theta_{i-1}\epsilon_{i-1} \quad (|\theta_{i-1}| < 1),$$

and from (9)
$$\sigma = f_1(x) + f_2(x) + \ldots + f_{i-1}(x) + \theta\epsilon \quad (|\theta| < 1). \qquad (11)$$

In the second case we have

$$\sigma = S^{(1)}(x, 0, n_i^{(1)}) + S^{(2)}(x, 0, n_i^{(2)}) + \ldots + S^{(i-1)}(x, 0, n_i^{(i-1)}) + S^{(i)}(x, 0, n), \qquad (12$$

where $n \leqslant n_i^{(i)}$.

We have from (4)

$$|S^i(x, 0, n)| < \frac{\phi(i)}{d},$$

where d is the distance of the point x from the set of points where $f_i(x)$ is different from zero. In the interval $(-i, +i)$ the function $f_i(x)$ is different from zero only

for $$\frac{i-1}{2} < x < \frac{i-1}{2} + 2 \quad (i \text{ odd})$$

or for $$-\frac{i}{2} < x < -\frac{i}{2} + 2 \quad (i \text{ even}).$$

Since from (9) $|x|<i/6-2$, we have $d>i/3$ in both cases, and consequently

$$|S^{(i)}(x, 0, n)|<\frac{3\phi(i)}{i}<\frac{\epsilon}{2}. \tag{13}$$

From (13), (12), (7), and (9) it follows that

$$\sigma=f_1(x)+f_2(x)+\ldots+f_{i-1}(x)+\theta\epsilon \quad (|\theta|<1), \tag{14}$$

the same formula as (11). This proves the convergence of the series (6).

On account of the irrationality of the quotient p_i/p_j for any i and j, all the terms of the series (6) are different, and thus the theorem is proved.

THE EXPRESSION IN STIELTJES INTEGRALS OF THE INVERSION FORMULAE OF FOURIER AND HANKEL

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1. *Introduction.*

Fourier's inversion theorem is that, if $f(x)$ satisfies certain conditions, then

$$f(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos xt\, \phi(t)\, dt, \tag{1.1}$$

where

$$\phi(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \cos tu\, f(u)\, du,$$

We establish the formulae

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xt}{t}\, d\Phi(t), \tag{1.2}$$

where

$$\Phi(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin tu}{u}\, dF(u),$$

assuming that $F(x)$ has bounded variation in any finite interval and for large x is such that

$$\int^\infty \frac{1}{u} |\, dF(u)\, |$$

exists*.

* The representation of $f(x)$ by an integral

$$\int_0^\infty \left(\cos \mu x\, d\Phi(\mu) + \sin \mu x\, d\Psi(\mu)\right)$$

has been studied by Hahn and Wiener. An abstract of Hahn's results appears in the *Jahresbericht der deutschen Mathematikervereinigung*, 33 (1925), 107. In Wiener's work, *Math. Zeitschrift*, 24 (1925), 575–616, the integrals converge in the mean, whereas we deal here only with ordinary convergence.

The representation of $f(x)$ by an integral

$$\int_{-a}^\infty e^{-x\mu}\, d\Phi(\mu),$$

has been treated by Fowler, *Proc. Royal Soc.* (A), 99 (1921), 462–471.

If, in particular, $F(x)$ and $\Phi(t)$ are indefinite integrals

$$F(x) = \int^x f(x)\,dx,$$

$$\Phi(t) = \int^t \phi(t)\,dt,$$

then equations (1 . 1) are formally obtained from (1 . 2) by differentiation under the integral sign.

We go on to express Hankel's inversion formulae in Stieltjes integrals

(1 . 3) $$F(x) = \int_0^\infty \left(\frac{x}{t}\right)^\nu J_\nu(xt)\,d\Phi(t),$$

where $$\Phi(t) = \int_0^\infty \left(\frac{t}{u}\right)^\nu J_\nu(tu)\,dF(u),$$

with certain restrictions on the behaviour of $F(x)$ at the origin and at infinity.

It will appear that any inversion formulae in Stieltjes integrals

$$F(x) = \int_a^b H(x, t)\,d\Phi(t),$$

$$\Phi(t) = \int_a^b K(t, u)\,dF(u),$$

arise from a discontinuous integral

$$\int_a^b H(x, t)\frac{\partial}{\partial t} K(t, u)\,dt,$$

which has the value 0 or 1 according as u is greater or less than x.

The integral of this type which yields the formulae (1 . 2) is Dirichlet's discontinuous factor

$$\frac{2}{\pi}\int_0^\infty \frac{\sin xt}{t}\cos tu\,dt.$$

The formulae (1 . 3) arise from the discontinuous integral

$$\int_0^\infty \left(\frac{x}{t}\right)^\nu J_\nu(xt)\frac{\partial}{\partial t}\left\{\left(\frac{t}{u}\right)^\nu J_\nu(tu)\right\}dt,$$

that is $$\frac{x^\nu}{u^{\nu-1}}\int_0^\infty J_\nu(xt)\,J_{\nu-1}(tu)\,dt,$$

which has the value 0, $\frac{1}{2}$, 1 according as* $u >, =, < x$.

* Watson, *Bessel functions*, 406 (8).

2. *Lemmas on Stieltjes integrals.*

LEMMA 2.1. *If for* $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$, $f(x, y)$ *is a continuous function of* (x, y) *and* $\phi(x)$, $\psi(y)$ *have bounded variation, then*

$$\int_a^b d\phi(x) \int_c^d f(x, y)\, d\psi(y) = \int_c^d d\psi(y) \int_a^b f(x, y)\, d\phi(x).$$

This is an immediate extension of the inversion theorem for repeated Riemann integrals*.

LEMMA 2.2. *If* $f(x)$ *decreases and* $\to 0$ *as* $x \to \infty$ *and*

$$\int^{\infty} f(x) |\, d\phi(x) \,|$$

exists, then $$f(x)\, \phi(x) \to 0$$

as $x \to \infty$.

If $w \geqslant v$ $$\int_v^w f(x) |\, d\phi(x) \,| \geqslant f(w) |\, \phi(w) - \phi(v) \,| \geqslant 0.$$

Given ϵ, choose v so that

$$\int_v^w f(x) |\, d\phi(x) \,| < \epsilon.$$

Let $w \to \infty$. Then

$$\epsilon \geqslant \overline{\lim} f(w) |\, \phi(w) \,| \geqslant 0,$$

giving the result.

It may be noticed that, if

$$\phi(x) = n \quad \text{for} \quad n-1 < x < n,$$

the lemma reduces to the elementary theorem in series.

If u_n *decreases and* Σu_n *converges, then* $nu_n \to 0$.

3. *The Fourier-Stieltjes formulae.*

3.1. THEOREM. *If*

(1) $F(x)$ *has bounded variation in* $0 \leqslant x \leqslant k$ *for any* k,

(2) $$F(x) = \tfrac{1}{2}\{F(x+0) + F(x-0)\},$$

(3) $$F(0) = 0,$$

(4) $$\int^{\infty} \frac{1}{u} |\, dF(u) \,| \text{ exists},$$

* Goursat, *Cours d'analyse*, 1 (1925), ch. 6.

and if

$$\Phi(t) = \int_0^\infty \frac{\sin tu}{u}\, dF(u),$$

then

$$F(x) = \int_0^\infty \frac{\sin xt}{t}\, d\Phi(t).$$

Each interval of integration $(0, \infty)$ is supposed closed at the origin, and the respective integrands for $u = 0$, $t = 0$ are taken to have their limiting values t, x.

It is sufficient to prove the result for $F(x)$ increasing.

LEMMA 3.2. *If $F(u)$ satisfies the condition (4) of the theorem, then*

$$\int_0^\infty \frac{\sin tu}{u}\, dF(u)$$

converges uniformly in any finite interval $0 \leqslant t \leqslant h$.

Given ϵ, there exists u_0 such that

$$\int_v^w \frac{1}{u}\, dF(u) < \epsilon \quad \text{if} \quad w \geqslant v \geqslant u_0.$$

Since

$$\left| \int_v^w \frac{\sin tu}{u}\, dF(u) \right| \leqslant \int_v^w \frac{1}{u}\, dF(u),$$

it follows that the convergence is uniform and that $\Phi(t)$ is continuous.

LEMMA 3.3. *If* $I = \int_0^T \frac{\sin xt}{t} \cos tu\, dt$, *then*

(1) *For all x, u, T,*

$$|I| \leqslant \pi.$$

(2) *If $u > x$,*

$$|I| \leqslant \frac{1}{T(u-x)}.$$

(3) *For fixed x and $T \to \infty$, I is uniformly convergent with respect to u in any finite interval for which $|u-x| \geqslant \delta$, (δ fixed).*

These are easily proved by expressing I as

$$\tfrac{1}{2} \int_{T(u-x)}^{T(u+x)} \frac{\sin v}{v}\, dv.$$

3.4. *Proof of the main theorem.* Integrating by parts, we have

$$(3.41)\quad \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^T \frac{\sin xt}{t}\,d\Phi(t)$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin xT}{T}\,\Phi(T) - \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^T \frac{\partial}{\partial t}\left\{\frac{\sin xt}{t}\right\}\Phi(t)\,dt.$$

The second term of the right-hand side

$$= -\frac{2}{\pi}\int_0^T \frac{\partial}{\partial t}\left\{\frac{\sin xt}{t}\right\}\left\{\lim_{U\to\infty}\int_0^U \frac{\sin tu}{u}\,dF(u)\right\}dt$$

$$= -\lim_{U\to\infty}\frac{2}{\pi}\int_0^T \frac{\partial}{\partial t}\left\{\frac{\sin xt}{t}\right\}\left\{\int_0^U \frac{\sin tu}{u}\,dF(u)\right\}dt,$$

since the inner integral is uniformly convergent (Lemma 3.2)

$$= -\lim_{U\to\infty}\frac{2}{\pi}\int_0^U dF(u)\int_0^T \frac{\partial}{\partial t}\left\{\frac{\sin xt}{t}\right\}\frac{\sin tu}{u}\,dt$$

by changing the order of integration (Lemma 2.1).

Integrating the inner integral by parts, we have

$$-\lim_{U\to\infty}\frac{2}{\pi}\left[\int_0^U \frac{\sin xT}{T}\,\frac{\sin Tu}{u}\,dF(u) - \int_0^U dF(u)\int_0^T \frac{\sin xt}{t}\cos tu\,dt\right]$$

$$= -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\sin xT}{T}\,\Phi(T) + \frac{2}{\pi}\int_0^\infty dF(u)\int_0^T \frac{\sin xt}{t}\cos tu\,dt.$$

Hence from (3.41)

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^T \frac{\sin xt}{t}\,d\Phi(t) = \int_0^\infty dF(u)\,\frac{2}{\pi}\int_0^T \frac{\sin xt}{t}\cos tu\,dt$$

$$= \int_0^\infty \psi(T, u)\,dF(u), \quad \text{say},$$

$$= \int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{k} + \int_k^\infty$$

$$= I_1 + I_2 + I_3 + I_4,$$

where k, δ are fixed later.

We now apply Lemma 3.3 to prove that, as $T\to\infty$, both $I_3\to 0$ and $I_4\to 0$, while, by choice of δ, I_1+I_2 can be made arbitrarily close to $F(x)$.

Given ϵ, choose k such that

$$\int_k^\infty \frac{1}{u}\,dF(u) < \epsilon$$

and $\psi(T,u) < K/u$ for $u \geqslant k$ and all sufficiently large T.

Then, for such values of T, $I_4 < K\epsilon$.

Now choose $\delta(\epsilon)$ such that

$$\text{(3.42)} \qquad \begin{cases} F(\xi)-F(x+0) < \epsilon & \text{for} \quad x < \xi \leqslant x+\delta \\ F(x-0)-F(\xi) < \epsilon & \text{for} \quad x-\delta \leqslant \xi < x. \end{cases}$$

Choose $T_0 = T_0(\delta, \epsilon) = T_0(\epsilon)$ such that, if $T \geqslant T_0$,

$$\text{(3.43)} \qquad \begin{cases} |\psi(T,u)-1| < \epsilon & \text{for} \quad 0 \leqslant u \leqslant x-\delta \\ |\psi(T,u)| < \epsilon & \text{for} \quad x+\delta \leqslant u \leqslant k \end{cases} \qquad \text{[Lemma 3.3 (3)]},$$

and

$$\text{(3.44)} \qquad |\psi(T,x)-\tfrac{1}{2}| < \epsilon.$$

Suppose that $T \geqslant T_0$.

Approximating to I_2 we have, using (3.42), (3.44), and the fact that $\psi(T,u)$ is bounded,

$$I_2 = \tfrac{1}{2}\{F(x+0)-F(x-0)\} + O(\epsilon),$$

the constant implied by the O being independent of T.

Again
$$I_1 = F(x-\delta)+O(\epsilon) \quad \text{from (3.43)}$$
$$= F(x-0)+O(\epsilon) \quad \text{from (3.42)},$$

and
$$I_3 = O(\epsilon) \quad \text{from (3.43)}.$$

Hence as $T \to \infty$,

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^T \frac{\sin xt}{t}\,d\Phi(t) \to F(x).$$

3.5. With the same restrictions on $F(x)$ we may prove the corresponding cosine formulae

$$F(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos xt}{t}\,d\Phi(t),$$

$$\Phi(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{1-\cos tu}{u}\,dF(u).$$

3.6. *Examples of the use of the formulae.*

(1) Take in the sine formulae

$$F(x) = x \quad \text{for} \quad x \leqslant 1,$$

$$F(x) = 1 \quad \text{for} \quad x > 1.$$

Then $$\Phi(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^1 \frac{\sin tu}{u}\, du = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^t \frac{\sin v}{v}\, dv.$$

Hence, from the reciprocal formula,

$$F(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin xt}{t} \frac{\sin t}{t}\, dt,$$

so that the value of the integral

$$\int_0^\infty \frac{\sin xt \sin t}{t^2}\, dt$$

is $\frac{1}{2}\pi x$ for $0 \leqslant x \leqslant 1$ and $\frac{1}{2}\pi$ for $x > 1$.

(2) Define an increasing function $F(x)$ thus—

$$F(0) = 0.$$

$F(x)$ is constant except for $x = b^n\pi$ $(n = 1, 2, \ldots)$. In passing through $x = b^n\pi$, $F(x)$ increases by $a^n b^n \pi$, where $a < 1$, $ab > 1$.

Then $\int_0^\infty \frac{1}{u}\, dF(u) = \sum_1^\infty a^n$ converges.

$$\Phi(t) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_1^\infty a^n \sin b^n \pi t$$

is Weierstrass's function, which has unbounded variation in any interval.

So that the integral formula for $F(x)$

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xt}{t}\, d\Phi(t)$$

is essentially a Stieltjes integral—it is not expressible as

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{\sin xt}{t}\, \Phi'(t)\, dt.$$

4. *The Hankel-Stieltjes formulae.*

4.1. We assume throughout that the order ν of the Bessel functions is greater than 0.

THEOREM. *If*

(1) $F(x)$ *has bounded variation in* $0 \leqslant x \leqslant k$ *for any* k,

(2) $F(x) = \frac{1}{2}\{F(x+0)+F(x-0)\}$,

(3) $F(0) = 0$,

(4) $\int^{\infty} u^{-\alpha}|dF(u)|$ *exists, where*

$$\alpha = 2\nu \quad \textit{for} \quad 0 < \nu \leqslant \tfrac{1}{2}$$

and

$$\alpha = \nu+\tfrac{1}{2} \quad \textit{for} \quad \nu > \tfrac{1}{2},$$

(5) $\int_0 u^{-\beta}|dF(u)|$ *exists, where*

$$\beta = 0 \quad \textit{for} \quad 0 < \nu \leqslant \tfrac{3}{2}$$

and

$$\beta = \nu-\tfrac{3}{2} \quad \textit{for} \quad \nu > \tfrac{3}{2},$$

and if

$$\Phi(t) = \int_0^{\infty} \left(\frac{t}{u}\right)^{\nu} J_{\nu}(tu)\, dF(u),$$

then

$$F(x) = \int_0^{\infty} \left(\frac{x}{t}\right)^{\nu} J_{\nu}(xt)\, d\Phi(t).$$

If $\nu < \frac{3}{2}$, the intervals of integration may be supposed closed at the origin and the respective integrands for $u = 0$, $t = 0$ taken to have their limiting values

$$\frac{t^{2\nu}}{2^{\nu}\Gamma(\nu+1)}, \quad \frac{x^{2\nu}}{2^{\nu}\Gamma(\nu+1)}.$$

If $\nu \geqslant \frac{3}{2}$, the intervals of integration are supposed open at the origin. It is sufficient to prove the result for $F(x)$ increasing.

LEMMA 4.2. *If* $F(u)$ *satisfies the condition* (4) *of the theorem, then*

$$\int_0^{\infty} \left(\frac{t}{u}\right)^{\nu} J_{\nu}(tu)\, dF(u)$$

converges uniformly in any finite interval $0 \leqslant t \leqslant h$.

From Lemma 2 . 2,

$$F(u) = o(u^{\alpha}).$$

Given ϵ, choose u_0 such that

$$\int_v^{\infty} u^{-\alpha} dF(u) < \epsilon$$

and $$F(v) < \epsilon v^{\alpha} \quad \text{for} \quad v \geqslant u_0.$$

There exist constants k, K such that

$$|J_\nu(x)| \leqslant Kx^{\nu} \quad \text{for} \quad 0 < x < k$$

and $$\leqslant Kx^{-\frac{1}{2}} \quad \text{for} \quad x \geqslant k.$$

If $h \geqslant t > k/v$, where $v \geqslant u_0$, then

$$\left| \int_v^{\infty} \left(\frac{t}{u}\right)^{\nu} J_\nu(tu)\, dF(u) \right| \leqslant K \int_v^{\infty} \left(\frac{t}{u}\right)^{\nu} (tu)^{-\frac{1}{2}}\, dF(u).$$

If $\nu \geqslant \frac{1}{2}$, this does not exceed $K_1 \int_v^{\infty} u^{-\nu-\frac{1}{2}} dF(u) \leqslant K_1 \epsilon$.

If $\nu < \frac{1}{2}$, the expression equals $K \int_v^{\infty} (tu)^{\nu-\frac{1}{2}} u^{-2\nu} dF(u)$

$$\leqslant K_1 \int_v^{\infty} u^{-2\nu} dF(u) \leqslant K_1 \epsilon.$$

If $0 < t \leqslant k/v$, then

$$\left| \int_v^{k/t} \left(\frac{t}{u}\right)^{\nu} J_\nu(tu)\, dF(u) \right| \leqslant K \int_v^{k/t} t^{2\nu} dF(u)$$

$$\leqslant Kt^{2\nu} F\left(\frac{k}{t}\right) \leqslant K_1 \epsilon.$$

And, as before,

$$\left| \int_{k/t}^{\infty} \right| \leqslant K_2 \epsilon.$$

Hence the convergence is uniform and $\Phi(t)$ is continuous.

4 . 3. The extension of Lemma 3 . 3 is more elaborate.

LEMMA 4 . 3. *If*

$$I = \frac{x^{\nu}}{u^{\nu-1}} \int_0^T J_\nu(xt)\, J_{\nu-1}(tu)\, dt,$$

where x is fixed and >0, then (1), (2), *and* (4) *hold for all $T \geqslant T_0$, the constants implied in the O's in* (1), (2), *and the bounds in* (4) *being independent of T.*

(1) *As* $u \to \infty$, $\qquad I = O(u^{-\nu-\frac{1}{2}})$.

(2) *As* $u \to 0$, $\qquad I = O(u^{-\beta})$,

where $\qquad \beta = 0 \quad \text{for} \quad 0 < \nu \leqslant \frac{3}{2}$

and $\qquad \beta = \nu - \frac{3}{2} \quad \text{for} \quad \nu > \frac{3}{2}$.

(3) *As* $T \to \infty$, *I converges uniformly with respect to u in the intervals*

$$0 < \delta \leqslant u \leqslant x - \delta,$$

$$x + \delta \leqslant u \leqslant k.$$

(4) *For fixed h, k, I is bounded for* $0 < h \leqslant u \leqslant k$.

Proofs. (1) If $u > x$,

$$I = -\frac{x^\nu}{u^{\nu-1}} \int_T^\infty = -I_1, \text{ say.}$$

In I_1 write

$$J_{\nu-1}(tu) = \left(\frac{2}{\pi tu}\right)^{\frac{1}{2}} \left\{-\sin\left(tu - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) + O\left(\frac{1}{tu}\right)\right\}.$$

Hence

$$I_1 = O\left| u^{-\nu+\frac{1}{2}} \int_T^\infty J_\nu(xt)\, t^{-\frac{1}{2}} \sin\left(tu - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) dt \right|$$

$$+ O\left\{ u^{-\nu-\frac{1}{2}} \int_T^\infty |J_\nu(xt)|\, t^{-\frac{3}{2}}\, dt \right\}.$$

The second term is $O(u^{-\nu-\frac{1}{2}})$. The first term integrated by parts is

$$O\left| u^{-\nu-\frac{1}{2}} J_\nu(xT)\, T^{-\frac{1}{2}} \cos\left(Tu - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) \right.$$

$$\left. + u^{-\nu-\frac{1}{2}} \int_T^\infty \frac{d}{dt}\left\{J_\nu(xt)\, t^{-\frac{1}{2}}\right\} \cos\left(tu - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi\right) dt \right| = O(u^{-\nu-\frac{1}{2}}).$$

(2) If $u < x$,

$$I = 1 - \frac{x^\nu}{u^{\nu-1}} \int_T^\infty = 1 - I_1, \text{ say.}$$

In I_1 write

$$(4.31)\quad J_\nu(xt) = \left(\frac{2}{\pi xt}\right)^{\frac{1}{2}}\left[\left\{1+\frac{a_2}{x^2t^2}+\frac{a_4}{x^4t^4}+\ldots\right\}\cos\left(xt-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right.$$
$$\left.+\left\{\frac{a_1}{xt}+\frac{a_3}{x^3t^3}+\ldots\right\}\sin\left(xt-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\right]$$
$$+O(xt)^{-s-\frac{1}{2}},$$

where a_1, a_2 ... are numerical coefficients and terms as far as that in $(xt)^{-s+1}$ are included in the square brackets, where $s > \nu-\frac{1}{2}$.

Consider the contribution to I_1 of a typical term of (4.31). Constants omitted, it is

$$u^{-\nu+1}\int_T^\infty \cos\left(xt-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)J_{\nu-1}(tu)\,t^{-r-\frac{1}{2}}dt$$
$$= u^{-\nu+r+\frac{1}{2}}\int_{Tu}^\infty \cos\left(\frac{xy}{u}-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)J_{\nu-1}(y)\,y^{-r-\frac{1}{2}}dy,$$

or, after integration by parts,

$$-x^{-1}u^{-\nu+r+\frac{3}{2}}\sin\left(xT-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)J_{\nu-1}(Tu)(Tu)^{-r-\frac{1}{2}}$$
$$-x^{-1}u^{-\nu+r+\frac{3}{2}}\int_{Tu}^\infty \sin\left(\frac{xy}{u}-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi\right)\left\{\frac{J_{\nu-2}(y)-J_\nu(y)}{2y^{r+\frac{1}{2}}}-(r+\tfrac{1}{2})\frac{J_{\nu-1}(y)}{y^{r+\frac{3}{2}}}\right\}dy.$$

If $r > \nu-\frac{3}{2}$, the first term is

$$O\left\{u^{-\nu+r+\frac{3}{2}}(Tu)^{\nu-r-\frac{3}{2}}\right\} = O(1).$$

For small y the integrand of the second term is

$$O(y^{\nu-r-\frac{3}{2}}),$$

and therefore the integral in the second term is

$$O\left\{(Tu)^{\nu-r-\frac{3}{2}}\right\},$$

and the second term is $O(1)$.

If $r \leqslant \nu-\frac{3}{2}$ (which can only occur if $\nu \geqslant \frac{3}{2}$), then

$$J_{\nu-1}(Tu)(Tu)^{-r-\frac{1}{2}}$$

is bounded, and the integral in the second term is bounded for all values of the lower limit. Therefore the contribution of the typical term is

$$O(u^{-\nu+r+\frac{3}{2}}) = O(u^{-\nu+\frac{3}{2}}).$$

Consider lastly the contribution of the error-term of (4.31) to I_1. It is

$$O\left\{u^{-\nu+1}\int_T^\infty |J_{\nu-1}(tu)|\,t^{-s-\frac{1}{2}}dt\right\} = O\left\{u^{-\nu+s+\frac{1}{2}}\int_{Tu}^\infty |J_{\nu-1}(y)|\,y^{-s-\frac{1}{2}}dy\right\}.$$

For small y, $\qquad |J_{\nu-1}(y)| = O(y^{\nu-1})$.

Hence the contribution of the error term is

$$O\{u^{-\nu+s+\frac{1}{2}}(Tu)^{\nu-s-\frac{1}{2}}\} = O(1).$$

Collecting results, we have (2).

(3) For values of u in the given intervals

$$\int_T^\infty J_\nu(xt)\, J_{\nu-1}(tu)\, dt$$

$$= O\int_T^\infty \{-t^{-1}\cos(xt-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)\sin(ut-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+O(t^{-2})\}\, dt,$$

which converges to zero uniformly with respect to u [as in Lemma 3.3 (3)].

Hence I converges uniformly.

(4) This follows since the integral in (3) is a bounded function of u, T [compare Lemma 3.3 (1)].

4.4. *Proof of the main theorem.* It is sufficient to give an outline of the proof, which is similar to that of the Fourier-Stieltjes theorem.

Integrating by parts and changing the order of integration, we find that

$$\int_0^T \left(\frac{x}{t}\right)^\nu J_\nu(xt)\, d\Phi(t) = \int_0^\infty dF(u)\, \frac{x^\nu}{u^{\nu-1}} \int_0^T J_\nu(xt)\, J_{\nu-1}(tu)\, dt$$

$$= \int_0^\infty \psi(T, u)\, dF(u), \text{ say.}$$

If $\nu < \frac{3}{2}$, the value for $u = 0$ of $\psi(T, u)$ is taken to be its limiting value

$$\frac{x^\nu}{2^{\nu-1}\Gamma(\nu)} \int_0^T J_\nu(xt)\, t^{\nu-1}\, dt,$$

and the integral with respect to $F(u)$ may be taken over the interval $(0, \infty)$ closed at 0.

If $\nu \geqslant \frac{3}{2}$, the interval is to be regarded as open at 0.

We complete the proof by splitting up

$$\int_0^\infty \psi(T, u)\, dF(u) = \int_0^h + \int_h^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^k + \int_k^\infty$$

and applying the results of Lemma 4.3.

WELL-POISED SERIES AND OTHER GENERALIZED HYPERGEOMETRIC SERIES

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1. *Introduction.*

In previous papers* it has been shown that there are two relations between the sums of certain generalized hypergeometric series of two classes. In the one class are "well-poised series," *i.e.* series in which the parameters occur in pairs, each pair with the same sum. In the other class are series which it is proposed to call Saalschützian; a Saalschützian series is a generalized hypergeometric series, which terminates and has parameters such that the sum of those which occur as factors in the denominator of the second term of the series exceeds by unity the sum of those which occur in the numerator. The relations in question are between well-poised series of the seventh order and Saalschützian series of the fourth order. When these relations were discovered it was noticed that they could be used to establish additional relations. These prove to be more numerous than was expected. The number established in the present paper is 22.

To make the search systematic an auxiliary notation is introduced. In the last part of the paper independent proofs of three of these new relations are obtained.

2. *The primary relations.*

The generalized hypergeometric series

$$\sum_{m=0}^{\infty} \frac{\Pi_{a,\,b,\,c,\,d,\,e,\,f,\,g}\{\Gamma(a+m)/\Gamma(a)\}}{\Pi_{p,\,q,\,r,\,s,\,t,\,u}\{\Gamma(p+m)/\Gamma(p)\}} \frac{x^m}{m!}$$

* F. J. W. Whipple, "On well-poised series, etc.", *Proc. London Math. Soc.* (2), 24 (1925), 247–263. This paper will be referred to as W.P.S. Shorter proofs of the two relations are given in "A fundamental relation between generalized hypergeometric series", *Journal London Math. Soc.*, 1 (1926), 138–145.

is denoted by $${}_7F_6\left[\begin{matrix}a, b, c, d, e, f, g\,; & x\\ p, q, r, s, t, u & \end{matrix}\right]$$

and the corresponding notation is used for series with other parameters. When x is not stated explicitly it is to be taken as $+1$. When

$$a+1 = b+p = c+q = \ldots,$$

the series is said* to be "well-poised".

The series ${}_3F_2\left[\begin{matrix}x, y, -n\\ u, v\end{matrix}\right]$, in which n is a positive integer, was shown† by Saalschütz to be summable when

$$u+v-x-y+n = 1,$$

the sum being equal to

$$\frac{\Gamma(u)\,\Gamma(1-v+x)\,\Gamma(1-v+y)\,\Gamma(1-v-n)}{\Gamma(1-v)\,\Gamma(u-x)\,\Gamma(u-y)\,\Gamma(u+n)}.$$

It is therefore appropriate to give the name Saalschützian to the more general series

$${}_4F_3\left[\begin{matrix}x, y, z, -n\\ u, v, w\end{matrix}\right],$$

in which the parameters satisfy the conditions, that n is a positive integer, and

$$u+v+w-x-y-z+n = 1.$$

The first of the known relations between Saalschützian series of the fourth order and well-poised series of the seventh order may be written‡

$$(2.1)\quad {}_4F_3\left[\begin{matrix}x, y, z, -n\\ u, v, w\end{matrix}\right]$$

$$= \frac{\Gamma(v+w-x)\,\Gamma(1+y-u)\,\Gamma(1+z-u)\,\Gamma(1-n-u)}{\Gamma(1+y-n-u)\,\Gamma(1+z-n-u)\,\Gamma(1+y+z-u)\,\Gamma(1-u)}$$

$$\times {}_7F_6\left[\begin{matrix}a, 1+\frac{1}{2}a, w-x, v-x, y, z, -n\\ \frac{1}{2}a, v, w, 1+z-n-u, 1+y-n-u, 1+y+z-u\end{matrix}\right],$$

* W.P.S., 247.

† *Zeitschrift für Math. und Phys.*, 35 (1890), 186. See also W.P.S., § 8, and F. J. W. Whipple, "A group of generalized hypergeometric series, etc.", *Proc. London Math. Soc.* (2), 23 (1924), 104–114, § 9. The latter paper will be referred to as G.G.H.S.

‡ W.P.S., Equation (16.1) with the substitutions $-n$, x for x and t respectively.

a being given by

$$a = y+z-n-u = w+v-x-1. \tag{2.11}$$

It will be noticed that the well-poised series terminates.

The second relation* is formally the same as the first with the x and $-n$ interchanged, but this interchange is of such importance that the relations must count as two in our classification. Thus we write

$$ {}_4F_3\left[\begin{matrix} x, & y, & z, & -n \\ u, & v, & w & \end{matrix}\right] \tag{2.2}$$

$$= \frac{\Gamma(v+w+n)\,\Gamma(1+x-u)\,\Gamma(1+y-u)\,\Gamma(1+z-u)}{\Gamma(1+y+z-u)\,\Gamma(1+z+x-u)\,\Gamma(1+x+y-u)\,\Gamma(1-u)}$$

$$\times {}_7F_6\left[\begin{matrix} a', & 1+\frac{1}{2}a', & w+n, & v+n, & x, & y, & z \\ \frac{1}{2}a', & v, & w, & 1+y+z-u, & 1+z+x-u, & 1+x+y-u & \end{matrix}\right],$$

where

$$a' = x+y+z-u = v+w+n-1. \tag{2.21}$$

In this case the well-poised series does not terminate. For the relation to be significant the series must be convergent. The necessary condition is

$$u < 1-n. \tag{2.22}$$

3. *The auxiliary notation for Saalschützian series and terminating well-poised series.*

To bring out the symmetry of these relations an auxiliary notation is required. The notation adopted is analogous to the one which has been used previously† in studying the generalized hypergeometric series of the third order, ${}_3F_2$.

Let $r_1, r_2, \ldots, r_6$ be six parameters such that

$$\Sigma r = 0, \tag{3.1}$$

and write ϕ for the fraction $\frac{1}{3}(n-1)$.

Further

$$e_{ij} = r_i+r_j-\phi, \tag{3.11}$$

* W.P.S., Equation (16.1) with the substitution $-n$ for t.

† G.G.H.S., § 3.

while

$$\delta_{ij} = r_i - r_j - n. \tag{3.12}$$

It will be seen that

$$\epsilon_{12} + \epsilon_{34} + \epsilon_{56} = 1 - n \tag{3.2}$$

and

$$\delta_{12} = \epsilon_{13} - \epsilon_{23} - n. \tag{3.21}$$

For the parameters which occur in the Saalschützian series on the left of (2.1) we write

$$x = \epsilon_{23}, \quad y = \epsilon_{13}, \quad z = \epsilon_{12}, \tag{3.3}$$

$$u = 1 - n - \epsilon_{56}, \quad v = 1 - n - \epsilon_{46}, \quad w = 1 - n - \epsilon_{45}. \tag{3.31}$$

The condition

$$u + v + w - x - y - z + n = 1 \tag{3.32}$$

is satisfied, since $3(1-n) + 6\phi + n = 1.$

For the parameters which occur on the right of (2.1) we have

$$a = y + z - n - u = r_1 - r_4 - n = \delta_{14}, \tag{3.33}$$

$$w - x = \epsilon_{16}, \quad v - x = \epsilon_{15}, \tag{3.34}$$

$$1 + z - n - u = 1 - n - \epsilon_{34}, \quad 1 + y - n - u = 1 - n - \epsilon_{24}, \tag{3.35}$$

$$1 + y + z - u = 1 + \delta_{14} + n. \tag{3.36}$$

It is also to be noted that

$$\frac{\Gamma(1-n-u)}{\Gamma(1-u)} = (-1)^n \frac{\Gamma(u)}{\Gamma(n+u)} = (-1)^n \frac{\Gamma(1-n-\epsilon_{56})}{\Gamma(1-\epsilon_{56})}. \tag{3.37}$$

The equation (2.1) can now be written

$$ {}_4F_3\left[\begin{matrix} \epsilon_{23}, \epsilon_{13}, \epsilon_{12}, -n \\ 1-n-\epsilon_{56}, 1-n-\epsilon_{46}, 1-n-\epsilon_{45} \end{matrix}\right] \tag{3.4}$$

$$= (-1)^n \frac{\Gamma(1+\delta_{14})\,\Gamma(1-\epsilon_{24})\,\Gamma(1-\epsilon_{34})\,\Gamma(1-n-\epsilon_{56})}{\Gamma(1+\delta_{14}+n)\,\Gamma(1-n-\epsilon_{24})\,\Gamma(1-n-\epsilon_{34})\,\Gamma(1-\epsilon_{56})}$$

$$\times {}_7F_6\left[\begin{matrix} \delta_{14}, 1+\tfrac{1}{2}\delta_{14}, \epsilon_{16}, \epsilon_{15}, \epsilon_{13}, \epsilon_{12}, -n \\ \tfrac{1}{2}\delta_{14}, 1-n-\epsilon_{46}, 1-n-\epsilon_{45}, 1-n-\epsilon_{34}, 1-n-\epsilon_{24}, 1+n+\delta_{14} \end{matrix}\right].$$

Now let

$$(3.5)\quad S(1,2,3) = (-1)^n \frac{\Gamma(1-\epsilon_{56})\,\Gamma(1-\epsilon_{46})\,\Gamma(1-\epsilon_{45})}{\Gamma(1-n-\epsilon_{56})\,\Gamma(1-n-\epsilon_{46})\,\Gamma(1-n-\epsilon_{45})} \times {}_4F_3\left[\begin{matrix}\epsilon_{23},\ \epsilon_{13},\ \epsilon_{12},\ -n\\ 1-n-\epsilon_{56},\ 1-n-\epsilon_{46},\ 1-n-\epsilon_{45}\end{matrix}\right],$$

while

$$(3.51)\quad W(1;4) = \frac{\Gamma(1+\delta_{14})\,\Gamma(1-\epsilon_{24})\,\Gamma(1-\epsilon_{34})\,\Gamma(1-\epsilon_{45})\,\Gamma(1-\epsilon_{46})}{\Gamma(1+n+\delta_{14})\,\Gamma(1-n-\epsilon_{24})\,\Gamma(1-n-\epsilon_{34})\,\Gamma(1-n-\epsilon_{45})\,\Gamma(1-n-\epsilon_{46})} \times {}_7F_6\left[\begin{matrix}\delta_{14},\ 1+\tfrac{1}{2}\delta_{14},\ \epsilon_{16},\ \epsilon_{15},\ \epsilon_{13},\ \epsilon_{12},\ -n\\ \tfrac{1}{2}\delta_{14},\ 1-n-\epsilon_{46},\ 1-n-\epsilon_{45},\ 1-n-\epsilon_{34},\ 1-n-\epsilon_{24},\ 1+n+\delta_{14}\end{matrix}\right],$$

all permutations of the characteristic numbers 1, 2, ..., 6 being allowed.

By the use of the abbreviation

$$x_p = x(x+1)\ldots(x+p-1),$$

the definitions may be written in the forms

$$(3.52)\quad S(1,2,3) = \sum_{p=0}^{n} {}_nC_p\,(\epsilon_{23})_p\,(\epsilon_{13})_p\,(\epsilon_{12})_p\,(\epsilon_{56})_{n-p}\,(\epsilon_{46})_{n-p}\,(\epsilon_{45})_{n-p}$$

and

$$(3.53)\quad W(1;4) = (-1)^n \sum_{p=0}^{n} {}_nC_p\,\frac{r_1-r_4+2p-n}{r_1-r_4} \times \frac{(\epsilon_{12})_p\,(\epsilon_{13})_p\,(\epsilon_{15})_p\,(\epsilon_{16})_p\,(\epsilon_{42})_{n-p}\,(\epsilon_{43})_{n-p}\,(\epsilon_{45})_{n-p}\,(\epsilon_{46})_{n-p}}{(1-r_1+r_4)_{n-p}\,(1+r_1-r_4)_p}.$$

The equation (3.4) is equivalent to

$$(3.54)\quad S(1,2,3) = W(1;4).$$

As the series involved in $S(1,2,3)$ and $W(1;4)$ terminate, each may be written out in the reverse order. With the help of (3.52) and (3.53)

it is easily verified that

$$S(1, 2, 3) \equiv S(4, 5, 6) \tag{3.6}$$

and

$$W(1; 4) \equiv W(4; 1). \tag{3.61}$$

To justify other permutations of the characteristic numbers equation (3.54) may be used repeatedly. Thus

$$S(1, 2, 3) = W(1; 4) = S(1, 2, 5) = W(2; 6) \tag{3.7}$$

$$= S(2, 3, 5) = \ldots.$$

Formally there are 20 S's and 30 W's, but since each series is counted twice the number of distinct expansions is 10 plus 15. The 25 expansions are all equal. The identities involved, in addition to (3.54), (3.6), and (3.61), are

$$S(1, 2, 3) = S(1, 2, 4) \equiv S(3, 5, 6), \tag{3.71}$$

$$W(1; 4) = W(2; 6) \equiv W(6; 2), \tag{3.72}$$

$$W(1; 4) = W(1; 5) \equiv W(5; 1), \tag{3.73}$$

$$S(1, 2, 3) = W(2; 1) \equiv W(1; 2). \tag{3.74}$$

In each of these identities permutations of the characteristics 1, 2, ..., 6 are allowed. Further consideration of the identities is postponed to § 10.

4. *The auxiliary notation for non-terminating well-poised series.*

Turning to equation (2.2) we note that the first of the parameters of the well-poised series on the right satisfies the equations

$$a' = v+w+n-1 = 1-n-\epsilon_{45}-\epsilon_{46} = -\phi-2r_4-r_5-r_6. \tag{4.1}$$

We write

$$\lambda_{4;56} = -\phi-2r_4-r_5-r_6. \tag{4.12}$$

The other parameters of the series are expressed at once in terms of

ϵ's and δ's by the use of (3 . 3), (3 . 31), and (3 . 36), so that (2 . 2) becomes

$$(4\,.\,2)\quad {}_4F_3\left[\begin{matrix}\epsilon_{23},\ \epsilon_{13},\ \epsilon_{12},\ -n\\ 1-n-\epsilon_{56},\ 1-n-\epsilon_{46},\ 1-n-\epsilon_{45}\end{matrix}\right]$$

$$= \frac{\Gamma(1+\lambda_{4;\,56})\,\Gamma(1-\epsilon_{14})\,\Gamma(1-\epsilon_{24})\,\Gamma(1-\epsilon_{34})}{\Gamma(1+n+\delta_{14})\,\Gamma(1+n+\delta_{24})\,\Gamma(1+n+\delta_{34})\,\Gamma(n+\epsilon_{56})}$$

$$\times {}_7F_6\left[\begin{matrix}\lambda_{4;\,56},\ 1+\tfrac{1}{2}\lambda_{4;\,56},\ 1-\epsilon_{45},\ 1-\epsilon_{46},\ \epsilon_{23},\ \epsilon_{13},\ \epsilon_{12}\\ \tfrac{1}{2}\lambda_{4;\,56},\ 1-n-\epsilon_{46},\ 1-n-\epsilon_{45},\ 1+n+\delta_{14},\ 1+n+\delta_{24},\ 1+n+\delta_{34}\end{matrix}\right].$$

We now introduce the definition,

$$(4\,.\,3)\quad W(4;\,5,\,6)$$

$$= \frac{\Gamma(1+\lambda_{4;\,56})\,\Gamma(1-\epsilon_{14})\,\Gamma(1-\epsilon_{24})\,\Gamma(1-\epsilon_{34})\,\Gamma(1-\epsilon_{45})\,\Gamma(1-\epsilon_{46})}{\Gamma(1+n+\delta_{14})\,\Gamma(1+n+\delta_{24})\,\Gamma(1+n+\delta_{34})\,\Gamma(1-n-\epsilon_{45})\,\Gamma(1-n-\epsilon_{46})\,\Gamma(\epsilon_{56})}$$

$$\times {}_7F_6\left[\begin{matrix}\lambda_{4;\,56},\ 1+\tfrac{1}{2}\lambda_{4;\,56},\ 1-\epsilon_{45},\ 1-\epsilon_{46},\ \epsilon_{23},\ \epsilon_{13},\ \epsilon_{12}\\ \tfrac{1}{2}\lambda_{4;\,56},\ 1-n-\epsilon_{46},\ 1-n-\epsilon_{45},\ 1+n+\delta_{14},\ 1+n+\delta_{24},\ 1+n+\delta_{34}\end{matrix}\right].$$

Remembering that

$$(4\,.\,31)\qquad (-1)^n\,\Gamma\epsilon_{56}\,\Gamma(1-\epsilon_{56}) = \Gamma(n+\epsilon_{56})\,\Gamma(1-n-\epsilon_{56}),$$

we find that (4 . 2) is equivalent to

$$(4\,.\,32)\qquad S(1,\,2,\,3) = W(4;\,5,\,6).$$

This equation is subject to the condition for the convergence of the series ${}_7F_6$ in $W(4;\,5,\,6)$, namely

$$(4\,.\,33)\qquad \epsilon_{56} > 0,$$

and it is worth noting that $\Gamma(\epsilon_{56})$ occurs in the denominator of the fraction multiplying the ${}_7F_6$.

As $S(4,\,5,\,6)$ is the same expansion as $S(1,\,2,\,3)$, though written in the reverse order, there are six non-terminating well-poised series corresponding with each Saalschützian series; not all of the six are convergent, however. The total number of non-terminating well-poised series derived from the ten equal Saalschützian series is 60.

The new identities which are found by combining (4 . 32) and (3 . 52) are set out below :—

(4 . 32) $W(4; 5, 6) = S(1, 2, 3) \equiv S(4, 5, 6)$,

(4 . 4) $W(4; 5, 6) = S(1, 2, 4) \equiv S(3, 5, 6)$,

(4 . 41) $W(4; 5, 6) = S(1, 2, 5) \equiv S(3, 4, 6)$,

(4 . 42) $W(4; 5, 6) = W(1; 4) \equiv W(4; 1)$,

(4 . 43) $W(4; 5, 6) = W(1; 5) \equiv W(5; 1)$,

(4 . 44) $W(4; 5, 6) = W(1; 2) \equiv W(2; 1)$,

(4 . 45) $W(4; 5, 6) = W(4; 5) \equiv W(5; 4)$,

(4 . 46) $W(4; 5, 6) = W(5; 6) \equiv W(6; 5)$,

(4 . 47) $W(4; 5, 6) = W(5; 4, 6)$, (4 . 48) $W(4; 5, 6) = W(5; 1, 6)$,

(4 . 49) $W(4; 5, 6) = W(5; 1, 2)$, (4 . 50) $W(4; 5, 6) = W(5; 1, 4)$,

(4 . 51) $W(4; 5, 6) = W(4; 1, 5)$, (4 . 52) $W(4; 5, 6) = W(4; 1, 2)$,

(4 . 53) $W(4; 5, 6) = W(1; 2, 3)$, (4 . 54) $W(4; 5, 6) = W(1; 2, 4)$,

(4 . 55) $W(4; 5, 6) = W(1; 2, 5)$, (4 . 56) $W(4; 5, 6) = W(1; 5, 6)$,

(4 . 57) $W(4; 5, 6) = W(1; 4, 5)$.

5. The results of this analysis are summarized in the following schedule. In the first column is given the number of equivalent series of each class. In the other three columns are given the numbers of types of relation, such as (3 . 52), (3 . 71)–(3 . 74), (4 . 32)–(4 . 56).

	No.	Relations with S.S.	T.W-P.S.	N-T.W-P.S.
Saalschützian Series	10	1	2	3
Terminating W-P. Series	15	2	2	5
Non-terminating W-P. Series	60	3	5	11

6. In the next stage of the work we pass back from the auxiliary notation and tabulate the parameters of the various series which are

associated with a given "master" series. In the case of well-poised series only the numerator-parameters will be stated, as the denominator-parameters can be found at once by the defining property of such series. When the series is non-terminating, the convergency-indicator (which must be positive if the series converges) is also tabulated. The number of different series obtained by permutation of the parameters of the master series is indicated in the last column of each schedule.

7. *The series allied with a given Saalschützian series.*

For the given series we take

$$ {}_4F_3\left[\begin{matrix} x, & y, & z, & -n \\ u, & v, & w & \end{matrix}\right]. $$

The equations (3 . 3) and (3 . 31) can be used at once, and with the aid of (3 . 2) we complete the following list :

(7.1) $\epsilon_{12} = z,$

$\epsilon_{13} = y, \qquad \epsilon_{23} = x,$

$\epsilon_{14} = u-x, \quad \epsilon_{24} = u-y, \quad \epsilon_{34} = u-z,$

$\epsilon_{15} = v-x, \quad \epsilon_{25} = v-y, \quad \epsilon_{35} = v-z, \quad \epsilon_{45} = 1-n-w,$

$\epsilon_{16} = w-x, \quad \epsilon_{26} = w-y, \quad \epsilon_{36} = w-z, \quad \epsilon_{46} = 1-n-v, \quad \epsilon_{56} = 1-n-u.$

The δ's and λ's can be written down as required, it being remembered that (3 . 21) and (4 . 1) give

$$ \delta_{12} = \epsilon_{13}-\epsilon_{23}-n $$

and $$ \lambda_{4;\,56} = 1-n-\epsilon_{45}-\epsilon_{46}, \quad \text{respectively.} $$

In Table I the parameter δ or λ is always shown first in its appropriate line and the parameter $1+\frac{1}{2}\delta$ or $1+\frac{1}{2}\lambda$ is omitted.

In the notation of (3 . 5) the parameters tabulated for $S(1, 2, 3)$ are ϵ_{23}, ϵ_{13}, ϵ_{12}, $-n$; $1-n-\epsilon_{56}$, $1-n-\epsilon_{46}$, $1-n-\epsilon_{45}$.

Similarly in the notation of (3 . 51) the parameters tabulated for $W(1;4)$ are δ_{14}; ϵ_{16}, ϵ_{15}, ϵ_{13}, ϵ_{12}, $-n$, and in the notation of (4 . 3) the parameters to be shown for $W(4;5,6)$ are $\lambda_{4;\,56}$; $1-\epsilon_{45}$, $1-\epsilon_{46}$; ϵ_{23}, ϵ_{13}, ϵ_{12}. The convergency indicator ϵ_{56} is shown in square brackets.

TABLE I.

Parameters of Associated Series. Master Series Saalschützian.

$$u+v+w-x-y-z+n=1.$$

$S(1,2,3)$	$x, y, z, -n; u, v, w$		1
$S(4,5,6)$	$1-n-u, 1-n-v, 1-n-w, -n; 1-n-x, 1-n-y, 1-n-z$		
$S(1,2,4)$	$v-z, w-z, 1-n-u, -n; 1-n-u+x, 1-n-u+y, 1-n-z$		9
$S(3,5,6)$	$u-x, u-y, z, -n; 1-n-v+z, 1-n-w+z, u$		
			10
$W(1;4)$	$v+w-x-1; v-x, w-x, y, z, -n$		9
$W(4;1)$	$1+x-v-w-2n; u-y, u-z, 1-n-w, 1-n-v, -n$		
$W(2;1)$	$x-y-n; x, u-y, v-y, w-y, -n$		3
$W(5;4)$	$v-u-n; v-x, v-y, v-z, 1-n-u, -n$		3
			15
$W(4;5,6)$	$v+w+n-1; n+w, n+v; x, y, z$	$[1-n-u]$	3
$W(1;2,3)$	$1-n-y-z; 1-y, 1-z; 1-n-u, 1-n-v, 1-n-w$	$[x]$	3
$W(4;2,3)$	$1-2u+y+z-n; 1-u+y, 1-u+z; v-x, w-x, 1-u-n$	$[x]$	9
$W(1;5,6)$	$1+2x-v-w-n; 1-v+x, 1-w+x; x, u-y, u-z$	$[1-n-u]$	9
$W(5;1,4)$	$x-v+w; 1-v+x, w+n; x, w-y, w-z$	$[u-x]$	18
$W(2;1,4)$	$1-u+y-z-n; 1-u+y, 1-z; v-z, w-z, 1-n-u$	$[u-x]$	18
			60

8. *The series allied with a given terminating well-poised series.*

Now let

$$(8.1)\quad {}_7F_6\left[\begin{matrix}\delta_{12}, & 1+\frac{1}{2}\delta_{12}, & \epsilon_{13}, & \epsilon_{14}, & \epsilon_{15}, & \epsilon_{16}, & -n\\ \frac{1}{2}\delta_{12}, & 1-n-\epsilon_{23}, & 1-n-\epsilon_{24}, & 1-n-\epsilon_{25}, & 1-n-\epsilon_{26}, & 1+n+\delta_{12}\end{matrix}\right]$$
$$={}_7F_6\left[\begin{matrix}a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & -n\\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a+n\end{matrix}\right].$$

With the aid of equation (3.2) we obtain the equations

$$(8.2)\quad \epsilon_{12}=c+d+e+f-2a-n-1,$$
$$\epsilon_{13}=c,\quad \epsilon_{23}=c-a-n,$$
$$\epsilon_{14}=d,\quad \epsilon_{24}=d-a-n,\quad \epsilon_{34}=1+a-e-f,$$
$$\epsilon_{15}=e,\quad \epsilon_{25}=e-a-n,\quad \epsilon_{35}=1+a-d-f,\quad \epsilon_{45}=1+a-c-f,$$
$$\epsilon_{16}=f,\quad \epsilon_{26}=f-a-n,\quad \epsilon_{36}=1+a-d-e,$$
$$\epsilon_{46}=1+a-c-e,\quad \epsilon_{56}=1+a-c-d.$$

As before, the δ's and λ's are to be obtained from equations such as

$$\delta_{12}=\epsilon_{15}-\epsilon_{23}-n$$

and $$\lambda_{4;56} = 1-n-\epsilon_{45}-\epsilon_{46}.$$

To simplify the table somewhat we write

$$c+d+e+f-2a-n-1 = s. \tag{8.3}$$

TABLE II.

Parameters of Associated Series. Master Series well-poised and terminating.

$s = c+d+e+f-2a-n-1.$

$W(1;2)$	$a;\ c, d, e, f, -n$	
$W(2;1)$	$-a-2n;\ c-a-n, d-a-n, e-a-n, f-a-n, -n$	1
$W(1;3)$	$s+a-c;\ s, d, e, f, -n$	
$W(3;1)$	$c-a-s-2n;\ c-a-n;\ 1+a-e-f, 1+a-d-f, 1+a-d-e, -n$	4
$W(2;3)$	$s-c-n;\ s, d-a-n, e-a-n, f-a-n, -n$	
$W(3;2)$	$c-s-n;\ c, 1+a-e-f, 1+a-d-f, 1+a-d-e, -n$	4
$W(3;4)$	$c-d-n;\ c, c-a-n, 1+a-d-f, 1+a-d-e, -n$	6
		15
$S(1,2,3)$	$s, c, c-a-n, -n;\ c+f-a-n, c+e-a-n, c+d-a-n$	
$S(4,5,6)$	$1+a-c-d, 1+a-c-f, 1+a-c-e, -n;\ 1-n-s, 1-n-c, 1+a-c$	4
$S(1,3,4)$	$c, d, 1+a-e-f, -n;\ 1+a-e, 1+a-f, c+d-a-n$	
$S(2,5,6)$	$e-a-n, f-a-n, 1+a-c-d, -n;\ 1-n-c, 1-n-d, e+f-a-n$	6
		10
$W(1;2,3)$	$1-s-n-c;\ 1-s, 1-c;\ 1+a-c-d, 1+a-c-e, 1+a-c-f$ $[c-a-n]$	4
$W(2;1,3)$	$1-s+a-c;\ 1-s, 1+n+a-c;\ 1+a-c-d, 1+a-c-e, 1+a-c-f$ $[c]$	4
$W(3;1,2)$	$1+a-2c;\ 1-c, 1+n+a-c;\ 1+a-c-d, 1+a-c-e, 1+a-c-f$ $[s]$	4
$W(1;3,4)$	$1-c-d-n;\ 1-c, 1-d;\ e-a-n, f-a-n, 1+a-c-d$ $[1+a-e-f]$	6
$W(2;3,4)$	$e+f-s;\ 1-c+a+n, 1-d+a+n;\ e, f, 1+a-c-d$ $[1+a-e-f]$	6
$W(3;1,4)$	$e+f-c-a-n;\ 1-c, e+f-a;\ e-a-n, f-a-n, 1+a-c-d$ $[d]$	12
$W(3;2,4)$	$e+f-c;\ 1-c+a+n, e+f-a;\ e, f, 1+a-c-d$ $[d-a-n]$	12
$W(3;4,5)$	$s-c;\ e+f-a, d+f-a;\ s, f, f-a-n$ $[1+a-c-f]$	12
		60

9. *The series allied with a given non-terminating well-poised series.*

Finally let

$$(9.1)\quad {}_7F_6\left[\begin{matrix}\lambda_{4;56},\ 1+\frac{1}{2}\lambda_{4;56},\ 1-\epsilon_{45},\ 1-e_{46},\ \epsilon_{23},\ \epsilon_{13},\ \epsilon_{12}\\ \frac{1}{2}\lambda_{4;56},\ 1-n-\epsilon_{46},\ 1-n-\epsilon_{45},\ 1+n+\delta_{14},\ 1+n+\delta_{24},\ 1+n+\delta_{34}\end{matrix}\right]$$

$$= {}_7F_6\left[\begin{matrix}c+d-n-1,\ 1+\frac{1}{2}(c+d-n-1),\ c,\ d,\ e,\ f,\ g\\ \frac{1}{2}(c+d-n-1),\ d-n,\ c-n,\ c+d-e-n,\ c+d-f-n,\ c+d-g-n\end{matrix}\right].$$

It is found convenient to write

$$t = c+d-e-f-g-2n. \tag{9.2}$$

By using relations such as (3.2)

$$\epsilon_{12}+\epsilon_{34}+\epsilon_{56} = 1-n,$$

the following list is built up :

$$(9.3)\quad \epsilon_{12}=g,\quad \epsilon_{13}=f,\quad \epsilon_{23}=e,\quad \epsilon_{56}=t,$$

$$\epsilon_{14}=f+g-c-d+n+1=1-n-e-t,$$

$$\epsilon_{24}=e+g-c-d+n+1=1-n-f-t,$$

$$\epsilon_{34}=e+f-c-d+n+1=1-n-g-t,$$

$$\epsilon_{15}=d-e-n,\quad \epsilon_{25}=d-f-n,\quad \epsilon_{35}=d-g-n,\quad \epsilon_{45}=1-e,$$

$$\epsilon_{16}=c-e-n,\quad \epsilon_{26}=c-f-n,\quad \epsilon_{36}=c-g-n,\quad \epsilon_{46}=1-d.$$

The parameters of the expansions, which occur in equations (4 . 32) to (4 . 56), can now be written down.

TABLE III.

Parameters of Associated Series. Master Series well-poised and non-terminating.

$$t=c+d-e-f-g-2n.$$

$W(4;5,6)$	$c+d-n-1;\ c, d;\ e, f, g$	$[t]$	1
$W(5;4,6)$	$c-t-n;\ c, 1-t;\ e, f, g$	$[1-d]$	2
$W(5;1,6)$	$1+e-t-d;\ 1-d+e+n, 1-t;\ e, 1-n-f-t, 1-n-g-t$	$[c-e-n]$	6
$W(5;1,2)$	$1+e+f-2d+n;\ 1-d+e+n, 1-d+f+n;\ 1-n-g-t, c-g-n, 1-d$	$[g]$	6
$W(5;1,4)$	$c-d+e;\ 1-d+e+n, c;\ e, c-f-n, c-g-n$	$[1-n-e-t]$	6
$W(4;1,5)$	$c+e+t-1;\ e+t+n, c;\ e, c-f-n, c-g-n$	$[d-e-n]$	6
$W(4;1,2)$	$e+f+2t+n-1;\ e+t+n, f+t+n;\ d-g-n, c-g-n, t$	$[g]$	3
$W(1;2,3)$	$1-f-g-n;\ 1-f, 1-g;\ 1-c, 1-d, t$	$[e]$	3
$W(1;2,4)$	$e-g+t;\ 1-g, e+t+n;\ d-g-n, c-g-n, t$	$[1-n-f-t]$	6
$W(1;2,5)$	$1-d+e-g;\ 1-g, 1-d+e+n;\ 1-g-t-n, c-g-n, 1-d$	$[d-f-n]$	12
$W(1;5,6)$	$1-c-d+2e+n;\ 1-d+e+n, 1-c+e+n;\ e, 1-n-f-t, 1-n-g-t$	$[t]$	3
$W(1;4,5)$	$c+e-f-g-n;\ e+t+n, 1-d+e+n;\ e, c-f-n, c-g-n$	$[1-c]$	6
			60
$S(1,2,3)$	$e, f, g, -n;\ c-n, d-n, 1-t-n$		1
$S(4,5,6)$	$1-c, 1-d, t, -n;\ 1-e-n, 1-f-n, 1-g-n$		
$S(1,2,4)$	$1-n-e-t, 1-n-f-t, g, -n;\ 1-d+g, 1-c+g, 1-t-n$		3
$S(3,5,6)$	$d-g-n, c-g-n, t, -n;\ e+t, f+t, 1-g-n$		
$S(1,2,5)$	$d-e-n, d-f-n, g, -n;\ g+t, 1-c+g, d-n$		6
$S(3,4,6)$	$1-n-g-t, c-g-n, 1-d, -n;\ 1-d+e, 1-d+f, 1-g-n$		
			10
$W(1;4)$	$c+d-e-2n-1;\ f, g, d-e-n, c-e-n, -n$		3
$W(4;1)$	$1-c-d+e;\ 1-n-f-t, 1-n-g-t, 1-c, 1-d, -n$		
$W(1;5)$	$f+g-d;\ f, g, 1-n-e-t, c-e-n, -n$		6
$W(5;1)$	$d-f-g-2n;\ d-f-n, d-g-n, 1-c, t, -n$		
$W(1;2)$	$f-e-n;\ f, 1-n-e-t, d-e-n, c-e-n, -n$		3
$W(4;5)$	$1-d-t-n;\ 1-n-e-t, 1-n-f-t, 1-n-g-t, 1-d, -n$		2
$W(5;4)$	$d+t-n-1;\ d-e-n, d-f-n, d-g-n, t, -n$		
$W(5;6)$	$d-c-n;\ d-e-n, d-f-n, d-g-n, 1-c, -n$		1
			15

10. *The relation between the Saalschützian series. Direct proof.*

All the relations which have been found have been deduced from the primary relations stated in § 2. It is of interest to look for direct proofs of some of the new relations.

Consider, in the first instance, the equation (3 . 71)

$$S(1, 2, 3) = S(3, 5, 6).$$

In the notation of Table I this is

$$(10.1)\quad \frac{\Gamma(u+n)\,\Gamma(v+n)\,\Gamma(w+n)}{\Gamma(u)\,\Gamma(v)\,\Gamma(w)}\,{}_4F_3\left[\begin{matrix}x, & y, & z, & -n\\ u, & v, & w\end{matrix}\right]$$

$$= \frac{\Gamma(1-v+z)\,\Gamma(1-w+z)\,\Gamma(u+n)}{\Gamma(1-n-v+z)\,\Gamma(1-n-w+z)\,\Gamma(u)}\,{}_4F_3\left[\begin{matrix}u-x, & u-y, & z, & -n\\ 1-n-v+z, & 1-n-w+z, & u\end{matrix}\right],$$

or

$$(10.11)\quad \frac{\Gamma(v-z)\,\Gamma(w-z)\,\Gamma(v+n)\,\Gamma(w+n)}{\Gamma(v)\,\Gamma(w)\,\Gamma(v-z+n)\,\Gamma(w-z+n)}\,{}_4F_3\left[\begin{matrix}x, & y, & z, & -n\\ u, & v, & w\end{matrix}\right]$$

$$= {}_4F_3\left[\begin{matrix}u-x, & u-y, & z, & -n\\ 1-v+z-n, & 1-w+z-n, & u\end{matrix}\right].$$

The parameters are, by hypothesis, subject to the condition,

$$(10.12)\qquad u+v+w-x-y-z+n = 1.$$

The equation (10 . 11) is symmetrical in z and $-n$, and the symmetry is not affected if w is eliminated by use of (10 . 12).

After the elimination the equation may be written.

$$(10.2)\quad \frac{\Gamma(v+n)}{\Gamma(v)}\,\frac{\Gamma(1-u-v+x+y+z)}{\Gamma(1-u-v+x+y+z-n)}\,{}_4F_3\left[\begin{matrix}x, & y, & z, & -n\\ u, & v, & 1-u-v+x+y+z-n\end{matrix}\right]$$

$$= \frac{\Gamma(1-v+z)\,\Gamma(u+v-x-y+n)}{\Gamma(1-v+z-n)\,\Gamma(u+v-x-y)}\,{}_4F_3\left[\begin{matrix}u-x, & u-y, & z, & -n\\ 1-v+z-n, & u+v-x-y, & u\end{matrix}\right],$$

This is an equation of the nth degree in z, as may be seen by expand-

ing the hypergeometric series. The coefficients of z^n on the two sides are

$$\frac{\Gamma(v+n)}{\Gamma v}\,{}_3F_2\left[\begin{matrix}x,\ y,\ -n\\ u,\ v\end{matrix}\right] \quad\text{and}\quad \frac{\Gamma(u+v-x-y+n)}{\Gamma(u+v-x-y)}\,{}_3F_2\left[\begin{matrix}u-x,\ u-y,\ -n\\ u+v-x-y,\ u\end{matrix}\right]$$

respectively. That these expressions are equal is well known*.

Accordingly, we can use Dougall's argument† to establish the identity in question.

Assume that the identity is valid for all integral values of n from 0 to $m-1$. Symmetry shows that on this assumption it is valid for any value of n provided that z has one of the values

$$0,\ -1,\ -2,\ \ldots,\ -(m-1).$$

In particular, it is valid for these values when $n = m$. But in that case the expressions on both sides of the proposed identity are of the m-th degree in z, the coefficients of z^m are equal, and the expressions are equal for m values of the variable. The identity is therefore valid for all values of z. This is for $n = m$. By mathematical induction the identity holds for all values of n.

10A. [*Added* 19 December, 1925.] Equation (10 . 1) may be brought into relation with the theory of ordinary hypergeometric series, the two sides of the equation being coefficients of the same power in two equal products.

Let $$z = 1-n-f, \quad v = 1-n-g, \quad w = 1-n-h.$$

The relation $$u+v+w-x-y-z+n = 1$$

is satisfied if

(10 . 3) $$x+y+g+h = u+f.$$

(10 . 1) becomes [by use of 3 . 37)]

(10 . 31) $$\frac{1}{n!}\,\frac{\Gamma(u+n)}{\Gamma(u)}\,\frac{\Gamma(g+n)}{\Gamma(g)}\,\frac{\Gamma(h+n)}{\Gamma(h)}\,{}_4F_3\left[\begin{matrix}x,\ y,\ 1-n-f,\ -n\\ u,\ 1-n-g,\ 1-n-h\end{matrix}\right]$$

$$= \frac{1}{n!}\,\frac{\Gamma(u+n)}{\Gamma(u)}\,\frac{\Gamma(f-g+n)}{\Gamma(f-g)}\,\frac{\Gamma(f-h+n)}{\Gamma(f-h)}\,{}_4F_3\left[\begin{matrix}u-x,\ u-y,\ 1-n-f,\ -n\\ u,\ 1-n-f+g,\ 1-n-f+h\end{matrix}\right].$$

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* J. Thomae, *Journal für Math.*, 87 (1879), 26–73, 52. See also G.G.H.S., Table IIA.

† J. Dougall, "On Vandermonde's theorem and some more general expansions", *Proc. Edinburgh Math. Soc.*, 25 (1907), 114–132.

The two sides of this equation are the coefficients of ζ^n on the two sides of

$$(10.4)\quad {}_2F_1\left[\begin{matrix}x, y;\\ u\end{matrix}\ \zeta\right] \times {}_2F_1\left[\begin{matrix}g, h;\\ f\end{matrix}\ \zeta\right]$$

$$= {}_2F_1\left[\begin{matrix}u-x, u-y;\\ u\end{matrix}\ \zeta\right] \times {}_2F_1\left[\begin{matrix}f-g, f-h;\\ f\end{matrix}\ \zeta\right].$$

This identity (10.4) is, however, an immediate deduction from the relations

$${}_2F_1\left[\begin{matrix}x, y;\\ u\end{matrix}\ \zeta\right] = (1-\zeta)^{u-x-y}\ {}_2F_1\left[\begin{matrix}u-x, u-y;\\ u\end{matrix}\ \zeta\right]$$

and

$${}_2F_1\left[\begin{matrix}g, h;\\ f\end{matrix}\ \zeta\right] = (1-\zeta)^{f-g-h}\ {}_2F_1\left[\begin{matrix}f-g, f-h;\\ f\end{matrix}\ \zeta\right].$$

the powers of $(1-\zeta)$ being reciprocals by virtue of the relation (10.3).

This analysis provides an alternative proof of the equation (10.1), the mutual transformation of Saalschützian series.

11. *A relation between two terminating well-poised series. Direct proof.*

As a typical relation between terminating well-poised series we take

$$W(1; 2) = W(3; 4).$$

From Table II and equation (3.51) we learn that this is equivalent to

$$(11.1)\quad \frac{\Gamma(1+a)}{\Gamma(1+a+n)}\,\frac{\Gamma(1+a-c+n)}{\Gamma(1+a-c)}\,\frac{\Gamma(1+a-e+n)}{\Gamma(1+a-e)}\,\frac{\Gamma(1+a-f+n)}{\Gamma(1+a-f)}$$

$$\times {}_7F_6\left[\begin{matrix}a, 1+\frac{1}{2}a, c, d, e, f, -n\\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a+n\end{matrix}\right]$$

$$= \frac{\Gamma(1+c-d-n)}{\Gamma(1+c-d)}\,\frac{\Gamma(1-d)}{\Gamma(1-d-n)}\,\frac{\Gamma(c+f-a)}{\Gamma(c+f-a-n)}\,\frac{\Gamma(c+e-a)}{\Gamma(c+e-a-n)}$$

$$\times {}_7F_6\left[\begin{matrix}a', 1+\frac{1}{2}a', c, c-a-n, 1+a-d-f, 1+a-d-e, -n\\ \frac{1}{2}a', 1-d-n, 1+a-d, c+f-a-n, c+e-a-n, 1+c-d\end{matrix}\right],$$

where $a' = c-d-n$.

For symmetry make the substitutions,

$$d = c-a'-n, \tag{11.2}$$

$$f = 1+a-d-e' = 1+a+a'+n-c-e'. \tag{11.21}$$

The identity to be verified is now

$$\frac{\Gamma(1+a)}{\Gamma(1+a+n)}\frac{\Gamma(1+a-c+n)}{\Gamma(1+a-c)}\frac{\Gamma(1+a-e+n)}{\Gamma(1+a-e)}\frac{\Gamma(c+e'-a')}{\Gamma(c+e'-a'-n)} \tag{11.3}$$

$$\times {}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ c,\ c-a'-n,\ e,\ 1+a+a'+n-c-e',\ -n \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a+a'-c+n,\ 1+a-e,\ c+e'-a'-n,\ 1+a+n \end{matrix}\right]$$

$$= \frac{\Gamma(1+a')}{\Gamma(1+a'+n)}\frac{\Gamma(1+a'-c+n)}{\Gamma(1+a'-c)}\frac{\Gamma(1+a'-e'+n)}{\Gamma(1+a'-e')}\frac{\Gamma(c+e-a)}{\Gamma(c+e-a-n)}$$

$$\times {}_7F_6\left[\begin{matrix} a',\ 1+\tfrac{1}{2}a',\ c,\ c-a-n,\ e',\ 1+a+a'+n-c-e,\ -n \\ \tfrac{1}{2}a',\ 1+a'-c,\ 1+a+a'-c+n,\ 1+a'-e';\ c+e-a-n,\ 1+a'+n \end{matrix}\right].$$

Noting that

$$\frac{\Gamma(c+e-a)}{\Gamma(c+e-a-n)} = (-1)^n \frac{\Gamma(1-c-e+a+n)}{\Gamma(1-c-e+a)},$$

we see that (11.3) is symmetrical in c and $-n$. If we multiply both sides of (11.3) by $(1+a+a'-c+n)_n$ and expand we have an equation in c of degree $3n$.

Now it is known that the expansion $W(3;4)$, when written in the reverse order is $W(4;3)$, so that c and d may be interchanged in equation (11.1). Thus, except for the order of the terms on the right, (11.3) is unaltered by the substitution of a for a, $-a'-2n$ for a', $c-a'-n$ for c, $1+a+a'+n-c-e'$ for e, and $1+a-c-e$ for e'. Similarly the terms on the left are reversed by the substitution of $-a-2n$ for a, a' for a', $c-a-n$ for c, $1+a'-c-e'$ for e, and $1+a+a'+n-c-e$ for e'. The terms on both sides are reversed by the substitution of $-a-2n$ for a, $-a'-2n$ for a', $c-a-a'-2n$ for c, $e-a-n$ for e, and $e'-a'-n$ for e'.

It follows that Dougall's method can be used again, the basis of the argument being that, if two expressions of degree $3n$ are equal for $4n$ values of the variable, they are identically equal. The $4n$ values of c are those which make either c, $c-a'-n$, $c-a-n$, or $c-a-a'-2n$ equal to zero or a negative integer not greater numerically than $n-1$.

12. *An important limiting case.*

The proposition which can be derived from (11 . 1) by letting $n \to \infty$ is interesting. As Dougall has shown* in an analogous case, the terms at the end of the series have to be considered as well as those at the beginning. It is found that, if the series in question are convergent, then

$$(12.1)\quad \frac{\Gamma(1+a)}{\Gamma(1+a-c)\,\Gamma(1+a-e)\,\Gamma(1+a-f)} \times {}_6F_5\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ c,\ d,\ e,\ f; \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f \end{matrix}\ -1\right]$$

$$= \frac{\Gamma(d-c)}{\Gamma(d)\,\Gamma(1+a-c-f)\,\Gamma(1+a-c-e)}\ {}_3F_2\left[\begin{matrix} c,\ 1+a-d-f,\ 1+a-d-e \\ 1+a-d,\ 1+c-d \end{matrix}\right]$$

$$+ \frac{\Gamma(c-d)}{\Gamma(c)\,\Gamma(1+a-d-f)\,\Gamma(1+a-d-e)}\ {}_3F_2\left[\begin{matrix} d,\ 1+a-c-f,\ 1+a-c-e \\ 1+a-c,\ 1+d-c \end{matrix}\right].$$

This equation can be deduced from the known fact† that the ${}_6F_5$ on the left is equal to

$$\frac{\Gamma(1+a-e)\,\Gamma(1+a-f)}{\Gamma(1+a)\,\Gamma(1+a-e-f)}\ {}_3F_2\left[\begin{matrix} 1+a-c-d,\ e,\ f \\ 1+a-c,\ 1+a-d \end{matrix}\right],$$

by use of one of the relations established‡ by Thomae for the ${}_3F_2$ functions.

13. *A relation between two well-poised series, terminating and non-terminating. Direct proof.*

To obtain one of the relations involving non-terminating well-poised series we apply a theorem§, due to Carlson, which states that, with

---

* *Loc. cit.*, § 12.

† W.P.S., Equation 6 . 3.

‡ Cf. G.G.H.S., Equation 10.

§ F. Carlson, "Sur une classe de séries de Taylor", *Dissertation*, Uppsala (1914). The theorem is quoted by G. H. Hardy. "A chapter from Ramanujan's notebook", *Proc. Camb. Phil. Soc.*, (1923), 492–503, 494.

certain limitations, a function of z, which vanishes when z has any of the values ζ, $\zeta+1$, $\zeta+2$, ..., is identically zero.

As a preliminary we make, in (11 . 1), the cyclic change of letters $c \to f \to e \to c$, and substitute z for n. At the same time, we bring the gamma-functions to one side of the equation. Thus we obtain

$$(13\,.\,1)\quad {}_7F_6\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ ,\ c,\ d,\ e,\ f,\ -z \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f,\ 1+a+z \end{matrix}\right]$$

$$= \frac{\begin{matrix}\Gamma(1+a+z)\,\Gamma(1+a-c)\,\Gamma(1+a-e)\,\Gamma(1+a-f)\,\Gamma(d+z) \\ \times\Gamma(d-f)\,\Gamma(1-e-f+a+z)\,\Gamma(1-c-f+a+z)\end{matrix}}{\begin{matrix}\Gamma(1+a)\,\Gamma(1+a-c+z)\,\Gamma(1+a-e+z)\,\Gamma(1+a-f+z)\,\Gamma(d) \\ \times\Gamma(d-f+z)\,\Gamma(1-e-f+a)\,\Gamma(1-c-f+a)\end{matrix}}$$

$$\times {}_7F_6\left[\begin{matrix} f-d-z,\ 1+\tfrac{1}{2}(f-d-z),\ f,\ f-a-z,\ 1+a-e-d,\ 1+a-c-d,\ -z \\ \tfrac{1}{2}(f-d-z),\ 1-d-z,\ 1+a-d,\ e+f-a-z,\ c+f-a-z,\ 1+f-d \end{matrix}\right].$$

The proof in § 11 shows that this equation is valid when z is a positive integer. We propose to show that, if $1+a-c-d$ is a negative integer, so that the series on the right terminates, then the equation is valid for any value of z, which makes the series on the left convergent.

In the first place let a, c, d, e, f be real and such that the series

$${}_6F_5\left[\begin{matrix} a,\ 1+\tfrac{1}{2}a,\ c,\ d,\ e,\ f \\ \tfrac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e,\ 1+a-f \end{matrix}\right]$$

is absolutely convergent. Then, so long as the real part of z exceeds a certain value k_1, the series on the left of (13 . 1) is also convergent and converges more rapidly than the ${}_6F_5$. The ${}_7F_6$ therefore represents an analytic function of z, and is regular and bounded as long as $R(z) > k_1$.

Again, since

$$\frac{\Gamma(1+a+z)\,\Gamma(d+z)\,\Gamma(1-e-f+a+z)\,\Gamma(1-c-f+a+z)}{\Gamma(1+a-c+z)\,\Gamma(1+a-e+z)\,\Gamma(1+a-f+z)\,\Gamma(d-f+z)} \to 1,$$

when $R(z) \to \infty$, it follows that, for all values of z large enough to exclude the poles of the gamma functions, say, when $R(z) > k_2$, this expression is less than some finite limit. Hence the multiplier of ${}_7F_6$ on the right of (13 . 1) is also finite as long as $R(z) > k_2$.

Since $1+a-c-d$ is a negative integer this ${}_7F_6$ terminates. More-

over, the individual terms are those of

$$ {}_3F_2\left[\begin{matrix} f, 1+a-e-d, 1+a-c-d \\ 1+a-d, 1+f-d \end{matrix}\right], $$

each multiplied by a finite number of expressions of the form

$$ X_r \equiv \frac{(f-d-z+r)(f-d-z+2r+2)(f-a-z+r)(r-z)}{(f-d-z+2r)(1-d-z+r)(e+f-a-z+r)(c+f-a-z+r)}. $$

When r is fixed and $R(z)\to\infty$, $X_r\to 1$ and, as long as $R(z)$ exceeds a certain value k_3, X_r will be finite. It follows that, with the same condition, the ${}_7F_6$ in question is a regular and bounded function of z. Hence, if $R(z)$ exceeds the greater of the quantities k_2 and k_3, the complete product on the right of (3 . 1) is also regular and bounded. It has been proved that the left of (3 . 1) satisfies that condition if $R(z) > k_1$.

Accordingly the difference between the two sides of (13 . 1) is regular and bounded if $R(z)$ exceeds the greatest of the three quantities k_1, k_2, and k_3.

This difference is known to vanish when z is a positive integer, and hence, by Carlson's theorem, it is identically zero, so that the equality is established.

For symmetry we write g in place of $-z$, and at the same time we substitute $-m$ for $1+a-d-c$, which is, by hypothesis, a negative integer. We get

(13 . 2)

$$ {}_7F_6\left[\begin{matrix} c+d-m-1,\ 1+\tfrac{1}{2}(c+d-m-1),\ c,\ d,\ e,\ f,\ g \\ \tfrac{1}{2}(c+d-m-1),\ d-m,\ c-m,\ c+d-e-m,\ c+d-f-m,\ c+d-g-m \end{matrix}\right] $$

$$ = \frac{\begin{matrix}\Gamma(c+d-g-m)\,\Gamma(d-m)\,\Gamma(c+d-e-m)\,\Gamma(c+d-f-m)\,\Gamma(d-g) \\ \times\Gamma(d-f)\,\Gamma(c+d-e-f-g-m)\,\Gamma(d-f-g-m)\end{matrix}}{\begin{matrix}\Gamma(c+d-m)\,\Gamma(d-g-m)\,\Gamma(c+d-e-g-m)\,\Gamma(c+d-f-g-m) \\ \times\Gamma(d)\,\Gamma(d-f-g)\,\Gamma(c+d-e-f-m)\,\Gamma(d-f-m)\end{matrix}} $$

$$ \times {}_7F_6\left[\begin{matrix} f+g-d,\ 1+\tfrac{1}{2}(f+g-d),\ f,\ f+g-c-d+m+1,\ c-e-m,\ g,\ -m \\ \tfrac{1}{2}(f+g-d),\ 1-d-g,\ c-m, \\ 1+f+g+e-c-d+m,\ 1+f-d,\ 1+f+g+m-d \end{matrix}\right]. $$

In the notation of § 9 and Table III, this is

$$ W(4;\ 5,\ 6) = W(1;\ 5). $$

It has been proved to be true when the parameters c, d, e, f are real and the ${}_6F_5$ obtained by dropping the parameters containing g in the ${}_7F_6$ on the left is an absolutely convergent series. By analytical continuation the result must be valid without these restrictions, so long as the ${}_7F_6$ on the left is convergent. This is what we set out to prove.

14. *Conclusion.*

The relation (13 . 2) has been established by an argument in which no reference is made to Saalschützian series. The theory of the transformation of well-poised series is therefore placed on an independent footing, just as that of Saalschützian series was in § 10.

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